Solution of Free-Particle Radial Dependant Schrödinger Equation Using He's Homotopy Perturbation Method

A. A. Mahasneh
Department of Applied Physics, Faculty of Science, Tafila Technical University
P. O. Box: 179, Tafila 66110 (Jordan)
Tel: 96-27-9626-2750 E-mail: d_aamahasneh@yahoo.com

A. M. Al-Qararah
Jordan Nuclear Regularity Commission
P. O. Box: 830283, Amman 11183 (Jordan)
Tel: 96-27-9654-2076 E-mail: Thegeant@yahoo.com

Abstract

Homotopy perturbation method (HPM) is one of the newest analytical methods to solve linear and nonlinear differential equations. In this paper, HPM is used to formulate a new analytic solution of free-particle radial dependent Schrödinger equation. In contrast to the traditional perturbation methods, the homotopy method does not require a small parameter in the equation. In this method, according to the homotopy technique, a homotopy with an embedding parameter $\delta \in [0,1]$ is constructed, and the embedding parameter is considered as a small parameter. The obtained result shows the evidence of simplicity, usefulness, and effectiveness of the homotopy perturbation method for obtaining approximate analytical solutions for the radial dependent Schrödinger equation.

Keywords: Homotopy perturbation method, Centrifugal barrier, Free radial schrödinger operator, Embedding parameter

1. Introduction

In order to solve radial dependant Schrödinger equation for a free-particle; V(r) = 0, we will examine the application of the Homotopy Perturbation Method (HPM), which was proposed first by He (1999). The HPM is designed for solving differential and integral equations, linear and nonlinear, and has been the subject of extensive analytical and numerical studies. The method, which is a coupling of a homotopy technique and a perturbation technique, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, In contrast to the traditional perturbation methods, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. This method doesn't need linearization, perturbation or un-justified assumptions. The HPM yields the solution in terms of a rapid convergent series with easily computable components (He, 2003).

In the last two decades with the rapid development of differential equations science, there has appeared ever-increasing interest of scientists and engineers in the analytical techniques for linear and nonlinear problems. The widely applied techniques are perturbation methods.

Latif (2005) applied the HPM to search for exact analytical solutions of linear differential equations with constant coefficients. In addition, based on the precise integration method, a coupling technique of the variational iteration method (VIM) and HPM is proposed to solve nonlinear matrix differential equations. Rezania et al (2009) used HPM and VIM to solve the heat equations which are functions on time and space. This type of equation governs on numerous scientific and engineering experimentations. Zhang et al (2006) obtained an explicit analytical solution for nonlinear Poisson-Boltzmann equation by the HPM. Wang et al (2007) applied HPM to solve reaction-diffusion equations which is governed by the nonlinear ordinary differential equation. Furthermore, HPM is also applied to solve the Helmholtz equation, and the results reveal that this method is very effective and simple (Bizzar et al, 2008).

In this work, HPM, in a realistic and efficient way, is proposed to provide approximate solutions for Free-particle radial dependent Schrödinger equation (spherical Bessel equation).

2. Basic Idea of HPM:

To illustrate the basic ideas of the new HPM, we consider the following nonlinear differential equation

$$A(u) - F(r) = 0, \quad r \in \Omega \tag{1}$$

with boundary conditions

$$B(u, \frac{du}{dn}) = 0, \quad r \in \Gamma \tag{2}$$

where A is a general differential operator , F(r) is a known analytic function , Γ is the boundary of the domain Ω . The operator A can , generally speaking, be divided into two parts L and N , where L is linear, while N is nonlinear. Eq. (1), therefore, can be written as follows (He, 2003):

$$L(u) + N(u) - F(r) = 0 \tag{3}$$

By the homotopy technique (Liao, 1997), we construct a homotopy v (r, δ): Ω x [0,1]→R which satisfies

$$H(v,\delta) = (1-\delta)[L(v) - L(u_o)] + \delta\,[A(v) - F(r)] = 0, \quad \delta \in [0,1]\ , r \in \Omega \tag{4}$$

or

$$H(v,\delta) = L(v) - L(u_O) + \delta\, L(u_O) + \delta\,[N(v) - F(r)] = 0, \quad \delta \in [0,1], r \in \Omega \tag{5}$$

