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Spirals and Cycles of Biological Systems via Extended Rosenzweig-MacArthur Model with Ratio-dependent Functional Response

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Abstract

This paper investigates stable proper nodes, stable spiral sinks and stable $\omega-$limit cycles of Extended Rosenzweig-MacAthur Model, which incorporates ratio-dependent functional response on predation mechanism. The ultimate boundedness condition has been used to predict extinction, co-existence, and exponential convergence scenarios of the model. The Poincare-Bendixson results guarantee existence of periodic cycles of the models. The system degenerate from stable spiral sinks to stable ω-limit cycles as control parameter varies. Numerical simulations are provided to support the validity of theoretical findings.

Keywords: spiral sinks, $\omega-$limit cycles, and stability

1. Introduction

The theory of nonlinear dynamical systems has been robustly explored in explaining, interpreting, and predicting the qualitative behaviors of ecological populations of interacting species. Rosenzweig and MacArthur (1963) formulated and studied the qualitative behaviors of a di-trophic food chain model, with Holling type II functional response given as;

$$\begin{cases} \frac{dx_1}{dt} = rx_1(t)\left(1 - \frac{x_1(t)}{K}\right) - a_2\frac{x_1(t)}{b_1+x_1(t)}x_2(t) \\ \frac{dx_2}{dt} = c_2a_2\frac{x_1(t)}{b_1+x_1(t)}x_2(t) - d_2x_2(t) \end{cases} \tag{1}$$

which shows stability behaviors of predator-prey populations. This two-dimensional model exhibits a unique global attractor which is either an equilibrium or limit cycle as well as dynamical behaviors with deviated arguments (Wrosek, 1990, Shi, 2013). In three-dimensional systems, a tri-trophic food chain model as an extension of the Rosenzweig-MacArthur predator-prey model was formulated and studied;

$$\begin{cases} \dfrac{dx_1}{dt_1} &= rx_1(t) - \dfrac{rx_1^2(t)}{K} - a_2\dfrac{x_1(t)}{b_1+x_1(t)}x_2(t) \\ \dfrac{dx_2}{dt_1} &= c_2a_2\dfrac{x_2(t)}{b_1+x_1(t)}x_3(t) - d_2x_2(t) - a_3\dfrac{x_2(t)}{b_2+x_2(t)}x_3(t) \\ \dfrac{dx_3}{dt_1} &= c_3a_3\dfrac{x_2(t)}{b_2+x_2(t)}x_3(t) - d_3x_3(t) \end{cases} \tag{2}$$

Qualitative behaviors such as stability of equilibrium points, local and global bifurcations, limit cycles, peak-to-peak dynamics on this model with constant or seasonal varying of parameters has been investigated in system (2) (Feo, & Rinaldi, 1997; Kutnetsov, & Rinaldi, 1996; Kutnestsov, Rinaldi, 2001; Candaten, & Rinaldi, 2003). Feng, Rocco, Freeze, and Lu (2014), modified the Rosenzweig-MacArthur model in three dimensional for a more general and complex, but realistic model as;

$$\begin{cases} \dfrac{dx_1}{dt_1} &= rx_1 - \dfrac{rx_1^2}{K} - a_2\dfrac{x_1}{b_1+x_1}x_2 - a_3\dfrac{x_1}{b_1+x_1}x_3 \\ \dfrac{dx_2}{dt_1} &= c_2a_2\dfrac{x_2}{b_1+x_1}x_3 - d_2x_2 - a_3\dfrac{x_2}{b_2+x_2}x_3 \\ \dfrac{dx_3}{dt_1} &= c_3a_3\dfrac{x_2}{b_2+x_2}x_3 - d_3x_3 + c_3a_3\dfrac{x_1}{b_1+x_1}x_3 \end{cases} \tag{3}$$

which exhibits rich dynamical complexity of ecological population species. A topological equivalence dynamical system of system (3) were formulated via non-dimensionalization of the state variables as follows;

$$\begin{cases} \dfrac{dx}{dt} &= \alpha x(t) - \dfrac{\alpha x^2(t)}{\kappa} - \dfrac{\eta x(t)y(t)}{1+x(t)} - \dfrac{x(t)z(t)}{1+x(t)} \\ \dfrac{dy}{dt} &= \dfrac{\varepsilon x(t)y(t)}{1+x(t)} - \xi y(t) - \dfrac{\sigma y(t)z(t)}{1+y(t)} \\ \dfrac{dz}{dt} &= \dfrac{\beta y(t)z(t)}{1+y(t)} - \mu z(t) + \dfrac{\beta x(t)z(t)}{1+x(t)} \end{cases} \tag{4}$$

where $x(t) = \dfrac{x_1(t_1)}{b_1}$, $y(t) = \dfrac{x_2(t_1)}{b_2}$, $z(t) = \dfrac{x_3(t_1)}{b_1}$, $\alpha = \dfrac{r}{a_3}$, $\kappa = \dfrac{K}{b_1}$, $\eta = \dfrac{a_2b_2}{a_3b_1}$, $\varepsilon = \dfrac{c_2a_2}{a_3}$, $\xi = \dfrac{d_2}{a_3}$, $\sigma = \dfrac{b_1}{b_2}$, $\mu = \dfrac{d_3}{a_3}$, $c_3 = \beta$, $t = a_3t_1$. The model shows dynamical behaviors such as stability, limit cycles, hopf-bifurcation, persistence and global stability,periodic solutions, and stability dynamics with deviated arguments or delays. (see., Joshua, Akpan, & Madubueze 2016; Joshua, & Akpan, 2016; Joshua, Akpan, Madubueze, Adebimpe, 2017, Joshua, & Akpan, 2018). In this article, system (4)is modified with a ratio-dependent functional response and its dynamical complexity is studied.

2. Model Description and Existence of Bounded Solutions

Consider the ratio-dependent functional response incorporated in the predation mechanism of Extended Rosenzweig-MacArthur Model as follows

$$\begin{cases} \dfrac{dx}{dt} &= \alpha x(t) - \dfrac{\alpha x^2(t)}{\kappa} - \dfrac{\eta x(t)y(t)}{x(t)+y(t)} - \dfrac{x(t)z(t)}{x(t)+z(t)} \\ \dfrac{dy}{dt} &= \dfrac{\varepsilon x(t)y(t)}{x(t)+y(t)} - \xi y(t) - \dfrac{\sigma y(t)z(t)}{y(t)+z(t)} \\ \dfrac{dz}{dt} &= \dfrac{\beta y(t)z(t)}{y(t)+z(t)} - \mu z(t) + \dfrac{\beta x(t)z(t)}{z(t)+x(t)} \end{cases} \tag{5}$$

