Modeling the Nonlinear Rheological Behavior of Materials with a Hyper-Exponential Type Function

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Abstract

This paper describes a nonlinear rheological model consisting of a modified and extended classical Voigt model for predicting the time dependent deformation of a variety of viscoelastic materials exhibiting elastic, viscous and inertial nonlinearities simultaneously. The usefulness of the model is illustrated by numerical examples.

Keywords: Hyper-exponential function, Lambert-type equation, Mathematical modeling, Nonlinear time dependent deformation, Viscoelasticity, Voigt model

1. Introduction

In viscoelasticity, mathematical models are required for studying the time dependent properties of materials under various loading conditions. In the characterization of materials, the well known established linear theory of viscoelasticity is only valid for small deformations or low stresses. When the material undergoes large deformations the linear theory becomes inapplicable, and nonlinear models are needed. Contrary to the linear theory of viscoelasticity that is usually described in the Boltzmann single integral or in the differential form, a standard framework does not exist in nonlinear viscoelasticity. Therefore, nonlinear mathematical rheological models are often constructed by through modifications and extensions to higher order stress or strain terms of the linear theory. From a mathematics point of view, the integral representation of viscoelastic constitutive equation is more difficult to perform than the differential form. Thus, several models with various complexities have been developed for describing the nonlinear behavior of these materials that are characterized by elastic, viscous and inertial nonlinear contributions (Bauer et al. 1979; Bauer 1984). However, in these models, due to the mathematical complications, only the elastic or viscous nonlinearity is often taken into account (Monsia 2011a,b,c) and the inertial contribution is ignored. Moreover, there are only a few theoretical models formulated with constant-value rheological material parameters. Therefore, nonlinear models with constant rheological coefficients are required. Following this viewpoint, by using a second-order elastic spring in series with a classical Voigt element, that is an extended form of the standard linear solid to finite strains, Monsia (2011a) formulated a hyperlogistic-type equation to reproduce the nonlinear time dependent stress response of some viscoelastic materials. Recently, in Monsia (2011b), a single differential constitutive equation derived from a standard nonlinear solid model consisting of a polynomial elastic spring in series with a classical Voigt element for the prediction of time dependent nonlinear stress of a class of viscoelastic materials is developed. More recently, in Monsia (2011c), a nonlinear four-parameter rheological Voigt model consisting of a nonlinear Voigt model in series with a classical linear Voigt element with constant material coefficients for representing the nonlinear stiffening response of the initial low-load portion and the softening behavior of some viscoelastic materials is formulated. In Monsia (2011d) the elastic and viscous nonlinearities are taken simultaneously into consideration through a simple nonlinear generalized Maxwell fluid model consisting of a nonlinear spring connected in series with a nonlinear dashpot obeying a power law with constant material coefficients. According to Bauer (1984), suitable constitutive equations of viscoelastic materials must relate stress, strain and their higher time derivatives, to say, must take into consideration the elastic, viscous and inertial nonlinearities simultaneously. To overcome the mathematical complexities in viscoelastic modeling, Bauer (1984) developed, for a complete characterization of rheological properties of arterial walls, a theory based on the classical Voigt model. The Bauer's theory (1984) is aimed to give satisfactorily and simultaneously account of strong elastic, viscous and inertial nonlinearities characterizing a viscoelastic material. The method consists essentially to decompose the total stress exciting the material as the sum of three components, that is to say, the elastic, viscous and inertial stresses and to express the pure elastic stress as a nonlinear function of deformation. The pure viscous and inertial stresses are then formulated as a first and second time derivatives of a similar function of deformation to the nonlinear elastic function, respectively. A fundamental theoretical difficulty in the use of the Bauer's theory (1984) consists of the determination of appropriate nonlinear elastic restoring force function that tends towards the expected linear elastic behavior for small deformations. In the Bauer's study (1984), the pure elastic stress is expanded in a power series of strain, the pure viscous stress is developed as a first time derivative of a similar power series of strain, and the pure inertial stress is expressed as a second time derivative of a similar power series of strain. The Bauer's stress decomposition method (1984), consisting to express the stress as a sum of three elementary stresses, has been after used by many authors (Armentano et al. 1995; Gamero et al. 2001) for a complete characterization of arterial behavior. Recently, Monsia (2011e), using the Bauer's method (1984), developed a hyperlogistic equation that is useful for representing the time dependent behavior of some viscoelastic materials. In Monsia (2011e), following the Bauer's theory (1984), the pure elastic stress is developed in an asymptotic expansions in powers of deformation, the pure viscous stress is expressed as a first time derivative of a similar asymptotic expansions in powers of deformation, and the inertial stress is formulated as a second time derivative of a similar asymptotic expansions in powers of deformation.