Where δ ∈ [0, 1] is an embedding parameter, u_o is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eq. (4) and Eq. (5), we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{6}$$

$$H(v,1) = A(v) - F(r) = 0 \tag{7}$$

The changing process of δ from zero to unity is just that of v(r, δ) from $u_o(r)$ to u(r). In topology, this is called deformation, and L(v) - L(u_o), A(v) – F(r) are called homotopic. We use the imbedding parameter δ as a “small parameter", and assume that the solution of Eq. (5) can be written as a power series in δ:

$$v = v_0 + \delta v_1 + \delta^2 v_2 + \delta^3 v_3 + \delta^4 v_4 + \dots\dots\dots\dots \tag{8}$$

Setting δ = 1 result in approximate solution of Eq. (1):

$$u = \lim_{\delta \to 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + \dots\dots\dots\dots \tag{9}$$

The coupling of the perturbation method is called the homotopy perturbation method, which has eliminated limitations of traditional methods. In the other hand, the proposed technique can take full advantage of the traditional perturbation techniques (He, 2004). The series (9) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator A(v):

1) The second derivative of N (v) with respect to v must be small, because the parameter δ may be relatively large, i.e. δ → 1.

2) The norm of $L^{-1}\left(\frac{\partial N}{\partial v}\right)$ must be smaller than one, in order that the series converges.

3. Free-Particle Radial Schrödinger Equation

In spherical coordinates, the Laplacian takes the form

$$\hat{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right) \tag{10}$$

The time-independent Schrödinger equation reads

$$\hat{H}_o\psi = E\psi \qquad \Leftrightarrow \qquad \left(\frac{-\hbar^2}{2m}\hat{\nabla}^2 + V(r)\right)\psi = E\psi \tag{11}$$

Where $\hat{H}_o$ is the Hamiltonian, V(r) is the potential energy, and E is the energy of the particle. In spherical coordinates, the Hamiltonian $\hat{H}_o$ takes the form

$$\hat{H}_o = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] - \frac{\hbar^2}{2m}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)\right] + V(r) \tag{12}$$

For a free particle, V(r) = 0, then $\hat{H}_o$ becomes

$$\hat{H}_o = \underbrace{-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]}_{1} \underbrace{- \frac{\hbar^2}{2m}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)\right]}_{2} \tag{13}$$

The first pat which involves only derivative with respect to r, describes the kinetic energy of the radial motion. The second part, describes the kinetic energy of the angular motion. The quantum angular momentum of the particle is given by the operator (Schiff, 1968)

$$\hat{L} = -i\hbar(\vec{r}\times\hat{\nabla}) \tag{14}$$

where $\hat{\nabla}$ is the expression for gradient in spherical coordinate and has the form

$$\hat{\nabla} = \hat{e}_r\frac{\partial}{\partial r}+\frac{1}{r}\left(\hat{e}_\theta\frac{\partial}{\partial\theta}+\hat{e}_\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{15}$$

Meanwhile, $\vec{r}=r\,\hat{r}$, so

$$\hat{L} = -i\hbar(\vec{r}\times\hat{\nabla}) = \left[\mathrm{r}\left(\hat{\mathrm{r}}\times\hat{\mathrm{r}}\right)\frac{\partial}{\partial r}+\left(\hat{\mathrm{r}}\times\hat{\theta}\right)\frac{\partial}{\partial\theta}+\left(\hat{\mathrm{r}}\times\hat{\varphi}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right] \tag{16}$$

But $\left(\hat{\mathrm{r}}\times\hat{\mathrm{r}}\right)=0$, $\left(\hat{\mathrm{r}}\times\hat{\theta}\right)=\hat{\varphi}$, and $\hat{\mathrm{r}}\times\hat{\varphi}=-\hat{\theta}$, and hence

$$\hat{L} = -i\hbar\left[\hat{\varphi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right] \tag{17}$$

the unit vectors $\hat{\theta}$ and $\hat{\varphi}$ are resolved into their Cartesian components (Tannoudji, 1977)

$$\begin{cases} L_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_x = -i\hbar\left(+\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_x = -i\hbar\left(\frac{\partial}{\partial\varphi}\right) \end{cases} \tag{18}$$

Therefore, the partial differential operator $\hat{L}^2 = \hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$ takes the form