subject to initial conditions $(x(0) = x_0, y(0) = y_0, z(0) = z_0)$, where $x(t), y(t), z(t)$ are the populations of interacting species; preys, predators and super-predators respectively. The ecological parameters; α is the preys growth rate, κ is the environmental carrying capacity of the prey, η is the maximum predation rate on prey, ε is the maximum biomass conversion efficiency of the predator, ξ is the natural death rate of the predator, β is the maximum super-predator biomass conversion efficiency, and μ is the natural death rate of the super-predators. The population density function of system (1) is continuously differentiable in the non-negative cone of the state space $\mathbb{R}^3_+ = (x(t), y(t), z(t)|x(t) \geq 0, y(t) \geq 0, z(t) \geq 0)\ \forall t \geq 0$. Assume that the long-term survival of super-predators is dependent on the prey, predator abundance as well as its biomass conversion efficiency, then the steady state behavior and co-existence fixed point $E^*(x^*, y^*, z^*)$ of system (5) satisfies the equations;

$$\begin{cases} \alpha x^* - \dfrac{\alpha x^{*2}}{\kappa} - \dfrac{\eta x^*y^*}{x^*+y^*} - \dfrac{x^*z^*}{x^*+z^*} = 0 \\ \dfrac{\varepsilon x^*y^*}{x^*+y^*} - \xi y^* - \dfrac{\sigma y^*z^*}{y^*+z^*} = 0 \\ \dfrac{\beta y^*z^*}{y^*+z^*} - \mu z^* + \dfrac{\beta x^*z^*}{z^*+x^*} = 0 \end{cases} \tag{6}$$

Using differential inequalities and standard comparison argument, the phase flows $\phi_t(t_0; x(t), y(t), z(t))$ of system (5) and (6) are ultimately bounded in the compact invariant region $\Omega \in Int(\mathbb{R}^3_+)\ \forall t \geq 0$ defined as

$$\Omega = \begin{pmatrix} 0 \leq x(t) \leq \frac{\kappa}{\alpha}(\alpha - \eta - 1) \\ 0 \leq y(t) \leq y(0)exp(\varepsilon - \xi + \sigma)t \\ 0 \leq z(t) \leq z(0) + |z^* - z_0|exp(2\beta t) \end{pmatrix} \tag{7}$$

Proposition 1(Agarwal,O'Regan, & Saker, 2014): Given that Ω is a closed, convex, and nonempty subset of a Banach space $\mathbb{R}^3_+$. Denote the system (5) as a vector differentials; $\dot{X}(t) = F(X(t), \rho^*)$; where ρ^* is a control parameter, $X(t) = ((x(t), y(t), z(t))^T$ and $F : \mathbb{R}^3_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^3_+$ is a continuous mapping with $F(\Omega)$ a relatively compact subset of $\mathbb{R}^3_+$. Then F has at least one fixed point(equilibrium point), and Ω is a global attractor of every phase flows $\phi_t(t_0; x(t), y(t), z(t))$ of system (5), if $\alpha > \eta + 1,\ \ \varepsilon + \sigma > \xi,\ \beta > 0$.

Remarks The proposition established the extinction, co-existence and exponential convergence scenarios of interacting population species of system (5)

Definition 1(Perko, 2001): A limit cycle γ of the dynamical system (5) in the plane is a periodic orbit which is a $\alpha\ or\ \omega-$ limit set of a trajectory γ' other than γ. If a limit cycle γ is the $\omega - limit$ set of every trajectory in a neighborhood of γ, γ is said to be an $\omega - limit$ cycle or stable limit cycle. Likewise, if γ is the $\alpha - limit$ set of neighboring trajectories of γ, γ is said to be an $\alpha - limit$ cycle or unstable limit cycle

Definition 2 (Wiggins, 2003): A four tuple dynamical system $(T, \mathbb{R}^3_+, \mathbb{A}, \phi^t_*)$ is **topologically equivalent** near an equilibrium point $E^*(x^*, y^*, z^*)$ to a dynamical system $(T, \mathbb{R}^3_+, \mathbb{A}, \phi^t)$ near an equilibrium point $E(x^*, y^*, z^*)$ if there exists a homeomorphism $h : \mathbb{R}^3_+ \rightarrow \mathbb{R}^3_+$ that is

- defined in a small neighborhood $U \subset \mathbb{R}^3_+$ of $E^*(x^*, y^*, z^*)$
- satisfies $E(x^*, y^*, z^*) = h(E^*(x^*, y^*, z^*))$
- maps orbits of the first system in U onto orbits of the second system in $V = \mathbf{F}(U) \subset \mathbb{R}^3_+$, preserving the direction of time.

Remarks Definition (2) provides us with a way of characterizing two vector fields of the same qualitative dynamics.

Definition 3 (Poincare-Bendixson conditions: Brauer & Castillo-Chavez, 2012) Given that Ω is a positively invariant region for the vector field function $\mathbf{F}$ containing a finite number of fixed points. Let $p \in \Omega$, and consider $\omega(p)$. Then one of the following possibilities holds;

- $\omega(p)$ is a fixed point.
- $\omega(p)$ is a periodic orbit
- $\omega(p)$ consist of a finite number of fixed points $p_1, p_2, ..., p_n$, and orbits γ with either $\omega(\gamma) = p_j,\ \ \alpha(\gamma) = p_i$

3. Equilibrium Points and Linearized Jacobian of Planar Subsystem

We obtain the steady state equilibrium points of system (6), by solving the planar sub-systems independent of time, and deduce the positivity conditions. The model exhibits the following trivial and semi-trivial equilibrium points.

$$\begin{cases} E_0(x^* = 0, y^* = 0, z^* = 0); \ \ E_1\left(u^* = \frac{1-\varepsilon+\eta-\alpha-\sigma}{\alpha+\sigma+\xi-\varepsilon-1},\ y^* = 0,\ v^* = 0\right) \\ E_2\left(x^* = \frac{\kappa(\alpha\varepsilon+\eta\xi-\mu\varepsilon)}{\alpha\varepsilon},\ y^* = \frac{\kappa(\varepsilon-\xi)(\alpha\varepsilon_+\eta\xi-\mu\varepsilon)}{\alpha\varepsilon\xi},\ z^* = 0\right),\ \frac{\varepsilon(\mu-\alpha)}{\mu} < \xi < \varepsilon \\ E_3\left(x^* = \frac{\kappa(\alpha\beta-\beta+\mu)}{\alpha\beta}, y^* = 0, z^* = \frac{\kappa(\beta-\mu)(\alpha\beta-\beta+\mu)}{\alpha\ \beta\mu}\ \right),\ \beta - \alpha\beta < \mu < \beta \end{cases} \tag{8}$$