In this work, using a hyperbolic function of deformation for the pure elastic constitutive relationship, we developed following the Bauer's theory (1984) a one-dimensional nonlinear theoretical rheological model with constant material coefficients taking into account all together elastic, viscous and inertial nonlinearities characterizing viscoelastic materials. The theoretical obtained results show that the model can be successfully applied to represent the nonlinear time deformation of some viscoelastic materials. Numerical examples are performed to illustrate the effects of rheological parameters action on the material response.

2. Formulation of the Mathematical Model

2.1 Theoretical considerations

In this part we develop the governing equations including the elastic, viscous and inertial nonlinearities with material constant coefficients. Most viscoelastic materials exhibit elastic, viscous and inertial nonlinearities, so that their mechanical responses are nonlinear time dependent, and require advanced mathematical models for their description. To that end, the use of nonlinear viscoelasticity is suggested (Bauer 1984). To construct then our proposed one-dimensional rheological model, in which the stresses and strain are scalar functions, we will use the Bauer's theory (1984) that brings significant modifications and extensions to the classical mechanical Kelvin-Voigt model for overcoming the preceding mentioned difficulties. The first significant modification herein consists to introduce a nonlinear restoring spring force function $\varphi(\varepsilon)$ of deformation ε for capturing the pure elastic component of the stress induced in the material studied. Then, for a general formulation, the pure elastic constitutive equation may be written in the form

$$\sigma_e = a\varphi(\varepsilon) \tag{1}$$

where a is a stiffness coefficient. The second important modification involves the fact that the pure nonlinear viscous constitutive equation is then directly given by

$$\sigma_v = \frac{d}{dt}\left[b\varphi(\varepsilon)\right] \tag{2}$$

where b is a viscosity module. The third significant modification consists to derive from $\varphi(\varepsilon)$ the nonlinear inertial stress component as

$$\sigma_i = \frac{d^2}{dt^2}\left[c\varphi(\varepsilon)\right] \tag{3}$$

in which c is an inertia module. The coefficients *a, b* and *c* are time independent material parameters. Thus, noting σ_t the total stress due to the external exciting force acting on the material, the superposition of elastic stress, viscous stress and inertial stress is given by

$$\ddot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\dot{\varepsilon}^2\frac{d^2\varphi}{d\varepsilon^2}+\frac{b}{c}\dot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\frac{a}{c}\varphi(\varepsilon)=\frac{1}{c}\sigma_t \tag{4}$$

where the dot over the symbol denotes a differentiation with respect to time and the inertial module c is different from zero. Equation (4) determines the differential constitutive relationship between the total external stress σ_t and the resulting strain $\varepsilon(t)$ for a given nonlinear function $\varphi(\varepsilon)$. It is required, at this stage of the model-building, to specify the nonlinear function $\varphi(\varepsilon)$. In this paper, the nonlinear spring force function $\varphi(\varepsilon)$ is chosen as the following hyperbolic law

$$\varphi(\varepsilon)=\frac{\varepsilon}{\varepsilon-1} \tag{5}$$

Thus, using Equation (5), Equation (4) may be written as

$$\ddot{\varepsilon}-\frac{2}{\varepsilon-1}\dot{\varepsilon}^2+\frac{b}{c}\dot{\varepsilon}-\frac{a}{c}\varepsilon(\varepsilon-1)=-\frac{1}{c}(\varepsilon-1)^2\sigma_t \tag{6}$$

Equation (6) denotes a second-order nonlinear ordinary differential equation in $\varepsilon(t)$ for a given total stress σ_t.