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\varphi}\right] \tag{19}$$

and this operator involves only derivatives with respect to θ and φ. Hence, it is easy to rewrite the free particle Hamiltonian of Eq. (13), which becomes

$$\hat{H}_o = -\frac{\hbar^2}{2m}\hat{\nabla}^2 = \frac{\hbar^2}{2m}\left(-\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2m}\frac{1}{r^2}\hat{L}^2 \tag{20}$$

Hence $\hat{L}^2$ commutes with the radial kinetic energy and hence with $\hat{H}_o$. The kinetic energy operator also commutes with $\hat{L}_z$, because $\hat{L}_z$ commutes with $\hat{L}^2$ and with the expression depending on r. This proves that the operator of the kinetic energy is invariant under rotations (Eisberg and Rinsik, 1985):

$$[\hat{H}_o,\hat{L}^2]=0 \tag{21}$$

The same commutation relation holds for the operators in Cartesian coordinates, which are related to the operator in spherical coordinate. Any eigenspace of $\hat{L}^2$ is left invariant by $\hat{H}_o$. If ψ_ℓ is an eigenvector of $\hat{L}^2$, then $\hat{H}_o\psi_\ell$ is an eigenvector of $\hat{L}^2$ belonging to same eigenvalue (Dirac, 1999). In order to solve the Schrödinger equation in spherical coordinates, the trial function

$$\Psi(r,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r}R_{\ell m}(r)Y_\ell^m(\theta,\varphi) \tag{22}$$

could be used . $Y_{\ell m}(\theta,\varphi)$ are the eigen-functions of $\hat{L}^2$ with eigen-values $\hbar^2\ell(\ell+1)$. We see immediately that the partial differential operator of $\hat{H}_o$ becomes an ordinary differential operator

$$\boldsymbol{h}_{o\ell} = -\frac{\hbar^2}{2m}\hat{\nabla}^2 = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}\right) \tag{23}$$

which is called the *free radial Schrödinger operator*. The angular kinetic energy appears in the form of a potential energy $\ell(\ell+1)/r^2$. This term is called *centrifugal potential energy* or centrifugal barrier, because it has the effect of a repulsive force in the radial direction (Flugge, 1971). Immediately it can be shown that $\hat{H}_o \Psi = E\Psi$ holds if $R_{\ell m}(r)$ is a solution of the radial Schrödinger equation

$$\frac{\hbar^2}{2m}\left(-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}\right)R(r)=E\,R(r) \tag{24}$$

After multiplying by r^2 and rearranging the terms, we get

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{d\,R(r)}{dr}+\left(\frac{2mE}{\hbar^2}r^2-\ell(\ell+1)\right)R(r)=0 \tag{25}$$

where $k^2=2mE/\hbar^2$, therefore Eq. (25) becomes

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{d\,R(r)}{dr}+\left(k^2r^2-\ell(\ell+1)\right)R(r)=0 \tag{26}$$

This is the *spherical Bessel differential* equation. It can be transformed by letting $\xi=kr$, where then

$$r\frac{d\,R(r)}{dr}=kr\,\frac{d\,R(r)}{k\,dr}=\xi\frac{d\,R(r)}{d\,\xi} \tag{27}$$

$$r^2\frac{d^2R(r)}{dr^2}=\xi^2\frac{d^2R(r)}{d\,\xi^2} \tag{28}$$

Substitution of Eq. (27) and Eq. (28) into Eq. (26) gives

$$\xi^2\frac{d^2R(r)}{d\,\xi^2}+2\xi\frac{d\,R(r)}{d\,\xi}+\left(\xi^2-\ell(\ell+1)\right)R(r)=0 \tag{29}$$

Now, we look for a solution of the form $R(r)=Y(\xi)\,\xi^{-1/2}$, therefore

$$\frac{dR(r)}{d\,\xi}=\frac{dY(\xi)}{d\,\xi}\xi^{-1/2}-\frac{1}{2}Y(\xi)\,\xi^{-3/2} \tag{30}$$

$$\frac{d^2R(r)}{d\,\xi^2}=\frac{d^2Y(\xi)}{d\,\xi^2}\xi^{-1/2}-\frac{1}{2}\frac{dY(\xi)}{d\,\xi}\xi^{-3/2}-\frac{1}{2}\frac{dY(\xi)}{d\,\xi}\xi^{-3/2}-\frac{1}{2}(-\frac{3}{2})\,Y(\xi)\,\xi^{-5/2} \tag{31}$$

$$\frac{d^2R(r)}{d\,\xi^2}=\frac{d^2Y(\xi)}{d\,\xi^2}\xi^{-1/2}-\frac{dY(\xi)}{d\,\xi}\xi^{-3/2}+\frac{3}{4}Y(\xi)\,\xi^{-5/2} \tag{32}$$