Thus, the linearized Jacobian of system (5) in the neighborhood of any equilibrium point $E(x^*, y^*, z^*)$ yields,

$$J = \begin{bmatrix} \frac{\alpha\kappa-2\alpha x^*}{\kappa} - \frac{\eta y^{*2}}{(x^*+y^*)^2} - \frac{z^2}{(x^*+z^*)^2} & -\frac{\eta x^{*2}}{(x^*+y^*)^2} & -\frac{\eta x^{*2}}{(x^*+z^*)^2} \\ \frac{\varepsilon x^{*2}}{(x^*+y^*)^2} & \frac{\varepsilon x^{*2}}{(x^*+y^*)^2} - \xi - \frac{\sigma z^{*2}}{(y^*+z^*)^2} & -\frac{\sigma y^{*2}}{(y^*+z^*)^2} \\ \frac{\beta z^{*2}}{(x^*+z^*)^2} & \frac{\beta z^{*2}}{(y^*+z^*)^2} & \frac{\beta y^{*2}}{(y^*+z^*)^2} - \mu + \frac{\beta x^{*2}}{(x^*+z^*)^2} \end{bmatrix} \tag{9}$$

4. Dynamical Behaviors of Stable Spirals and $\omega-$Limit Cycles of Planar Subsystem

Consider the transformation $u = \frac{x}{y},\ v = \frac{y}{z}$, the dynamical system (5) is topologically equivalent to system (10) define as follows;

$$\begin{cases} \frac{du}{dt} = \left(\frac{u^2(t)(\xi+\alpha-\varepsilon)+u(t)(\xi-\eta+\alpha)}{u(t)+1}\right) + \frac{\sigma u(t)}{v(t)+1} - \frac{\alpha u^2(t)y(t)}{\kappa} - \frac{u(t)}{u(t)v(t)+1} \\ \frac{dy}{dt} = \frac{\varepsilon u(t)y(t)}{u(t)+1} - \xi y(t) - \frac{\sigma y(t)}{v(t)+1} \\ \frac{dv}{dt} = (\mu - \xi)v(t) - \frac{v(t)(\beta v(t)+\sigma)}{v(t)+1} - \frac{\beta u(t)v^2(t)}{u(t)v(t)+1} + \frac{\varepsilon u(t)v(t)}{u(t)+1} \end{cases} \tag{10}$$

and subject to initial conditions $u(0) > 0,\ y(0) > 0,\ v(0) > 0$. The topologically equivalence systems satisfy the domino effects at extinction equilibrium point $E_0(0,0,0)$ and prey-free equilibrium point $E_1(u^*,0,0)$;

$$\begin{cases} \begin{pmatrix} u(t) \\ y(t) \\ v(t) \end{pmatrix}_{t\to\infty} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ as\ x(t) \to 0\ faster\ than\ y(t) \to 0,\ y(t) \to 0\ faster\ than\ z(t) \to 0\ at\ E_0(0,0,0) \\ \begin{pmatrix} u(t) \\ y(t) \\ v(t) \end{pmatrix}_{t\to\infty} \to \begin{pmatrix} \frac{1-\varepsilon+\eta-\alpha-\sigma}{\alpha+\sigma+\xi-\varepsilon-1} \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ as\ y(t) \to 0\ faster\ than\ z(t) \to 0,\ x(t) \to 0\ at\ finite\ rate\ as\ y(t) \to 0\ at\ E_1(u^*,0,0) \end{cases} \tag{11}$$

The extinction equilibrium point $E_0(0,0,0)$ and prey-free equilibrium are always presents. The Jacobian matrix of system (10) evaluated at $E_0(0,0,0)$ yields,

$$J_{E_0} = \begin{bmatrix} \xi - \eta + \alpha + \sigma - 1 & 0 & 0 \\ 0 & -\xi - \sigma & 0 \\ 0 & 0 & \mu - \xi - \sigma \end{bmatrix} \tag{12}$$

Thus, $E_0(0,0,0)$ is asymptotically stable and a proper nodal sink with negative eigenvalues $(\lambda_i\ i = 1,2,3) = diag(J_{E_0}) < 0$ and unstable for at least one positive eigenvalues of $(\lambda_i\ i = 1,2,3) = diag(J_{E_0})$.

Proposition 2: The origin admits stable spiral sink and proper node if $\mu < \sigma < -\eta - \alpha + 1$, otherwise an unstable saddle node.

Also, the Jacobian matrix of system (10) evaluated at prey-free equilibrium point E_1 yields,

$$J_{E_1} = \begin{bmatrix} -\frac{\xi+\alpha+\sigma-\varepsilon-1}{\eta-\varepsilon} & -\frac{\alpha(\varepsilon-\eta+\alpha+\sigma-1)^2}{\kappa(\xi+\alpha+\sigma-\varepsilon-1)^2} & \frac{(\sigma^2+\xi\sigma+\alpha\sigma-\varepsilon\sigma+\xi-\eta+\alpha-1)(\xi-\eta+\alpha+\sigma-1)}{(\xi+\alpha+\sigma-\varepsilon-1)^2} \\ 0 & \frac{\varepsilon(\eta-\alpha+1)-\eta(\xi+\sigma)}{\eta-\varepsilon} & 0 \\ 0 & 0 & \frac{\varepsilon(\eta-\mu-\alpha+1)+\eta(\mu-\xi-\sigma)}{\eta-\varepsilon} \end{bmatrix} \tag{13}$$

and admits a locally asymptotic stability with negative eigenvalues, $\lambda_i(i = 1,2,3)$ defined as follows;

$$\begin{cases} \lambda_1 = & -\frac{\alpha^2-\alpha\eta+2\alpha\sigma+2\alpha\xi-\alpha\varepsilon-\eta\sigma-\eta\xi+\eta\varepsilon+\sigma^2+2\sigma\xi-\sigma\varepsilon+\xi^2-\xi\varepsilon-2\alpha+\eta-2\sigma-2\xi+\varepsilon+1}{\eta-\varepsilon} \\ \lambda_2 = & -\frac{\alpha\varepsilon-\eta\mu+\eta\sigma+\eta\xi-\eta\varepsilon+\mu\varepsilon-\varepsilon}{\eta-\varepsilon} \\ \lambda_3 = & -\frac{\alpha\varepsilon-\mu\eta+\eta\sigma+\eta\xi-\eta\varepsilon+\mu\varepsilon-\varepsilon}{\eta-\varepsilon} \end{cases} \tag{14}$$

The Jacobian matrix (9) of system (5) evaluated at the prey-predator equilibrium E_2 yields

$$J_{E_2} = \begin{bmatrix} \frac{(\eta-\alpha)\varepsilon^2-\eta\xi}{\varepsilon} & -\frac{\xi^2\eta}{\varepsilon} & -1 \\ \frac{(\xi-\varepsilon)^2}{\varepsilon} & \frac{(\xi-\varepsilon)\xi}{\varepsilon} & -\sigma \\ 0 & 0 & 2\beta-\mu \end{bmatrix} \tag{15}$$

