Almost Sure Convergence of Pair-wise NQD Random Sequence

Yanchun Wu
College of Science, Guilin University of Technology
Guilin 541004, China
Tel: 86-137-377-26466 E-mail: wyc@glite.edu.cn

Dawang Wang
College of Science, Guilin University of Technology
Guilin 541004, China
Tel: 86-150-773-09717 E-mail: nytcdw@163.com

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Abstract

In this paper, considering the sequences of pairwise NQD that is applied broadly through introducing the slowly varying function, we extend a series of conclusions.

Keywords: Pairwise NQD random sequences, Almost sure, Slowly varying function

1. Introduction and Lemmas

Definition 1(Lehmann E L. 1966): Let random variables X and Y are said to be NQD (Negatively Quadrant Dependent), if $\forall x, y \in R$,

$$P(X < x, Y < y) \le P(X < x)P(Y < y).$$

Where $\forall\ i \ne j$, X and Y are said to be NQD (Negatively Quadrant Dependent), A sequence of random variables $\{X_n; n \ge 1\}$ is said to be pairwise NQD.

This definition was introduced by Lehmann. Obviously, Pairwise NQD random sequences contains a kind of negatively correlative sequences of NA, LNQD, ND random variables. For NA random sequences, a number of writers have obtained as same as the convergence property in many independent conditions, the properties of limit behavior of LNQD, ND sequences seldom appear in literature. Matula (Matula P. 1992) obtained strong law of large numbers for Kolmogorov as same as them under the independent, the distinguishing theorem of the Complete Convergence of Baum and Kata(WANG Y B, Chun Su, & Xuguo Liu. 1998) was obtained by author Yuebao Wang who considered $\varphi^*(1) < 1$, Qunying Wu (WU Q Y. 2005) obtained the weak law of large numbers and the criteria theorem of Complete Convergence of Baum and Kata (WU Q Y. 2002) on the condation that pairwise NQD. These properties achieved the results of independence condition, and the results of properties of Jamison(Jamsion B, O Rey S, & Pruitt W. 1965) Weighted Sums is obtained.

Currently, a number of writers have studied a series of useful results of the limit of pairwise NQD random Sequence. Yuebao Wang (WANG Y B, YAN Ji-gao, & CHENG Feng-yang etal. 2001) studied the strong stability of different distribution pairwise NQD, Wancheng Gao(WAN Cheng-gao. 2005) studied f the weak law of large numbers of pairwise NQD and in condition of rank 2 Cesàro's uniformly integrability is the convergence properties of L_r and Yanping Chen(CHEN Ping-yan. 2008) studied the convergence properties of L_r satisfied pairwise NQD in the condition of uniformly integrability is $r(1 < r < 2)$ Cesàro's uniformly integrability as same as them under the condition of independent, and the teacher Yuebao Wang Wang(WANG Y B, YAN J G, & CAI X Z. 2001) studied different distributions strong stability of pair-wise NQD, and Qunying Wu(WU Q Y. 2002) studied the three series theorem of pair-wise NQD.

In the following, Almost Sure Convergence of NQD pair-wise random Sequences are extened on the condation that slowly varying function.

Definition 2(WU Q Y. 2002): Let $l(x)$ be a Positive Function $\forall x \in [0, \infty)$, which satisfies $x \to +\infty$, then

$\lim_{x\to+\infty}\frac{l(cx)}{l(x)}=1$, for all $c>0$.

For positive functions, as we know, if there exist a Positive Function such as $l(x)>0$ with $x\to+\infty$, then

(1) $\lim_{x\to+\infty}\frac{l(tx)}{l(x)}=1,\ t>0,\ \lim_{x\to+\infty}\frac{l(x+u)}{l(x)}=1,\ \forall u\ge 0$.

(2) $\lim_{k\to+\infty}\sup_{2^k\le x<2^{k+1}}\frac{l(x)}{l(2^k)}=\lim_{k\to+\infty}\inf_{2^k\le x<2^{k+1}}\frac{l(x)}{l(2^k)}=1$.

