Odd Graceful Labeling of Some New Graphs

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Abstract

In this work some new odd graceful graphs are investigated. We prove that the graph obtained by joining two copies of even cycle C_n with path P_k and two copies of even cycle C_n sharing a common edge are odd graceful graphs. In addition to this we derive that the splitting graph of $K_{1,n}$ as well as the tensor product of $K_{1,n}$ and P_2 admits odd graceful labeling.

Keywords: Odd graceful labeling, Splitting graph, Tensor product

1. Introduction

We begin with simple, finite and undirected graph $G=(V(G),E(G))$ with p vertices and q edges. For standard terminology and notations we follow (Harary F., 1972). We will provide brief summary of definitions and other information which serve as prerequisites for the present investigations.

Definition 1.1 If the vertices are assigned values subject to certain conditions then it is known as *graph labeling*. Labeled graphs have many diversified applications in coding theory particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. A systematic study of variety of applications of graph labeling is carried out by (Bloom G.S. and Golomb S.W., 1977). For detailed survey on graph labeling and related results we refer to (Gallian J.A., 2009). Most of the graph labeling schemes found their origin with graceful labeling which was introduced by (Rosa A., 1967) .

Definition 1.2 A function f is called *graceful labeling* of graph G if $f:V\rightarrow\{0,1,\dots,q\}$ is injective and the induced function $f^*:E\rightarrow\{1,2,\dots,q\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ is bijective. A graph which admits graceful labeling is called a *graceful graph*.

Many researchers have carried out significant work on graceful labeling. For e.g. In (Golomb S.W.,1972) it has been proved that complete graph K_n is not graceful for $n\geq 5$. In (Drake A and Redl T.A., 2006) the non graceful Eulerian graphs are enumerated.

The famous Ringel-Kotzig graceful tree conjecture and illustrious work in (Kotzig A., 1973) brought a tide of labeling problems having graceful theme. The present work is targeted to discuss one such labeling known as odd graceful labeling which is defined as follows.

Definition 1.3 A graph $G = (V(G),E(G))$ with p vertices and q edges is said to admit *odd graceful labeling* if $f:V(G)\rightarrow\{0,1,\dots,2q-1\}$ is injective and the induced function $f^*:E(G)\rightarrow\{1,3,5,\dots,2q-1\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ is bijective. A graph which admits odd graceful labeling is called an *odd graceful graph*.

(Gnanajothi R.B.,1991) introduced the concept of odd graceful graphs and she has proved many results on this newly defined concept. (Kathiresan K.M.,2008) has discussed odd gracefulness of ladders and graphs obtained from them by subdividing each step exactly once. (Sekar C.,2002) has proved that the splitting graph of path P_n and the splitting graph of even cycle C_n are odd graceful graphs.

Definition 1.4 For a graph G the *splitting graph* is obtained by adding to each vertex v, a new vertex v' so that v' is adjacent to every vertex that is adjacent to v in G.

Definition 1.5 The *tensor product* of two graphs G_1 and G_2 denoted by $G_1(T_p)G_2$ has vertex set $V(G_1(T_p)G_2) = V(G_1) \times V(G_2)$ and the edge set

$E(G_1(T_p)G_2) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$

Here we prove that the graph obtained by joining two copies of even cycles by path P_k, two copies of even cycles sharing a common edge, splitting graph of $K_{1,n}$ are graceful graphs. We also show that tensor product of $K_{1,n}$ and P_2 admits odd graceful labeling.

2. Main Results

Theorem 2.1 The graph obtained by joining two copies of cycle C_n of even order with the path P_k admits odd graceful labeling.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n and $u_1, u_2, ..., u_k$ be the vertices of path P_k. Consider two copies of C_n of even order. Let G be the graph obtained by connecting two copies of C_n with path P_k. Let $v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, ..., v_{[n+(k-2)]+n-1}, v_{2n+(k-2)}$ be the vertices of G and these vertices form a spanning path in G. In this spanning path the vertex v_n is the vertex common to the first copy of C_n and path P_k as well as the vertex $v_{[n+(k-2)]+1}$ is the vertex common to the second copy of C_n and path P_k. Define $f : V(G) \to \{0, 1, 2, ..., 2q-1\}$ as follows.