. It satisfies the characteristic polynomial,

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{16}$$

where

$$\begin{pmatrix} A_1 = & \frac{\alpha\varepsilon^2-2\beta\varepsilon^2+\eta\xi^2-\eta\varepsilon^2+\mu\varepsilon^2-\xi^2\varepsilon+\xi\varepsilon^2}{\varepsilon^2} \\ A_2 = & -\frac{2\alpha\beta\varepsilon^2-\alpha\mu\varepsilon^2+\alpha\xi^2\varepsilon^2+2\beta\eta\xi^2-2\beta\eta\varepsilon^2-2\beta\xi^2\varepsilon+2\beta\xi\varepsilon^2-\eta\mu\xi^2+\eta\mu\varepsilon^2-\eta\xi^3-2\eta\varepsilon\xi^2+\eta\xi\varepsilon^2+\mu\xi^2\varepsilon-\mu\xi\varepsilon^2}{\varepsilon^2} \\ A_3 = & \frac{\xi(2\beta-\mu)(\xi-\varepsilon)(\alpha\varepsilon+\eta\xi-\mu\varepsilon)}{\varepsilon^2} \end{pmatrix} \tag{17}$$

with eigenvalues, $\lambda_i(i = 1,2,3)$ defined as follows:

$$\begin{pmatrix} \lambda_1 = & 2\beta-\mu \\ \lambda_2 = & -\frac{\alpha\varepsilon^2+\eta\xi^2-\eta\varepsilon^2-\varepsilon\xi^2+\varepsilon^2\xi}{2\varepsilon^2} + \\ & \frac{\alpha^2\varepsilon^4+2\alpha\eta\xi^2\varepsilon^2-2\alpha\eta\varepsilon^4+2\alpha\xi^2\varepsilon^3-2\alpha\xi\varepsilon^4+\eta^2\xi^4-2\eta^2\xi^2\varepsilon^2+\eta^2\varepsilon^4-2\eta\varepsilon^4\xi+6\eta\xi^3\varepsilon^2-6\eta\xi^2\varepsilon^3+2\eta\xi\varepsilon^4+\varepsilon^2\xi^4-2\varepsilon^3\xi^3+\varepsilon^4\xi^2}{2\varepsilon^2} \\ \lambda_3 = & -\frac{\alpha\varepsilon^2+\eta\xi^2-\eta\varepsilon^2-\varepsilon\xi^2+\varepsilon^2\xi}{2\varepsilon^2} - \\ & \frac{\alpha^2\varepsilon^4+2\alpha\eta\xi^2\varepsilon^2-2\alpha\eta\varepsilon^4+2\alpha\xi^2\varepsilon^3-2\alpha\xi\varepsilon^4+\eta^2\xi^4-2\eta^2\xi^2\varepsilon^2+\eta^2\varepsilon^4-2\eta\varepsilon^4\xi+6\eta\xi^3\varepsilon^2-6\eta\xi^2\varepsilon^3+2\eta\xi\varepsilon^4+\varepsilon^2\xi^4-2\varepsilon^3\xi^3+\varepsilon^4\xi^2}{2\varepsilon^2} \end{pmatrix} \tag{18}$$

By Routh-Hurwitz conditions and Descartes rule of sign, the next proposition follows.

Proposition 3: The population of prey-predator species in the neighborhood of the equilibrium point E_2 exhibits stable spiral sink, if $A_1A_2 - A_3 > 0$ and eigenvalues (18) $\lambda_i(i = 1, 2, 3)$ have negative real parts, otherwise unstable. It degenerates to a stable $\omega-$limit cycle if $A_1A_2 - A_3 = 0$, eigenvalues $\lambda_i(i = 1, 2, 3)$ have purely imaginary parts. and $\eta^* = \frac{\alpha\varepsilon^2+\xi\varepsilon^2-\xi^2\varepsilon}{\varepsilon^2-\xi^2}$.

Similarly, the Jacobian matrix (9) evaluated at the prey super-predator equilibrium point E_3 yields;

$$J_{E_3} = \begin{bmatrix} \frac{\beta^2(1-\alpha)-\mu^2}{\beta^2} & -\eta & -\frac{\mu^2}{\beta^2} \\ 0 & \varepsilon-\xi-\sigma & 0 \\ \frac{(\beta-\mu)^2}{\beta} & \beta & \frac{\mu(\mu-\beta)}{\beta} \end{bmatrix} \tag{19}$$

Its satisfies the characteristic polynomial,

$$P^{'}(\lambda) = \lambda^3 + A'_1\lambda^2 + A'_2\lambda + A'_3 = 0 \tag{20}$$

where

$$\begin{pmatrix} A'_1 = & \frac{\alpha\beta^2+\beta^2\mu+\beta^2\sigma+\beta^2\xi-\beta^2\varepsilon-\beta\mu^2-\beta^2+\mu^2}{\beta^2} \\ A_2' = & \frac{\beta^2(\alpha\mu+\alpha\sigma+\alpha\xi-\alpha\varepsilon+\mu\sigma+\mu\xi-\mu\varepsilon-\mu-\sigma-\xi+\varepsilon)-\alpha\beta\mu^2-\beta\mu^2\sigma-\beta\mu^2\xi-\beta\mu^2\varepsilon+2\beta\mu^2-\mu^3+\mu^2\sigma+\mu^2\xi-\mu^2\varepsilon}{\beta^2} \\ A_3' = & \frac{\mu(\beta-\mu)(\alpha\beta\sigma+\alpha\beta\xi-\alpha\beta\varepsilon-\beta\sigma-\beta\xi+\beta\varepsilon+\mu\sigma+\mu\xi-\mu\varepsilon)}{\beta^2} \end{pmatrix} \tag{21}$$

with eigenvalues, $\lambda'_i(i = 1, 2, 3)$ defined as follows:

$$\begin{pmatrix} \lambda_1' = & \varepsilon-\xi-\sigma \\ \lambda_2' = & -\frac{\alpha\beta^2+\beta^2\mu-\beta\mu^2-\beta^2+\beta^2\mu^2}{2\beta^2}+ \\ & \frac{\alpha^2\beta^4-2\alpha\beta^4\mu+2\alpha\beta^3\mu^2+\beta^4\mu^2-2\beta^3\mu^3+\beta^2\mu^4-2\alpha\beta^4+2\alpha\beta^2\mu^2+2\beta^4\mu-6\beta^3\mu^2+6\beta^2\mu^3-2\beta\mu^4+\beta^4-2\beta^2\mu^2+\mu^4}{2\beta^2} \\ \lambda_3' = & -\frac{\alpha\beta^2+\beta^2\mu-\beta\mu^2-\beta^2+\beta^2\mu^2}{2\beta^2}- \\ & \frac{\alpha^2\beta^4-2\alpha\beta^4\mu+2\alpha\beta^3\mu^2+\beta^4\mu^2-2\beta^3\mu^3+\beta^2\mu^4-2\alpha\beta^4+2\alpha\beta^2\mu^2+2\beta^4\mu-6\beta^3\mu^2+6\beta^2\mu^3-2\beta\mu^4+\beta^4-2\beta^2\mu^2+\mu^4}{2\beta^2} \end{pmatrix} \tag{22}$$

By Routh-Hurwitz conditions and Descartes rule of sign, the next proposition follows.