(3) $\lim_{x\to+\infty}x^{\delta}l(x)=\infty,\ \lim_{x\to+\infty}x^{-\delta}l(x)=0,\ \forall\delta>0$.

Definition 3(WU Q Y. 2005): Suppose that $\{X_n,X;n\ge 1\}$ is random variables of L_p, if $\lim_{n\to+\infty}E|X_n-X|^p=0$, then a sequence of random variables $\{X_n\}$ is advanced at a p-th average of Convergence where $L_p=\left\{X;X\ \ is\ \ \text{r.v.},and\ \ E|X|^p<\infty\right\}$, for $\forall 0<p<\infty$. In other words $X_n\xrightarrow{L_p}X$.

In this paper, c is usually said to be different real constants and $l(x)$ is the slowly varying function.

Lemma 1(Lehmann E L. 1966) Assume that random variables X and Y are NQD ,
Then

(1) $EXY\le EXEY$

(2) $P(X>x,Y>y)\le P(X>x)P(Y>y),\ x,y\in R$

(3)If the functions of r and s with non descending(non incremental),then $r(X)$ and $s(Y)$ are NQD.

Lemma 2: Assume that $g(a,k)$ is the function of Joint Distribution $(a\ge 0,k\ge 1)$ of $X_{a+1},X_{a+2},\cdots,X_{a+k}$ that satisfied:

$$f(a,k)+f(a+k,m)\le f(a,k+m),1\le k<k+m,a\ge 0,\qquad\ldots\ldots\ldots\ldots(1)$$

$$E(\sum_{i=a+1}^{a+n}X_i)^2\le f(a,n),n\ge 1,a\ge 0,\qquad\ldots\ldots\ldots\ldots(2)$$

If there exists the slowly varying function $l(x)$

Such that $l(t)+l(n)\le l(tn)$, for $\forall t>0$, we can easily get the following results such that

$$E(\max_{a\le k\le n}\left|\sum_{i=a+1}^{a+n}X_i\right|)^2\le[\frac{l(tn)}{l(n)}]^2f(a,n)$$ (t is even)

Proof. Because of randomicity of a, using a fixed. If t is even, either let t=2,we have Mathematic Induction , if n=1, we get

$E(X_i)^2\le f(a,1)$

Assume that there exists $n<N$, then two conditions of $1\le n\le\frac{N}{2}$ and $\frac{N}{2}\le n\le N$ are satisfied by N is even .

Set $M_{a,n}=\max_{a\le k\le n}\left|\sum_{i=a+1}^{a+k}X_i\right|$.

If $1\le n\le\frac{N}{2}$, then

$$(\sum_{i=a+1}^{a+n} X_i)^2 = M^2_{a,\frac{N}{2}}$$

If $\frac{N}{2} \le n \le N$, then

$$(\sum_{i=a+1}^{a+n} X_i)^2 = (\sum_{i=a+1}^{a+\frac{N}{2}} X_i + \sum_{i=a+\frac{N}{2}+1}^{a+n} X_i)^2$$

$$= (\sum_{i=a+1}^{a+\frac{N}{2}} X_i)^2 + 2(\sum_{i=a+1}^{a+\frac{N}{2}} X_i)(\sum_{i=a+\frac{N}{2}+1}^{a+n} X_i) + (\sum_{i=a+\frac{N}{2}+1}^{a+n} X_i)^2$$

$$\le M^2_{a,\frac{N}{2}} + 2\left|\sum_{i=a+1}^{a+\frac{N}{2}} X_i\right| M_{a+\frac{N}{2},\frac{N}{2}} + M^2_{a+\frac{N}{2},\frac{N}{2}}$$

Therefore

$$M^2_{a,N} \le M^2_{a,\frac{N}{2}} + 2\left|\sum_{i=a+1}^{a+\frac{N}{2}} X_i\right| M_{a+\frac{N}{2},\frac{N}{2}} + M^2_{a+\frac{N}{2},\frac{N}{2}}$$

Applying on both sides of expectations by inequation of Cauchy-Schwarz.