2.2 Dimensionalization

Noting that the strain $\varepsilon(t)$ is a dimensionless quantity, the above material parameters used in Equation (6) have then the following dimensions. Noting also *M, L* and *T* the mass, length and time dimension respectively, the dimension of the stress becomes $ML^{-1}T^{-2}$. Therefore, the dimension of *a* is given by $ML^{-1}T^{-2}$, that of *b* varies as $ML^{-1}T^{-1}$, and that of *c* varies as ML^{-1} (mass per unit length).

2.3 Solution using a stress $\sigma_t=0$

2.3.1 Evolution equation of the deformation $\varepsilon(t)$

In the absence of external exciting stress ($\sigma_t=0$), the internal dynamics of the viscoelastic material studied can be represented by the following nonlinear ordinary differential equation

$$\ddot{\varepsilon}-\frac{2}{\varepsilon-1}\dot{\varepsilon}^2+\frac{b}{c}\dot{\varepsilon}-\frac{a}{c}\varepsilon(\varepsilon-1)=0$$

or

$$\ddot{\varepsilon}-\frac{2}{\varepsilon-1}\dot{\varepsilon}^2+\lambda\dot{\varepsilon}-\omega_o^2\varepsilon(\varepsilon-1)=0 \tag{7}$$

where $\lambda=\frac{b}{c}$, and $\omega_o^2=\frac{a}{c}$.

Equation (7) represents the time dynamics equation of the strain $\varepsilon(t)$ induced in the material under the external exciting stress fixed to zero. By introducing the auxiliary variable

$$x=\varepsilon-1 \tag{8}$$

Equation (7) transforms, after some algebraic manipulations, into

$$\ddot{x}-2\frac{\dot{x}^2}{x}+\lambda\dot{x}-\omega_o^2x-\omega_o^2x^2=0 \tag{9}$$

In Equation (9), the first term is proportional to the basic inertial stress, the second to a nonlinear quadratic viscous stress, the third term to the linear viscous stress, the fourth term to the classical linear elastic stress and the last term to a quadratic nonlinear elastic stress. Equation (9) is a Lambert-type differential equation which can be analytically solved by using an appropriate change of variable and suitable boundary and initial conditions that satisfy the time dynamics of the viscoelastic material considered.

2.3.2 Solving time dependent deformation equation

For solving Equation (9), a novel change of variable is required. Making the following substitution

$$x = y^{-1} \tag{10}$$

with $y \neq 0$, Equation (9) transforms, after a few algebraic manipulations as

$$\ddot{y} + \lambda \dot{y} + \omega_o^2 y = -\omega_o^2 \tag{11}$$

Equation (11) is the well-known second-order linear ordinary differential equation with a right-hand member different from zero. Integration yields for $y(t)$ the following solution

$$y(t) = A_1 \exp(r_1 t) + A_2 \exp(r_2 t) - 1 \tag{12}$$

where

$$r_1 = -\frac{\lambda}{2}(1-\delta)$$

and

$$r_2 = -\frac{\lambda}{2}(1+\delta)$$

are the two negative real roots of the characteristic equation

$$r^2 + \lambda r + \omega_o^2 = 0$$

with

$$\delta = \sqrt{1 - 4\frac{\omega_o^2}{\lambda^2}}$$

A_1 and A_2 are two integration constants determined by the initial conditions. Thus, using the suitable initial conditions

$$t = 0,\ \varepsilon(t) = 0$$

and

$$t = 0,\ \dot{\varepsilon}(t) = f_o$$

where the coefficient f_o denotes the initial strain rate, and taking into consideration Equation (8) and Equation (10), the desired strain versus time relationship can be written as