Proposition 4: The population of prey super-predator species in the neighborhood of the equilibrium point E_3 exhibits stable spiral sink, if $A_1'A_2' - A_3' > 0$ and eigenvalues (22) $\lambda_i'(i = 1, 2, 3)$ have negative real parts, otherwise unstable. It degenerates to a stable $\omega-$limit cycle if $A_1'A_2' - A_3' = 0$ and eigenvalues (22) $\lambda_i'(i = 1, 2, 3)$ have purely imaginary parts.

5. Dynamical Behaviors of Positive Coexistence Equilibrium Point

System (10) has a unique positive coexisting equilibrium point say, $E_4(u^*, y^*, v^*)$ defined as follows;

$$\begin{cases} v^* = & -\frac{\sigma u^*+\xi u^*-\varepsilon u^*+\sigma+\xi}{\xi u^*-\varepsilon u^*+\xi} \\ v^* = & \frac{\kappa[(\alpha\sigma+\alpha\xi-\alpha\varepsilon)u^{*^3}+(\xi+\alpha\xi-\eta\xi+2\alpha\sigma+\eta\varepsilon-\varepsilon-\eta\sigma)u^{*^2}+(2\xi-\alpha\xi+\alpha\sigma+\varepsilon\alpha-\eta\varepsilon-\varepsilon-\eta\sigma)u^*+\xi\eta-\xi\alpha+\varepsilon]}{\alpha u^*(u^*+1)((\sigma+\xi-\varepsilon)u^{*^2}+\sigma u^*+\varepsilon u^*-\xi)} \\ p(u^*) = & u^{*^4}+A_3u^{*^3}+A_2u^{*^2}+A_1u^*+A_0 = 0, \quad \forall A_i(i = 0, 1, 2, 3) > 0 \\ A_3 = & \frac{(\sigma-\xi)(\sigma+\xi-\varepsilon)(\mu\sigma-2\beta\sigma-\beta\xi+\beta\varepsilon)+(\sigma+\xi-\varepsilon)(\mu\sigma^2-2\beta\sigma^2-\beta\xi\sigma-\mu\xi\sigma+\varepsilon\mu\sigma-\varepsilon\beta\sigma+\beta\xi^2-2\beta\varepsilon)}{(\mu\sigma-2\beta\sigma-\beta\xi+\beta\varepsilon)(\sigma+\xi-\varepsilon)^2} \\ A_2 = & \frac{(\sigma+\xi)(2\mu\sigma^2-4\beta\sigma^2+\mu\xi\sigma-5\beta\xi\sigma+2\beta\varepsilon\sigma-\beta\xi^2+\beta\varepsilon^2)+(\sigma+\xi-\varepsilon)(\mu\sigma^2-2\beta\sigma^2-2\beta\xi\sigma-\mu\xi\sigma+\varepsilon\mu\sigma-\varepsilon\beta\sigma+\beta\xi^2-2\beta\xi\varepsilon)}{(\mu\sigma-2\beta\sigma-\beta\xi+\beta\varepsilon)(\sigma+\xi-\varepsilon)^2} \\ A_1 = & \frac{(\sigma+\xi)(\mu\sigma^2-2\beta\sigma^2-\beta\xi\sigma-\mu\xi\sigma+\varepsilon\mu\sigma-\beta\varepsilon\sigma+\beta\xi^2-2\beta\varepsilon)-(\sigma+\xi-\varepsilon)(\xi\mu\sigma-\xi\beta\sigma-\beta\xi^2)}{(\mu\sigma-2\beta\sigma-\beta\xi+\beta\varepsilon)(\sigma+\xi-\varepsilon)^2} \\ A_0 = & \frac{-(\sigma+\xi)(\xi\mu\sigma-\xi\beta\sigma-\beta\xi^2)}{(\mu\sigma-2\beta\sigma-\beta\xi+\beta\varepsilon)(\sigma+\xi-\varepsilon)^2} \end{cases} \tag{23}$$

Also, the Jacobian matrix $J_{i\,j}(i, j = 1, 2, 3)$ of system (10) evaluated at the coexistence equilibrium point and the corresponding characteristic polynomial yields,

$$\begin{cases} J_{E_4} = \begin{bmatrix} J_{11} & -\frac{\alpha {u^*}^2}{\kappa} & \frac{{u^*}^2}{(u^*v^*+1)^2} - \frac{\alpha u^*}{(v^*+)1^2} \\ \frac{\varepsilon y^*}{(y^*+1)^2} & \frac{\varepsilon u^*}{u^*+1} - \xi - \frac{\sigma}{v^*+1} & \frac{\sigma y^*}{(v^*+1)^2} \\ \frac{\varepsilon v^*}{(u^*+1)^2} - \frac{\beta {v^*}^2}{u^*v^*+1} & 0 & J_{33} \end{bmatrix} \\ J_{11} = \left(\frac{(\xi-\kappa-\varepsilon){u^*}^2+(2\xi+2\alpha-2\varepsilon)u^*+\xi-\mu+\alpha}{(u^*+1)^2} - \frac{y^*+\alpha+1}{(u^*v^*+1)(y^*+1)} - \frac{2\alpha^* y^*}{\kappa} \right) \\ J_{33} = (\mu-\xi) - \frac{\beta {v^*}^2+2\beta v^*\sigma}{(v^*+1)^2} - \frac{\beta u^* v^*(u^*v^*+2)}{(u^*v^*+1)^2} + \frac{\varepsilon u^*}{u^*+1} \\ P_{E_4}(\lambda) = \lambda^3 - Trace(J_{E_4})\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - det(J_{E_4}) = 0, \\ \text{where } A_{ij}\ \forall\ i = j(1,2,3) \text{ are cofactors of } J_{E_4} \end{cases} \tag{24}$$

Using Routh-Hurwitz conditions (see, Wiggins, 2003), the dynamical behaviors of system (5) at coexisting equilibrium point is established in the following proposition.