Substitution of Eq. (30) and Eq. (32) into Eq. (26) leads to

$$\begin{aligned}&\xi^2\left(\frac{d^2Y(\xi)}{d\,\xi^2}\xi^{-1/2}-\frac{dY(\xi)}{d\,\xi}\xi^{-3/2}+\frac{3}{4}Y(\xi)\,\xi^{-5/2}\right)+\\ &2\xi\left(\frac{dY(\xi)}{d\,\xi}\xi^{-1/2}-\frac{1}{2}Y(\xi)\,\xi^{-3/2}\right)+\left(\xi^2-\ell(\ell+1)\right)\,Y(\xi)\,\xi^{-1/2}=0\end{aligned} \tag{33}$$

If we multiply Eq. (33) by $\xi^{-1/2}$, we will get

$$\xi^2\left(\frac{d^2Y(\xi)}{d\,\xi^2}-\frac{dY(\xi)}{d\,\xi}\xi^{-1}+\frac{3}{4}Y(\xi)\,\xi^{-2}\right)+2\xi\left(\frac{dY(\xi)}{d\,\xi}-\frac{1}{2}Y(\xi)\,\xi^{-1}\right)+\left(\xi^2-\ell(\ell+1)\right)\,Y(\xi)\;=0 \tag{34}$$

Summing up similar terms yields

$$\xi^2\frac{d^2Y(\xi)}{d\,\xi^2}+\xi\,\frac{dY(\xi)}{d\,\xi}+\left(\xi^2-(\ell+\tfrac{1}{2})^2\right)\,Y(\xi)\;=0 \tag{35}$$

But the solutions to this equation are *the ordinary Bessel functions* of half integral order; $J_{\ell+1/2}$ and $N_{\ell+1/2}$ (Griffiths, 1995) therefore

$$Y(\xi)=A_\ell\,J_{\ell+1/2}(\xi)+B_\ell\,N_{\ell+1/2}(\xi) \tag{36}$$

$J_{\ell+1/2}$ and $N_{\ell+1/2}$ are related to the spherical Bessel and Neumann functions, j_ℓ and n_ℓ, respectively by:

$$j_\ell(\xi)=\sqrt{\frac{\pi}{2}}\frac{J_{\ell+1/2}(\xi)}{\sqrt{\xi}} \tag{37}$$

$$n_\ell(\xi)=\sqrt{\frac{\pi}{2}}\frac{N_{\ell+1/2}(\xi)}{\sqrt{\xi}} \tag{38}$$

Substitution for $J_{\ell+1/2}$ and $N_{\ell+1/2}$ in Eq. (36) leads to

$$Y\,(\xi)=A_\ell\sqrt{\frac{2}{\pi}}\sqrt{\xi}\,j_\ell(\xi)+B_\ell\sqrt{\frac{2}{\pi}}\sqrt{\xi}\,n_\ell(\xi) \tag{39}$$

The Bessel functions j_ℓ are finite at the origin, but the Neumann functions blows up at the origin. Accordingly, we must have $B_\ell=0$, and hence the normalized solutions of Eq. (26) are

$$R(r)=\xi^{-1/2}Y\,(kr)=A_\ell^{'}\;j_\ell(kr) \tag{40}$$

The function $j_\ell(kr)$ is most commonly given in series form by

$$j_\ell(kr)=\sum_{p=0}^{\infty}A_\ell^{'}\frac{(-1)^p\,2^\ell\,k^{2p}\,(\ell+p)!}{p\,!(2\ell+2p+1)!}\,r^{\ell+2p} \tag{41}$$

4. Solution of Radial Dependent Schrödinger Equation for a Free-Particle Using HPM

In this section, we implement the HPM, in a realistic and efficient way, to provide approximate solutions for the radial dependence Schrödinger equation (spherical Bessel equation) subject to the condition that R(r) is bounded at r = 0. For the sake of continuity, this equation is rewritten here