$$EM^2_{a,N} \le (\frac{L(N)}{L(2)})^2 f(a,\frac{N}{2}) + 2E(\left|\sum_{i=a+1}^{a+\frac{N}{2}} X_i\right| M_{a+\frac{N}{2},\frac{N}{2}}) + (\frac{L(N)}{L(2)})^2 f(a+\frac{N}{2},\frac{N}{2})$$

$$\le (\frac{L(N)}{L(2)})^2 f(a,\frac{N}{2}) + 2E^{\frac{1}{2}}(\sum_{i=a+1}^{a+\frac{N}{2}} X_i)^2 E^{\frac{1}{2}}(M^2_{a+\frac{N}{2},\frac{N}{2}}) + (\frac{L(N)}{L(2)})^2 f(a+\frac{N}{2},\frac{N}{2})$$

$$\le (\frac{L(N)}{L(2)})^2 f(a,\frac{N}{2}) + 2\frac{L(N)}{L(2)} f^{\frac{1}{2}}(a,\frac{N}{2}) f^{\frac{1}{2}}(a+\frac{N}{2},\frac{N}{2}) + (\frac{L(N)}{L(2)})^2 f(a+\frac{N}{2},\frac{N}{2})$$

$$\le (\frac{L(N)}{L(2)})^2 f(a,\frac{N}{2}) + \frac{L(N)}{L(2)}(f(a,\frac{N}{2}) + f(a+\frac{N}{2},\frac{N}{2})) + (\frac{L(N)}{L(2)})^2 f(a+\frac{N}{2},\frac{N}{2})$$

$$\le [\frac{L(N)}{L(2)} + (\frac{L(N)}{L(2)})^2][f(a,\frac{N}{2}) + f(a+\frac{N}{2},\frac{N}{2})]$$

$$\le [\frac{L(N)}{L(2)} + (\frac{L(N)}{L(2)})^2] f(a,N)$$

$$= \frac{L(N)[L(2)+L(N)]}{L^2(2)} f(a,N)$$

$$\le \frac{L(N)L(2N)}{L^2(2)} f(a,N)$$

$$\to \frac{L^2(2N)}{L^2(2)} f(a,N), (N \to \infty)$$

Therefore , if N =even, then

$$E(\max_{a\le k\le n}\left|\sum_{i=a+1}^{a+k}X_i\right|)^2\le[\frac{L(2N)}{L(2)}]^2 f(a,N)$$

or conditions of $1\le n\le\frac{N+1}{2}$ and $\frac{N+1}{2}\le n\le N$ are satisfied by N is even.

Consequence, the conclusion is satisfied by mathematical Induction,
where $\forall t\in$ even, $a\ge 0, n\ge 1$.

Infer. Let $\{X_n;n\ge 1\}$ are pairwise NQD random sequences, $EX_n=0, EX_n^2<\infty,\ T_j(k)\square\sum_{i=j+1}^{j+k}EX_i^2$,for $j\ge 0$

Then

$$E(T_j(k))^2\le\sum_{i=j+1}^{j+k}EX_i^2$$

$$E(\max_{1\le k\le n}(T_j(k))^2)\le cl^2(n)\sum_{i=j+1}^{j+n}EX_i^2 .$$

Proof . Since (1) holds

$$E(T_j(k))^2\le\sum_{i=j+1}^{j+k}EX_i^2+2\sum_{j+1\le i<r\le j+k}EX_iEX_r$$

$$=\sum_{i=j+1}^{j+k}EX_i^2\square f(j,k)$$

and $f(j,k)+f(j+k,m)=f(j,k+m), m\ge 1$ holds by lemma 2,
we have

$$E(\max_{1\le k\le n}(T_j(k))^2)\le(\frac{l(tn)}{l(t)})^2\sum_{i=j+1}^{j+n}EX_i^2\le cl^2(n)\sum_{i=j+1}^{j+n}EX_i^2$$

2. Main results and the proofs.

Theorem 1: Let $f(a,k)$ and $g(a,k)$ are the functions of Joint Distribution $(a\ge 0, k\ge 1)$ of $X_{a+1}, X_{a+2},\cdots,X_{a+k}$ that satisfied and $f(a,k)$ is satisfied with supposed that(1)and(2). Assume:

$$g(a,k)+g(a+k,m)\le g(a,k+m), 1\le k<k+m, a\ge 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$g(a,n)\le c<\infty, n\ge 1, a>0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

There exist that $l(x)$ is the slowly varying function,such that

$$f(a,n)\le c\frac{g(a,n)}{l^2(a)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\sum_{k=1}^{\infty}\frac{1}{l^2(k)}<\infty \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Therefore S_n is Almost Sure Convergence.