For $1 \le i \le n-1$

$f(v_i) = i-1$; i is odd.

$= (4n+2k)-(i+1)$; i is even.

For $n \le i \le \frac{3n}{2} + k - 1$

$f(v_i) = i-1$; i is even.

$= (4n+2k)-(i-1)$; i is odd

For $\frac{3n}{2} + k \le i \le 2n + k - 2$.

$f(v_i) = (4n+2k)-(i+1)$; i is odd

$= i+1$; i is even.

In accordance with the above labeling pattern the graph under consideration admits odd graceful labeling.

Illustration 2.2 Consider the graph obtained by attaching two copies of C_{10} by P_5. The labeling pattern is as shown in *Fig 1*.

Theorem 2.3 Two copies of even cycle C_n sharing a common edge is an odd graceful graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n of even order. Consider two copies of cycle C_n. Let G with $|V(G)| = 2n-2$ and $|E(G)| = 2n-1$ denotes the graph for two copies of even cycle C_n sharing a common edge. Without loss of generality let this edge be $e = v_{\frac{n+2}{2}} v_{\frac{3n}{2}}$. To define $f : V(G) \to \{0, 1, 2, ..., 2q-1\}$ two cases are to be considered.

Case 1: $n \equiv 0 (\mathrm{mod}\, 4)$.

For $1 \le i \le n+1$

$f(v_i) = i-1$; i is odd.

$= 4n-(i+1)$; i is even.

For $n+2 \le i \le \frac{3n}{2}$

$f(v_i) = i-1$; i is even.

$= 4n-(i+1)$; i is odd

For $\frac{3n+2}{2} \le i \le 2n-2$

$f(v_i) = i-1$; *i* is even.

$= 4n-i-3$; *i* is odd.

Case 2: $n \equiv 2(\mathrm{mod}\,4)$

For $1 \le i \le n$

$f(v_i)=i-1$; *i* is odd.

$= 4n-(i+1)$; *i* is even.

For $n+2 \le i \le \frac{3n}{2}$

$f(v_i) = i-1$; *i* is even.

$= 4n-(i+1)$; *i* is odd

For $\frac{3n+2}{2} \le i \le 2n-2$

$f(v_i) = i+1$; *i* is even.

$=4n-(i+1)$; *i* is odd.

Above defined labeling pattern exhausts all possibilities and in each case the graph under consideration admits graceful labeling.

Illustration 2.4 Consider two copies of cycle C_8 sharing a common edge. The labeling pattern is as shown in *Fig 2*.

Theorem 2.5 Splitting graph of a star admits odd graceful labeling.

Proof: Let $v, v_1, v_2, ..., v_n$ be the vertices of star $K_{1,n}$ with *v* be the apex vertex. Let *G* be the splitting graph of $K_{1,n}$ and v' , $v_1', v_2', ..., v_n'$ be the newly added vertices in $K_{1,n}$ to form *G*. Define $f : V(G) \to \{0,1,2,...,2q-1\}$ by

$f(v) = 0$

$f(v_i) = 6n-4i+3$; for $1 \le i \le n$

$f(v') = 2$

$f(v_i') = 2i-1$; for $1 \le i \le n$

In view of the above defined labeling pattern *G* admits odd graceful labeling.

Illustration 2.6 Fig 3 shows the labeling pattern of the splitting graph of $K_{1,4}$.

Theorem 2.7 $K_{1,n}(T_p)P_2$ is an odd graceful graph.

Proof: Let $u_1, u_2, ..., u_n, u_{n+1}$ be the vertices of star $K_{1,n}$, with u_1 be the apex vertex. Let v_1, v_2 be the vertices of P_2. Let *G* be the graph $K_{1,n}(T_p)P_2$. We divide the vertex of *G* into two disjoint sets $T_1 = \{(u_i, v_1) / i = 1,2,...,n+1\}$ and $T_2 = \{(u_i, v_2) / i = 1,2,...,n+1\}$.

Define $f : V(G) \to \{0,1,2,...,2q-1\}$ as follows.