$$\varepsilon(t) = 1 - \frac{1}{1 - \dfrac{f_o}{\lambda\sqrt{1 - 4\dfrac{\omega_o^2}{\lambda^2}}}\left[\exp(-\dfrac{\lambda}{2}(1+\sqrt{1-4\dfrac{\omega_o^2}{\lambda^2}})t) - \exp(-\dfrac{\lambda}{2}(1-\sqrt{1-4\dfrac{\omega_o^2}{\lambda^2}})t)\right]} \tag{13}$$

Equation (13) describes mathematically the time dependent strain in the viscoelastic material under study. It predicts the strain versus time relationship of the material studied as a hyper-exponential type function that asymptotically approaches a maximum value with increasing time.

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3. Numerical Results and Discussion

In this section some numerical examples are performed to illustrate the predictive quality of the model for representing the mechanical behavior of the material considered and the influence of rheological coefficients action on the material response.

Since viscoelastic materials are characterized by high elastic, viscous and inertial nonlinearities, advanced analytical formulations are needed for determining and predicting accurately their properties. In this regard the Bauer's theory (1984) becomes an important mathematical tool in viscoelastic modeling. For the present model, the nonlinear restoring spring force function is expressed as a hyperbolic law. The hyperbolic law has been extensively used in soils constitutive modeling. The hyperbolic function proposed here is inspired by the work (Prevost and Keane 1990). This law gives the advantage to be an exact analytical expression, compared with the asymptotic series used in Monsia (2011e). Moreover, the choice of Equation (5) proceeds from the fact that when $\varepsilon << 1$, Equation (5) reduces to a geometric series formula, so that for small deformations, $\varphi(\varepsilon)$ shows linear behavior as expected. In this respect, the proposed hyperbolic law agrees well with the power series of deformation used by Bauer (1984). Another feature that involves the advantage of Equation (5) is that $\varphi(\varepsilon)$ becomes infinite force for a finite deformation , that is, for $\varepsilon \rightarrow 1$, in other words, for a strain range of about 100 %, like the nonlinear elastic FENE (Finitely Extensible Nonlinear Elastic) spring force, contrary to polynomial restoring force in which the deformation ε should tend also towards infinity. For $\varepsilon \rightarrow +\infty$, that is, for $\varepsilon >> 1$, $\varphi(\varepsilon)$ tends towards a finite value. By applying the Bauer's theory (1984) and using the preceding hyperbolic restoring force function, the governing equation obtained is a Lambert-type nonlinear differential equation. By considering then suitable change of variable and initial conditions, this equation led to obtain the time versus strain relationship as a hyper-exponential type function, demonstrating that the present model is applicable to represent mathematically the mechanical behavior of a variety of viscoelastic materials. However, more experimental results are required to further verify the feasibility of the model.

4. Conclusions

A nonlinear rheological model with constant material coefficients, taking into account all together elastic, viscous and inertial nonlinearities, is developed. The theoretical obtained results showed that the model can be successfully applied to represent the nonlinear time deformation behavior of some viscoelastic materials. Numerical results illustrated also the effects of rheological parameters action on the deformation response of the material studied, and particularly the sensitivity of the model to the initial rate of application of the strain. The present uniaxial model can act as a starting point to a more general three dimensional model, but this study will be done as future work.

