Analytic Properties of the Quaternion Function

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Abstract

Many properties of complex functions are pretty difficult to be generalized in the field of quaternion function, as the commutative law of multiplication fails in the latter. The derivative of quaternion function is defined in this paper. Moreover, by the similar method in judging the analytic property of complex function, Cauchy-Riemann equation is used to determine the analytic property of quaternion function. Furthermore, several concrete examples are discussed, and certain errors in (P. W. Yang. 2009) are pointed out as well.

Keywords: Quaternion function, Analytic function, Cauchy-Riemann equation, Derivative formula

1. Introduction

The quaternion function is an important aspect of functions theory. With a similar method of Cauchy-Riemann equation in complex functions, (P. W. Yang. 2009) defines the analytic of quaternion function directly. However, the definition may lead to some mistakes. We define the derivative of quaternion function and give a different definition of its analytic property. Moreover, we also present the derivative formulas as well as some examples.

Throughout the whole paper, let R and C denote the fields of real numbers and complex numbers respectively. Set $Q=\left\{q=x_1+x_2i+x_3j+x_4k \,\middle|\, x_1,x_2,x_3,x_4\in R\right\}$ be a real quaternion field. Assume D is a region in R^4 and $f(q)$ is a quaternion function defined in D. Let $x=\left\{x_1,x_2,x_3,x_4\right\}$ be some element in R^4, such that $U:D\to Q$; $w=f(q)=u_1+u_2i+u_3j+u_4k$, where $u_i(x_1,x_2,x_3,x_4), i=1,2,3,4$ are real functions defined on D.

Definition 1.1. Let $w=f(q)$ be the quaternion function defined in the certain neighborhood of q_0, or some region D containing q_0, we could consider the quotient

$$\frac{\Delta w}{\Delta q}=\frac{f(q)-f(q_0)}{q-q_0}=\frac{f(q_0+\Delta q)-f(q_0)}{\Delta q}\quad,\ \Delta q\neq 0$$

If the limit of quotient exists whenever a point q approaches to the point q_0 arbitrarily, or equivalently speaking, the difference Δq vanishes in any arbitrary direction, the limit is called the derivative of the function $f(q)$ at a given point q_0, and denoted by $f'(q_0)$ or $\frac{dw}{dq}$,

$$f'(q_0)=\lim_{\Delta q\to 0}\frac{\Delta w}{\Delta q}=\lim_{q\to q_0}\frac{f(q)-f(q_0)}{q-q_0} \tag{1.1}$$

Let the function $f(q)$ is derivative at a point q_0, then $\lim\limits_{\Delta q\to 0}\frac{\Delta w}{\Delta q}=f'(q)$ holds, i.e.

$\frac{\Delta w}{\Delta q}=f'(q)+\eta, \lim\limits_{\Delta q\to 0}\eta=0$. Hence, $\Delta w=f'(q)\Delta q+\varepsilon$, where the remainder $|\varepsilon|=|\eta\cdot\Delta q|$ is a higher order infinitesimal of the difference $|\Delta q|$.

Definition 1.2. Define $f'(q)\Delta q$ is the difference of $f(q)$, and denote to be dw or $df(q)$,i.e.

$$dw=f'(q)\Delta q \tag{1.2}$$

Taking $f(q)=q$ specifically, $dq=\Delta q$ holds. Then $dw=f'(q)dq$,(i.e.) $\frac{dw}{dq}=f'(q)$ follows by (1.2). Therefore, this implies the equivalence between differentiability and derivative of some certain function $f(q)$.

Example1.1. Let the function $f(q)=q^n$ (n should be integers) be differentiable everywhere in R^4 , so that

$$f'(q)=nq^{n-1} \tag{1.3}$$

Example1.2. Let the function $f(q)=\overline{q}=x_1-x_2i-x_3j-x_4k$, then it isn't differentiable anywhere in R^4 .

It's clear that the differentiability of the function $f(q)$ at the given point q should imply its continuity. However, the converse may not hold according to the second example. Speaking more precisely, to a given point q , the continuity of the function could not deduce the differentiability at that point.

Let two functions $f_1(q), f_2(q)$ both be differentiable at the point q , one can easily verify that

$$[f_1(q)\pm f_2(q)]]) = f_1'(q)\pm f_2'(q) \tag{1.4}$$

$$[f_1(q)f_2(q)]' = f_1'(q)f_2(q)+f_1(q)f_2'(q) \tag{1.5}$$

$$\left[\frac{f_1(q)}{f_2(q)}\right]' = \frac{f_1'(q)f_2(q)-f_1(q)f_2'(q)}{f_2^2(q)}, f_2(q)\neq 0 \tag{1.6}$$

2. Main Results

Definition2.1 The quaternion function $f(q)$ is an analytical function in D , if $f(q)=u_1+u_2i+u_3j+u_4k$ is defined properly in $D(\subset R^4)$, $u_i(x_1,x_2,x_3,x_4)$ are differentiable in this region, and the following holds

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1}-\dfrac{\partial u_2}{\partial x_2}+\dfrac{\partial u_3}{\partial x_3}-\dfrac{\partial u_4}{\partial x_4}=0 \\ \dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}-\dfrac{\partial u_3}{\partial x_4}-\dfrac{\partial u_4}{\partial x_3}=0 \\ \dfrac{\partial u_1}{\partial x_3}-\dfrac{\partial u_2}{\partial x_4}+\dfrac{\partial u_3}{\partial x_1}-\dfrac{\partial u_4}{\partial x_2}=0 \\ \dfrac{\partial u_1}{\partial x_4}-\dfrac{\partial u_2}{\partial x_3}-\dfrac{\partial u_3}{\partial x_2}+\dfrac{\partial u_4}{\partial x_1}=0 \end{cases} \text{otherwise,} \begin{pmatrix} \dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_4} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_4} & -\dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_4} & \dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_4} & -\dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = O \tag{2.1}$$

The previous system of partial differential equations could appropriately be viewed as a generalized version of the famous Cauchy-Riemann equation in complex functions.