$f(u_i, v_1) = 2(i-1)$; for $1 \le i \le n+1$

$f(u_1, v_2) = 1$

$f(u_i, v_2) = 2(n+i)-3$; for $2 \le i \le n+1$

The above defined function *f* provides graceful labeling for tensor product of $K_{1,n}$ and path P_2. That is, $K_{1,n}(T_p)P_2$ is an odd graceful graph.

Illustration 2.8 In Fig *4* the graph $K_{1,4}(T_p)P_2$ and its odd graceful labeling is shown.

3. Concluding Remarks

Gracefulness and odd gracefulness of a graph are two entirely different concepts. A graph may possess one or both of these or neither. In the present work we investigate four new families of odd graceful graphs. To investigate similar results for other graph families and in the context of different labeling techniques is an open area of research.

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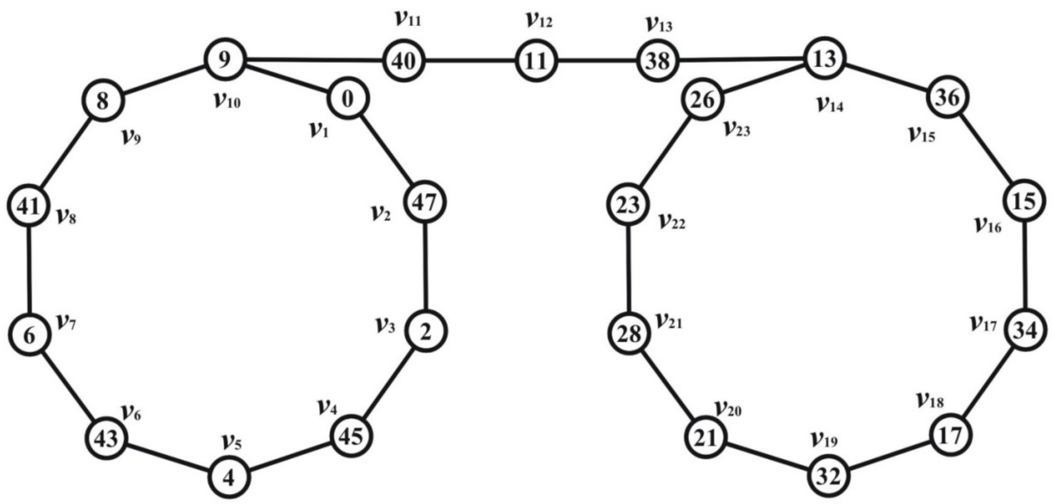

Figure 1. Odd graceful labeling of the graph obtained by joining two copies of cycle C_{10} by path P_5.

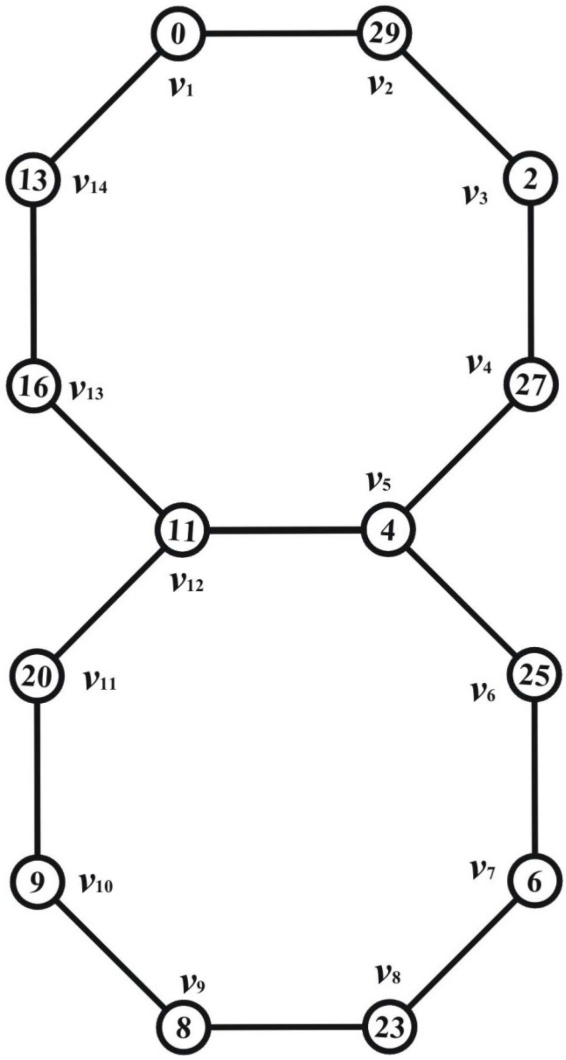

Figure 2. Odd graceful labeling of two copies of cycle C_8 sharing a common edge.