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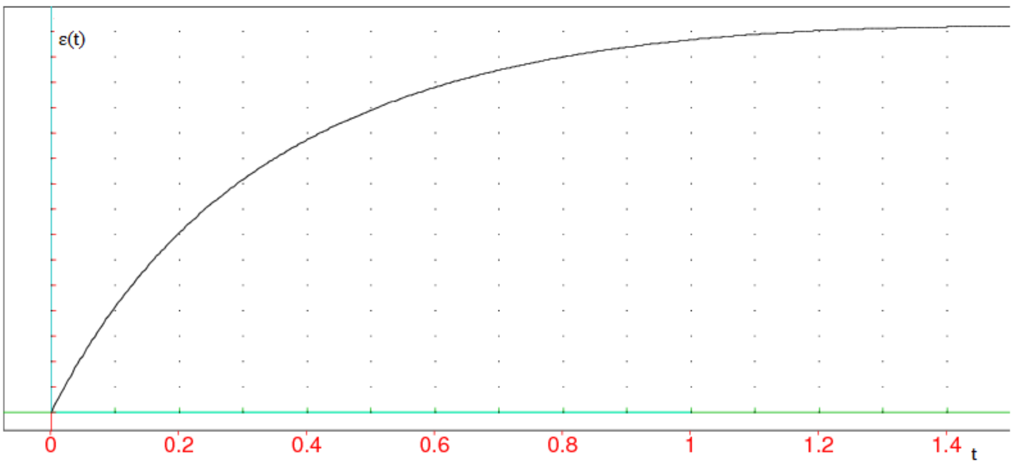

Figure 1. Typical strain versus time curve

Figure 1 shows the typical time versus strain variation with an increase until a maximum value, obtained from Equation (13) with the fixed value of parameters at $\lambda = 2$, $\omega_o = 0.5$, $f_o = 1$. It can be seen from Figure 1 that the proposed model is able to reproduce mathematically and accurately the typical exponential deformation response of a variety of viscoelastic materials, for example, soft living tissues (Lesecq et al., 1997). The model predicts a time dependent response in which the slope declines gradually with increasing time until the failure point at which the slope reduces to zero.

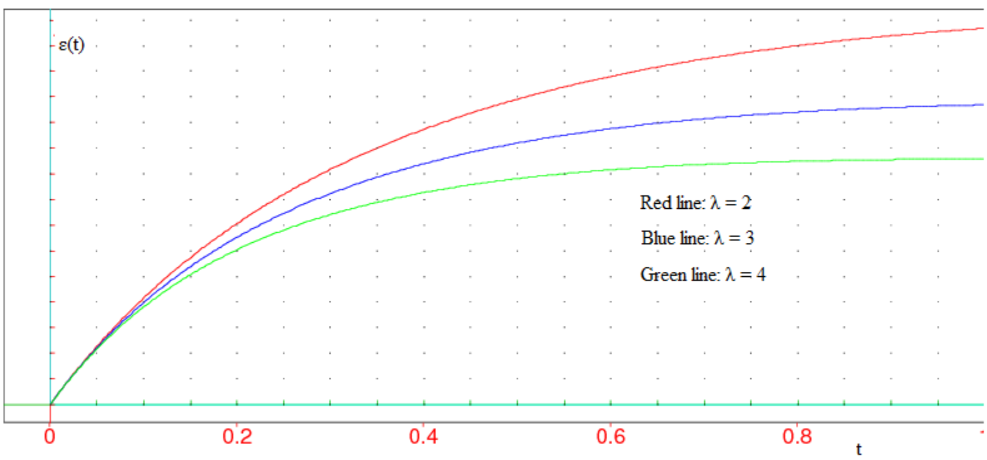

Figure 2. Strain versus time curves showing the effect of the coefficient λ

Figure 2, 3 and 4, illustrates the effects of material coefficients on the time dependent strain response of the material studied. These effects are studied by varying one coefficient while the other two are kept constant. Figure 2 exhibits the effect of the damping coefficient λ change on the strain-time relationship. The graph shows that an increasing λ, reduces the value of the strain on the time period considered. The slope also decreases with increase λ. The red line corresponds to $\lambda = 2$, the blue line to $\lambda = 3$, and the green line to $\lambda = 4$. The other parameters are $\omega_o = 0.5$, $f_o = 1$.