Proposition 5: The coexistence equilibrium point $E_4(u^*, y^*, v^*)$ for the model (5) or (10) is asymptotically stable if and only if the ecological parameters satisfies the condition (25), and degenerates to a stable $\omega - limit$ cycle for some ecological parameters, if condition (26) holds.

$$Trace(J_{E_4}) < 0,\ det(J_{E_4}) < 0,\ Trace(J_{E_4})(Trace(J_{E_4})^2 - Trace(J_{E_4}^2)) < 2det(J_{E_4}) \tag{25}$$

$$Trace(J_{E_4})[(A_{11}+A_{22}+A_{33})] = det(J_{E_4}) \tag{26}$$

6. Numerical Simulations and Applications in Ecological Population

Dynamical Behaviors of the Model at the Origin: Consider the ecological parameters; $\alpha = 0.267,\ \kappa = 5.92,\ \xi = 2.734,\ \sigma = 1.674,\ \mu = 1.53,\ \eta = 4.36,\ \varepsilon = 2.734,$ subject to initial conditions $u(0) = 2.30,\ y(0) = 2.5,\ v(0) = 1.5,$ then system (10) admits a stable spiral and proper node at the origin $E_0(u^* = 0, y^* = 0, v^* = 0)$ as seen in fig. 1. Observe that proposition (2) holds for the given ecological parameters with eigenvalues ($\lambda_1 = -0.685,\ \lambda_2 = -4.408,\ \lambda_3 = -2.878$).

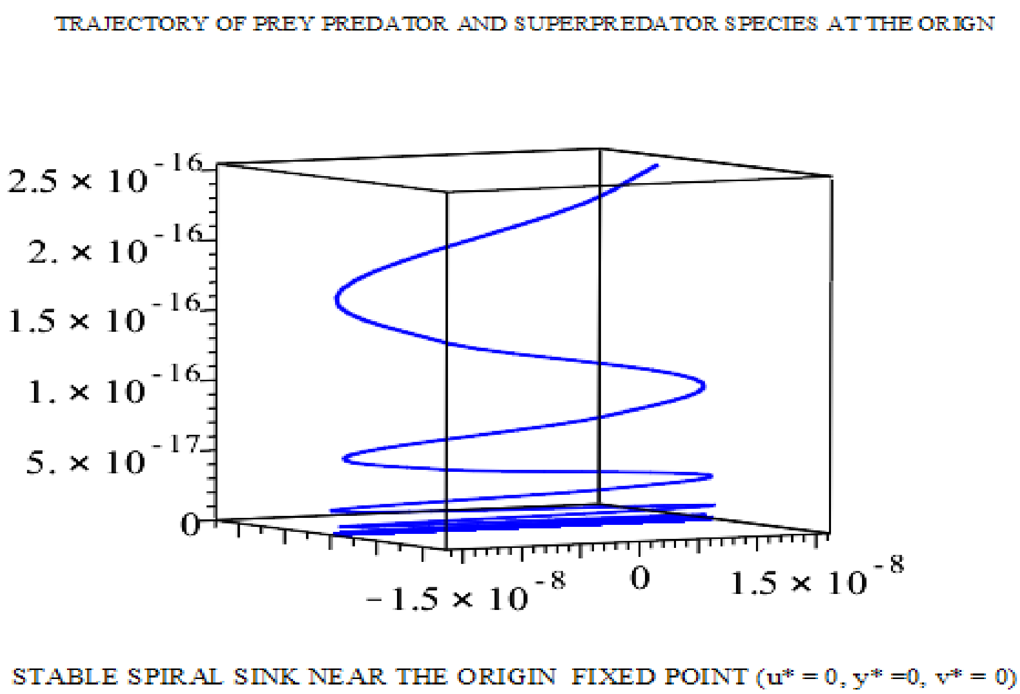

Figure 1. Phase space diagram at origin

Behavior of the Model at Prey-free Equilibrium Point: Consider the ecological parameters; $\alpha = 0.0031$, $\kappa = 5.92$, $\xi = 0.50$, $\sigma = 1.50$, $\mu = 0.42$, $\eta = 0.40$, $\varepsilon = 1.20$, then system (10) has a prey-free equilibrium point in the absent of predators and super-predators interactions at $E_1(u^* = 3.0630, y^* = 0, v^* = 0)$ with negative eigenvalues $(-0.1484, -1.0954, -0.6753)$ satisfying equation (14). Thus, every trajectory of system (10) converges, and asymptotically stable to the fixed point E_1 as seen in the phase portrait of figure 2.

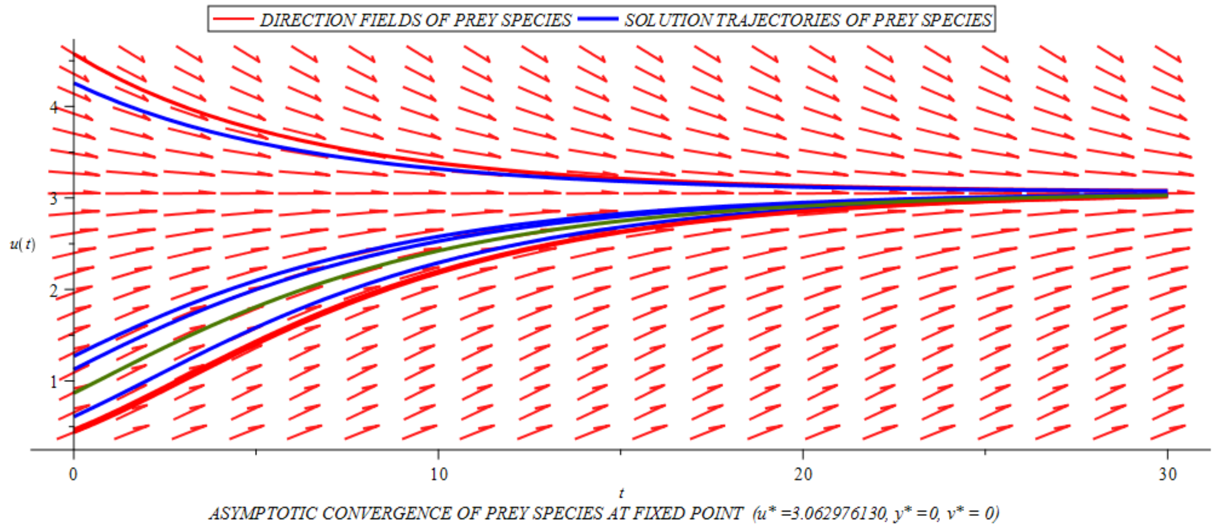

Figure 2. Phase Portrait of Prey Population

Behavior of the Model at Prey-Predator Equilibrium Point: Consider the ecological parameters; $\alpha = 4.2031$, $\kappa = 5.92$, $\xi = 1.60$, $\sigma = 1.0$, $\mu = 2.567$, $\eta = 5.40$, $\varepsilon = 2.960$, $\beta = 0.2563$ then system (5) has a prey-predator equilibrium point in the absent of super-predators interactions at $E_2(x^* = 2.4254, y^* = 2.0616, z^* = 0)$ It has negative complex eigenvalues $(-0.5580 \pm 0.9770i; -2.0544 + 0i)$ satisfying equation (18).Also, proposition 3 is satisfied for $A_1A_2 - A_3 = 8.681865534 > 0$. Thus, every trajectory of system (5) converges spirally and is asymptotically stable to the fixed point E_2 as seen in the phase portrait of figure 3.