Note if $\exists r.v.S$, then $S_n\to S.a.s.$ where $S_n=\sum_{k=1}^{n}X_i$, $S=\sum_{k=1}^{\infty}X_i$.

Proof. From(6), we have $\frac{1}{l^2(k)}\to 0$, $k\to\infty$.

From(2)(4)(5), we get

$$E(\sum_{i=a+1}^{a+n}X_i)^2\le f(a,n)\le c\frac{g(a,n)}{l^2(a)}\le\frac{c^2}{l^2(a)}\to 0 \ , a\to\infty .$$

Hence, $\{S_n; n \geq 1\}$ is a sequence of Cauchy satisfying L_2, because of completeness of L_2, there exist a $r.v.S$ satisfied that

$$ES^2 < \infty, E(S_n - S)^2 \to 0$$

For sub-sequence method ,we firstly prove that $S_{2^k} - S.a.s.$

By Markov inequation

$$P(\left|S_{2^k} - S\right| > \varepsilon) \leq E(S_{2^k} - S)^2 / \varepsilon^2$$

According for the necessary conditions of progression convergence,

then we prove $\sum_{k=1}^{\infty} E(S_{2^k} - S)^2 < \infty$.

Applying the properties of Lemma2 and (4)(5)and the slowly varying function, we have

$$E(S_{2^k} - S)^2 = \limsup_{n \to +\infty} E(S_n - S_{2^k})^2$$

$$\leq \limsup_{n \to +\infty} g(2^k, n-2)$$

$$\leq c \limsup_{n \to +\infty} g(2^k, \frac{n-2}{l^2(2^k)})$$

$$\leq \frac{c^2}{l^2(2^k)}$$

Then we have

$$\sum_{k=1}^{\infty} E(S_{2^k} - S)^2 \leq c^2 \sum_{k=1}^{\infty} \frac{1}{l^2(2^k)} < \infty$$

by the B-C lemma , we obtain

$$E(S_{2^k} - S)^2 \to 0$$

Therefore

$$S_{2^k} \to S.a.s.$$

Thus , according Sub-sequence method, we shall prove

$$\max_{2^{k-1} < n \leq 2^k} \left|S_n \to S_{2^{k-1}}\right| \to 0.a.s. \ \ , k \to \infty$$

Hence

$$\sum_{k=1}^{\infty} E(\max_{2^{k-1} < n \leq 2^k} (S_n - S_{2^{k-1}})^2) < \infty$$

Applying the properties of Lemma2 and (3)(4)(5)and the slowly varying function,we have

$$\sum_{k=1}^{\infty} E(\max_{2^{k-1} < n \leq 2^k} (S_n - S_{2^{k-1}})^2) \leq \sum_{k=1}^{\infty} (\frac{l(2^k)}{l(2)})^2 f(2^{k-1}, 2^{k-1})$$

$$\leq c \sum_{k=1}^{\infty} (\frac{l(2^k)}{l(2)})^2 \frac{g(2^{k-1}, 2^{k-1})}{l^2(2^{k-1})}$$

$$\square \ c \sum_{k=1}^{\infty} g(2^{k-1}, 2^{k-1})$$

$$= \lim_{n \to \infty} \sum_{k=1}^{\infty} g(2^{k-1}, 2^{k-1})$$

$$\leq \lim_{n \to \infty} g(1, 2^{n-1})$$

$$\leq c < \infty \quad (n \geq 1)$$

This compeletes the proof of Theorem.

Infer. Let a sequence of constant $\{\rho_i; i \geq 0\}$ satisfies

$$EX_iY_i \leq \rho_{j-i}(EX_i^2Y_i^2)^{\frac{1}{2}}, 0 \leq \rho_i \leq 1, \forall j \geq i > 0$$

$$\sum_{i=0}^{\infty} \rho_i < \infty$$

Then S_n is a.s.