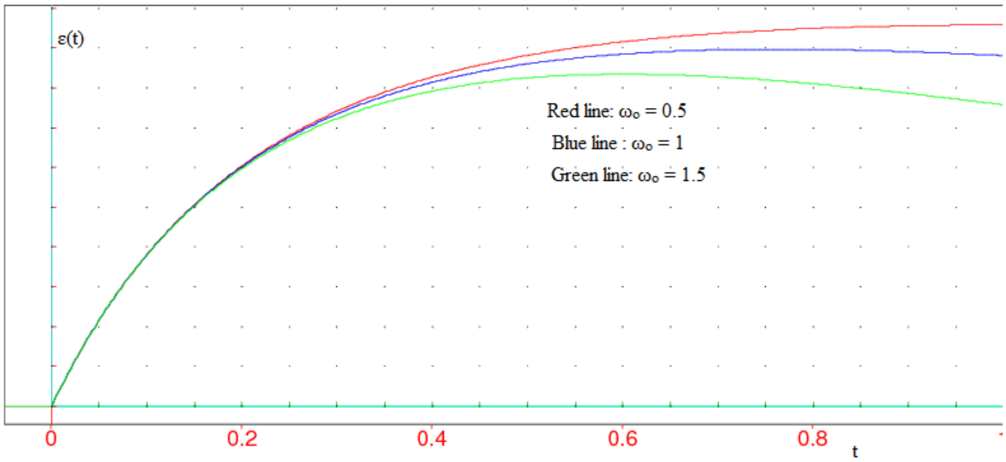

Figure 3. Strain-time curves at three various values of coefficient ω_o

As shown in Figure 3, an increase of the natural frequency coefficient ω_o, decreases also the value of the strain on the time period considered. The slope decreases slowly in the early periods of time with increase ω_o. The red line corresponds to $\omega_o = 0.5$, the blue line to $\omega_o = 1$, and the green line to $\omega_o = 1.5$. The other parameters are $\lambda = 4$, $f_o = 1$.

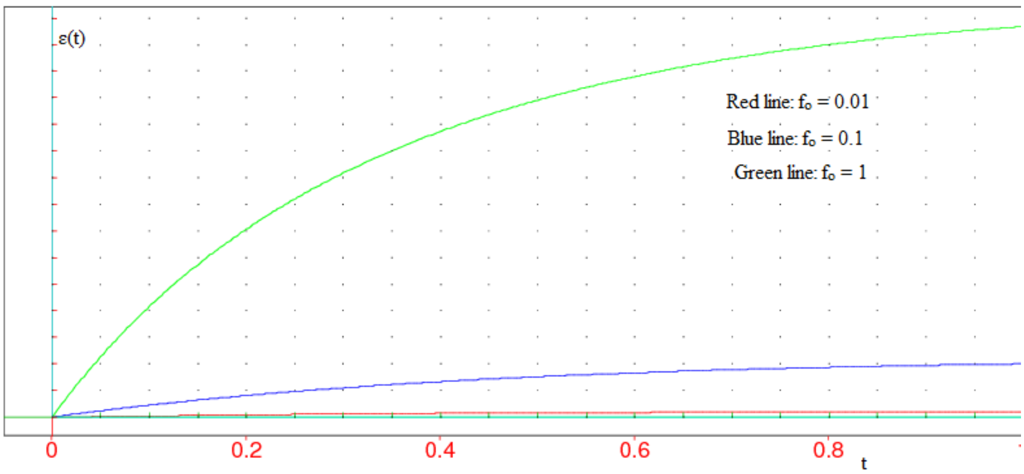

Figure 4. Strain-time curves for three different values of the parameter f_o

Figure 4 exhibits how the initial rate of application of the strain f_o affects the time dependent response of the material studied. The graph shows that the curves become more linear with decreasing f_o. The slope decreases also with decreasing f_o. The red line corresponds to $f_o = 0.01$, the blue line to $f_o = 0.1$, and the green line to $f_o = 1$. The other parameters are $\lambda = 2$, $\omega_o = 0.5$.