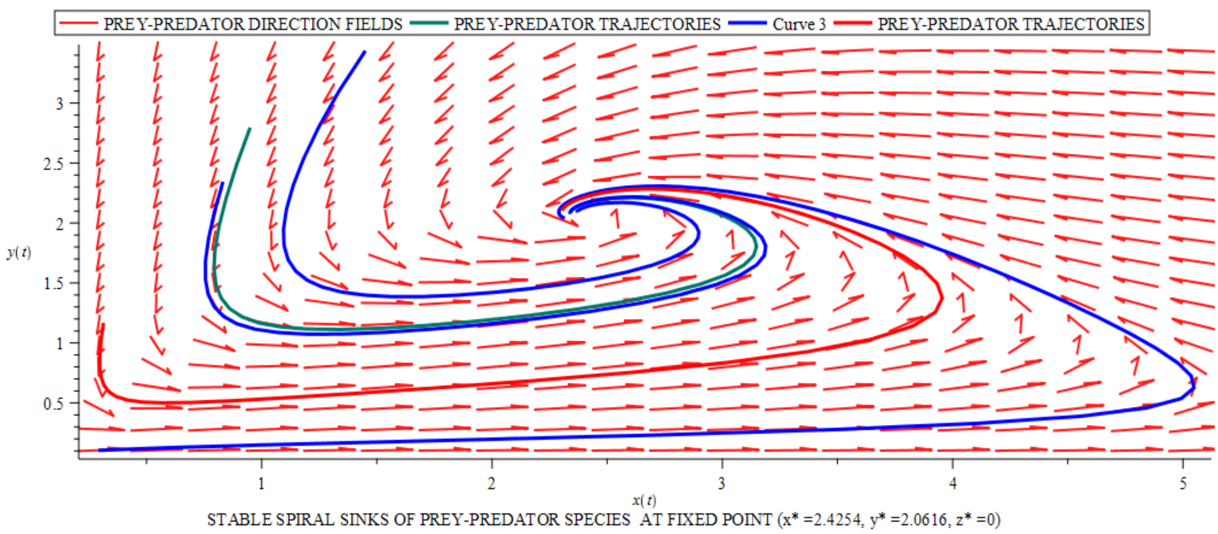

Figure 3. Phase Portrait of Prey Predator Populations

Observe that, the model degenerates to a stable $\omega - limit$ cycle which satisfies conditions of proposition 2; $A_1A_2 -$

$A_3 = 0$, $\lambda_{1,2} = \pm 0.85635i$, $\lambda_3 = -2.0544$ and $\eta^* = 6.97672291$. Thus, every trajectory of the model encloses an $\omega - limit$ point $x^* = 1.405074202$, $y^* = 1.194313072$, $z^* = 0$ as seen in figure 4.

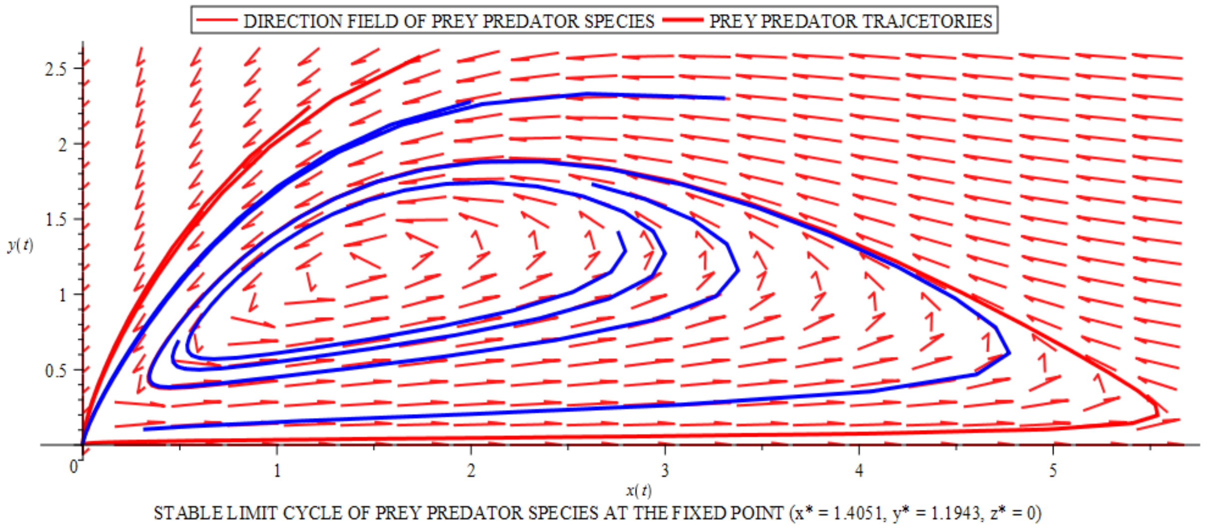

Figure 4. Phase portrait of prey predator stable $\omega-$ limit cycle

Behaviors of the Model at Prey-Superpredator Equilibrium point: Consider the ecological parameters; $\alpha = 1.5$, $\kappa = 0.92$, $\xi = 1.6$, $\sigma = 3.5$, $\mu = 1.97$, $\varepsilon = 2.960$, $\beta = 6.67$ then system (5) has a prey-Superpredator equilibrium point in the absent of predators interactions at $E_2(x^* = 0.4873, y^* = 0, z^* = 1.1675)$. It has negative complex eigenvalues $(-0.9884 \pm 0.3571i; -2.1340 + 0i)$ satisfying equation (22).Also, proposition (4) is satisfied for some $A_1^{'}A_2^{'} - A_3^{'} = 19.15956 > 0$. Thus, every trajectories of system (5) converges spirally and is asymptotically stable to the fixed point E_3 as seen in the phase portrait of figure 5.