Proof. Since lemma 2, fixing $a \geq 0$, $n \geq 1$.

First using the estimate of $E(\sum_{i=a+1}^{a+n} X_i)^2$ to check f, g.

$$E(\sum_{i=a+1}^{a+n} X_i)^2 = E(\sum_{i=a+1}^{a+n} X_i^2 + 2\sum_{a+1 \leq i < j \leq a+n} X_iX_j)$$

$$\leq \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{a+1 \leq i < j \leq a+n} \rho_{j-i} E^{\frac{1}{2}} X_i^2 E^{\frac{1}{2}} X_j^2$$

$$= \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{i=a+1}^{a+n-1} \sum_{j=i+1}^{a+n} \rho_{j-i} E^{\frac{1}{2}} X_i^2 E^{\frac{1}{2}} X_j^2$$

$$= \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{k=1}^{n-1} \rho_k \sum_{i=a+1}^{a+n-k} E^{\frac{1}{2}} X_i^2 E^{\frac{1}{2}} X_j^2$$

$$\leq \sum_{i=a+1}^{a+n} EX_i^2 + \sum_{k=1}^{n-1} \rho_k \sum_{i=a+1}^{a+n-k} (EX_i^2 + EX_{k+i}^2)$$

$$\leq \sum_{i=a+1}^{a+n} EX_i^2 + 2\sum_{k=1}^{n-1} \rho_k \sum_{i=a+1}^{a+n} EX_i^2$$

$$\leq 2\sum_{i=a+1}^{a+n} EX_i^2 (1 + \sum_{k=1}^{n-1} \rho_k)$$

$$\leq 2\sum_{i=a+1}^{a+n} EX_i^2 (\sum_{k=0}^{n} \rho_k)$$

Definition :

$$f(a,n) = 2\sum_{i=a+1}^{a+n} EX_i^2 (\sum_{k=0}^{n} \rho_k), g(a,n) = 2\sum_{i=a+1}^{a+n} l^2(i) EX_i^2 (\sum_{k=0}^{n} \rho_k).$$

It is easy to check that(1) (2) (3) (4) and (5) hold,using Theorem , we obtain the infer.

References

CHEN Ping-yan. (2008). Lr Convergence for Pairwise NQD Random Variables[J]. Acta Math Scientia. 28A(3):447-453.

Jamsion B, O Rey S, & Pruitt W. (1965). Convergence of Weighted of in dependent random variables. Z. *Wahrsch Verb Gebiete*. (4):40-44.

Lehmann E L. (1966). *Some concep ts of dependence*. Ann Math Statist. 43: 1137-1153.

Matula P. (1992). A note on the almost sure convergence of sums of negatively dependent random variables. *Statist. Probab. Lett.*.pp.15:209-213

WAN Cheng-gao. (2005). Law of Large Numbers and Complete Convergence for Pairwise NQD Random Sequences. Acta Math Appl Sinica, 28(2): 253-261.

WANG Y B, Chun Su, & Xuguo Liu. (1998). Some limit properties of pairwise NQD Random sequences, *Applied Mathematics*, 21(3):404-414.

WANG Y B, YAN J G, & CAI X Z. (2001). On the strong stability for Jamison type weighted product sums of pairwise NQD series with different distribution. Chinese Ann Math. 22A:701-706.

WANG Y B, YAN Ji-gao, & CHENG Feng-yang etal. (2001). On the Strong Stability for Jamison Type Weighted Product Sums of Pair-wise NQD Series with Different Distribution. Chinese Ann Math: Ser A.22(6): 701-706.

WU Q Y. (2002). Convergence properties of pairwise NQD random sequences. Acta Math Sinica, 45 (3):617-624.

WU Q Y. (2002). The convergence properties of pairwise NQD Random sequences. *Mathematics* ,45(3):617-624.

WU Q Y. (2005). *Probability limit theory of mixing random sequences*. Beijing:Science Press. *(Chapter 2-3).*