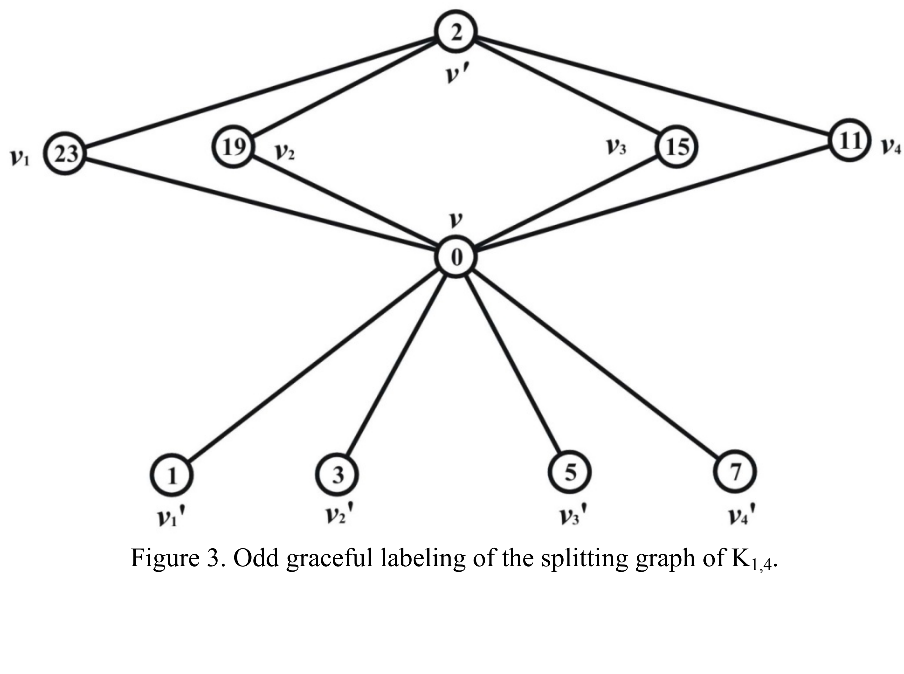

Figure 3. Odd graceful labeling of the splitting graph of $K_{1,4}$.

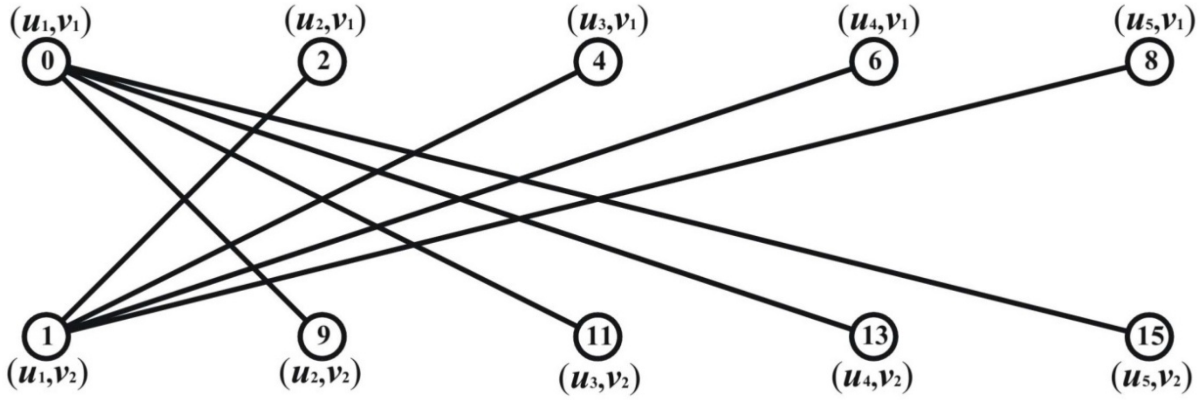

Figure 4. Odd graceful labeling of the tensor product of $K_{1,4}$ and P_2.