Theorem2.1. Let the function $f(q)=u_1+u_2i+u_3j+u_4k$ be properly defined and analytic in region $D(\subset R^4)$, then

$$\begin{aligned} f'(q) &= \frac{\partial u_1}{\partial x_1}+i\frac{\partial u_2}{\partial x_1}+j\frac{\partial u_3}{\partial x_1}+k\frac{\partial u_4}{\partial x_1}=\frac{\partial u_2}{\partial x_2}-i\frac{\partial u_1}{\partial x_2}+j\frac{\partial u_4}{\partial x_2}-k\frac{\partial u_3}{\partial x_2} \\ &= \frac{\partial u_3}{\partial x_3}-i\frac{\partial u_4}{\partial x_3}-j\frac{\partial u_1}{\partial x_3}+k\frac{\partial u_2}{\partial x_3}=\frac{\partial u_4}{\partial x_4}+i\frac{\partial u_3}{\partial x_4}-j\frac{\partial u_2}{\partial x_4}-k\frac{\partial u_1}{\partial x_4} \end{aligned} \tag{2.2}$$

Proof. Since $f(q)=u_1+u_2i+u_3j+u_4k$ is analytic in region D , and $q\in D$, then $f(q)$ is differentiable at same point q , and the following limit holds

$$\lim_{\Delta q\to 0}\frac{f(q+\Delta q)-f(q)}{\Delta q}=f'(q) \tag{2.3}$$

Let the differences of quaternion functions be

$\Delta q=\Delta x_1+i\Delta x_2+j\Delta x_3+k\Delta x_4$, $f(q+\Delta q)-f(q)=\Delta u_1+i\Delta u_2+j\Delta u_3+k\Delta u_4$,

and denote $\Delta u_i=u_i(x_1+\Delta x_1,x_2+\Delta x_2,x_3+\Delta x_3,x_4+\Delta x_4)-u_i(x_1,x_2,x_3,x_4)$, then we could rewrite down the form of the limit(2.3)in a concise way

$$\lim_{\substack{\Delta x_m \to 0 \\ m=1,2,3,4}} \frac{\Delta u_1 + i\Delta u_2 + j\Delta u_3 + k\Delta u_4}{\Delta x_1 + i\Delta x_2 + j\Delta x_3 + k\Delta x_4} = f'(q) \tag{2.4}$$

The limit(2.3)always exists whenever the difference of quaternion functions Δq vanishes in any arbitrary direction. With the preceding assumption, we could set $\Delta x_1 \to 0,$ and $\Delta x_2 = \Delta x_3 = \Delta x_4 = 0$. Then we could derive another elegant formula from (2.4)and(2.3)

$$\lim_{\Delta x_1 \to 0}\left(\frac{\Delta u_1}{\Delta x_1} + i\frac{\Delta u_2}{\Delta x_1} + j\frac{\Delta u_3}{\Delta x_1} + k\frac{\Delta u_4}{\Delta x_1}\right) = f'(q) \quad ,$$

Therefore, four partial derivatives $\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_3}{\partial x_1}, \frac{\partial u_4}{\partial x_1}$ all exist, and such that

$$f'(q) = \frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} + j\frac{\partial u_3}{\partial x_1} + k\frac{\partial u_4}{\partial x_1}$$

Other formulas in(2.2)follows in a similar way.

Example 2.1 Discuss the analyticity of the function $f(q) = q^2 = (x_1 + ix_2 + jx_3 + kx_4)^2$.

Solution: The function $f(q)$ is well defined in entire R^4, and with some computations, we could obtain that $q^2 = (x_1^2 - x_2^2 - x_3^2 - x_4^2) + 2x_1x_2i + 2x_1x_3j + 2x_1x_4k$.

By the definition(2.1), it's easy to verify this function is analytic in R^4, and such that

$$\left(q^2\right)' = \frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} + j\frac{\partial u_3}{\partial x_1} + k\frac{\partial u_4}{\partial x_1} = 2x_1 + 2x_2 i + 2x_3 k + 2x_4 k = 2q$$

Therefore, this completes the claim that the function q^2 is analytic in R^4. By a similar method, we could prove that the function $f(q) = q^n$ is analytic in R^4 and $\left(q^n\right)' = nq^{n-1}$.

Theorem 2.2 Let two functions $f_1(q), f_2(q)$ be analytic in region D, then $f_1(q) \pm f_2(q),\quad f_1(q)\cdot f_2(q),\quad \frac{f_1(q)}{f_2(q)}, \left(f_2(q) \neq 0\right)$ are all analytic functions in the same region D.

Example2.2. Recalling Theorem2.2 and Example2.1, the polynomial function $f(q) = a_n q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0, a_i \in R(i = 0,1,\cdots,n)$ is analytic in R^4.

Thus, the fractional polynomial function

$$f(q) = \frac{f_1(q)}{f_2(q)} = \frac{a_n q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0}{b_m q^m + b_{m-1}q^{m-1} + \cdots + b_1 q + b_0}, a_i, b_j \in R(i = 0,1,\cdots,n, j = 1,2,\cdots,m)$$

is analytic, where $f_2(q) \neq 0$.

By the way, the errors of definitions of analytic functions via two operators(4)and(8)are unavoidable in (P. W. Yang. 2009), because even a simple function $f(q) = q^2$ could not be valid by the definition described in it.

3. Further open problems.

In the successive articles, we would like to discuss the analyticity of some other elementary functions of a quaternion variable, i.e. exponential function and logarithmic function etc.

References

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