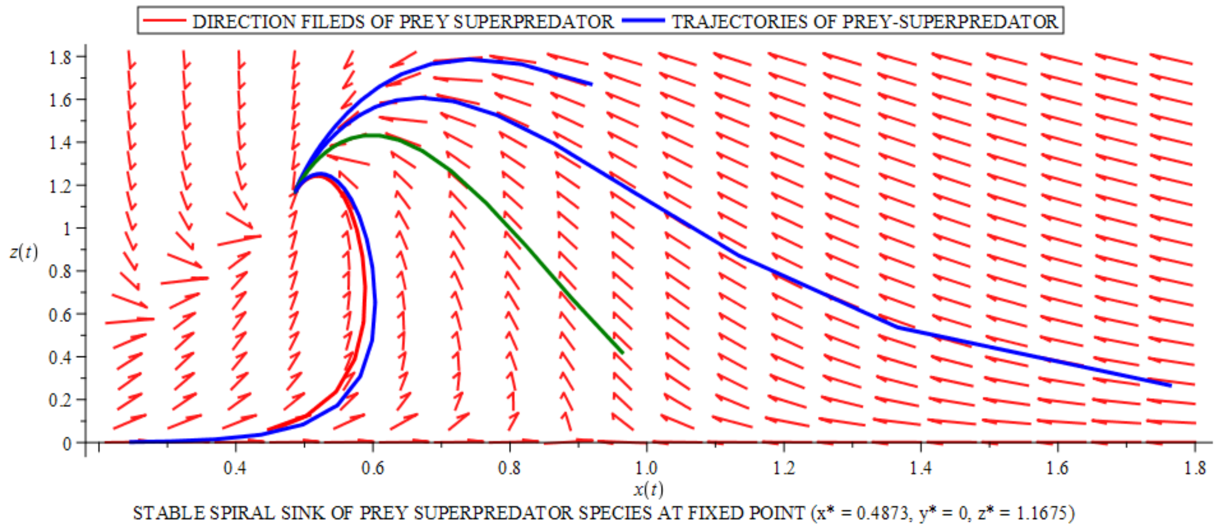

Figure 5. Phase portrait of prey superpredator population

Observe that, the model degenerates from a stable spiral sink to a stable proper node for ecological parameters; $\alpha = 1.5$, $\kappa = 0.92$, $\xi = 1.6$, $\sigma = 3.5$, $\mu = 0.97$, $\varepsilon = 2.960$, $\beta = 1.3532$ which satisfies a condition of proposition 4 for $A_1^{'}A_2^{'} - A_3^{'} = 9.8845 > 0$, $\lambda_1 = -0.360$, $\lambda_2 = -0.9285$, $\lambda_3 = -2.140$ and $\beta^* = 1.3532$, $\mu^* = 0.97$. Thus, every trajectory of the model converges to the stable proper node $(x^* = 0.7463,\ y^* = 0,\ z^* = 0.2949)$ as seen in figure 6.

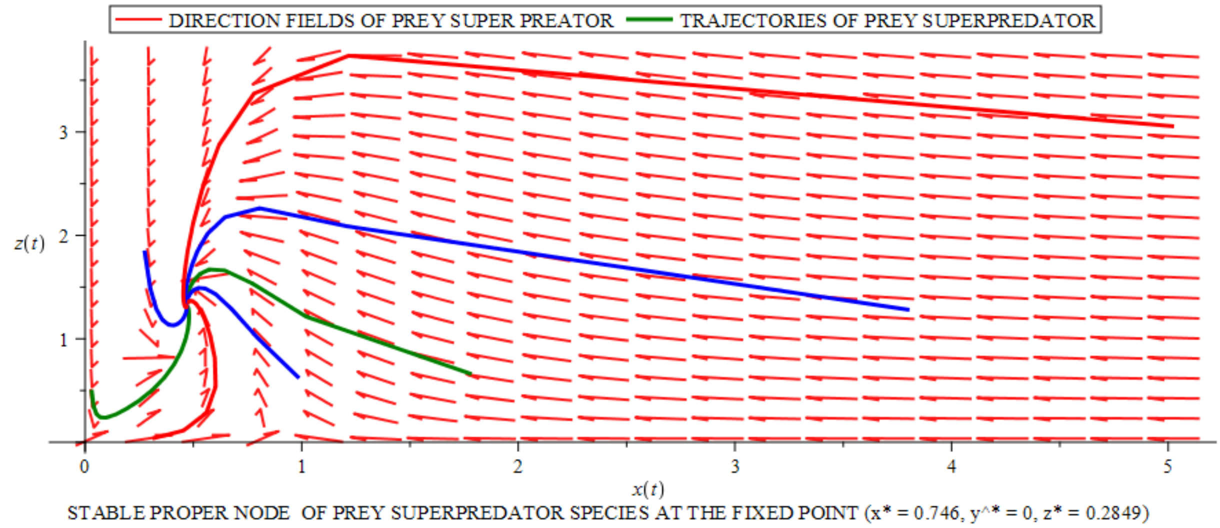

Figure 6. Phase portrait of prey superpredator population with a stable proper node

Behavior at the Coexistence Equilibrium Point: Consider the ecological parameters; $\alpha = 4.2031$ $\kappa = 5.92$, $\varepsilon = 2.96$ $\xi = 1.60$, $\sigma = 1.00$, $\mu = 2.567$, $\eta = 0.2473$, $\beta = 2.2563$ subject to initial conditions $u(0) = 0.23$, $y(0) = 1.23$, $v(0) = 0.22$, then the population of the interacting species coexist at the fixed point point ($u^* = 2.6454, y^* = 2.0344, v^* = 0.8248$). The equilibrium point satisfies conditions of proposition (5) in equation (25). The Jacobian of system (10) evaluated at the coexistence equilibrium point($u^* = 2.64545378939, y^* = 2.034410208, v^* = 0.8247739476$) yields;

$$\begin{cases} J_{E_4} = \begin{bmatrix} -4.14548078 & -4.968482898 & -0.103231176 \\ -0.453153018 & 4E-10 & 0.6109706985 \\ 0.0321097726 & 0 & -0.797432804 \end{bmatrix} \\ P(\lambda) = \lambda^3 + 4.942913584\lambda^2 + 5.56054011\lambda + 1.892878761 \\ \lambda_{1,2,3} = \begin{pmatrix} -0.7146064606 \pm 0.1674853075i \\ -3.513700663 \end{pmatrix} \end{cases} \tag{27}$$

Using, proposition (5) the coexisting equilibrium point is asymptotically stable satisfying the conditions, $Trace(J_{E_4}) = -4.942913585 < 0$, $det(J_{E_4}) = -1.892878766 < 0$; $Trace(J_{E_4})(Trace(J_{E_4})^2 - Trace(J_{E_4}^2)) - 2det(J_{E_4}) = -46.24435679 < 0$. Thus,the neighborhood of the fixed point ($u^* = 2.64545378939, y^* = 2.034410208, v^* = 0.8247739476$) is an attractor set and interacting species coexist in the long-run.

7. Conclusion

The study investigated qualitative dynamical behaviors of an extended Rosenzweig-MacArthur model with ratio-dependent functional response on predation mechanism. Using the theory of nonlinear dynamical systems, we established existence and boundedness of solutions in real parameter space. Some pseudo-codes in maple 16 on dynamical systems were used to reduce the algebraic complexity of the model. The result of numerical simulations were plotted as phase portraits, and phase space diagrams to verify the propositions established. The model shown stable spirals, stable proper nodes and stable limit cycles via appropriate variations of the ecological parameters. Thus, incorporating ratio-dependent functional response unfolded a robust and realistic dynamics of the models.

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