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{d\,R(r)}{dr}+\left(k^2r^2-\ell(\ell+1)\right)R(r)=0 \tag{42}$$

In view of Eq. (4) or (5), the homotopy for Eq. (42) can be constructed as

$$H(R,\delta)=r^2\frac{d^2R(r)}{dr^2}+2r\frac{d\,R(r)}{dr}+\left[\delta k^2r^2-l(l+1)\right]R(r)=0\ \ ,\delta\in[0,1] \tag{43}$$

The basic assumption of HPM is that the solution R(r) can be expressed as a power of series in δ.

$$R(r)=\sum_{n=0}\delta^n\,R_n(r)=R_0(r)+\delta R_1(r)+\delta^2R_2(r)+\delta^3R_3(r).......... \tag{44}$$

The terms up to δ^3 are considered, where

$$R(r)\approx R_o(r)+\delta R_1(r)+\delta^2R_2(r)+\delta^3R_3(r) \tag{45}$$

$$\frac{dR(r)}{dr}\approx\frac{dR_o(r)}{dr}+\delta\frac{dR_1(r)}{dr}+\delta^2\frac{dR_2(r)}{dr}+\delta^3\frac{dR_3(r)}{dr} \tag{46}$$

$$\frac{d^2R(r)}{dr^2}\approx\frac{d^2R_o(r)}{dr^2}+\delta\frac{d^2R_1(r)}{dr^2}+\delta^2\frac{d^2R_2(r)}{dr^2}+\delta^3\frac{d^2R_3(r)}{dr^2} \tag{47}$$

Substitution of Eqs. (45-47) into Eq. (42) yields

$$\begin{aligned}&r^2\left(\frac{d^2R_o(r)}{dr^2}+\delta\frac{d^2R_1(r)}{dr^2}+\delta^2\frac{d^2R_2(r)}{dr^2}+\delta^3\frac{d^2R_3(r)}{dr^2}\right)+\\&2r\left(\frac{dR_o(r)}{dr}+\delta\frac{dR_1(r)}{dr}+\delta^2\frac{dR_2(r)}{dr}+\delta^3\frac{dR_3(r)}{dr}\right)+\\&\left[\delta k^2r^2-\ell(\ell+1)\right]\left(R_0(r)+\delta R_1(r)+\delta^2R_2(r)+\delta^3R_3(r)\right)=0\end{aligned} \tag{48}$$

Summing up the coefficient of like power of δ gives

$$\begin{aligned}&\delta^o\left(r^2\frac{d^2R_o(r)}{dr^2}+2r\frac{dR_o(r)}{dr}-\ell(\ell+1)R_o(r)\right)+\\&\delta^1\left(r^2\frac{d^2R_1(r)}{dr^2}+2r\frac{dR_1(r)}{dr}+k^2r^2R_o(r)-\ell(\ell+1)R_1(r)\right)+\\&\delta^2\left(r^2\frac{d^2R_2(r)}{dr^2}+2r\frac{dR_2(r)}{dr}+k^2r^2R_1(r)-\ell(\ell+1)R_2(r)\right)+\\&\delta^3\left(r^2\frac{d^2R_3(r)}{dr^2}+2r\frac{dR_3(r)}{dr}+k^2r^2R_2(r)-\ell(\ell+1)R_3(r)\right)=0\end{aligned} \tag{49}$$

where,

$$\begin{cases} \delta^o : r^2 \dfrac{d^2R_o(r)}{dr^2} + 2r\dfrac{dR_o(r)}{dr} - \ell(\ell+1)R_o(r) = 0 & (a_o) \\ \delta^1 : r^2 \dfrac{d^2R_1(r)}{dr^2} + 2r\dfrac{dR_1(r)}{dr} - \ell(\ell+1)R_1(r) = -k^2r^2R_o(r) & (a_1) \\ \delta^2 : r^2 \dfrac{d^2R_2(r)}{dr^2} + 2r\dfrac{dR_2(r)}{dr} - \ell(\ell+1)R_2(r) = -k^2r^2R_1(r) & (a_2) \\ \delta^3 : r^2 \dfrac{d^2R_3(r)}{dr^2} + 2r\dfrac{dR_3(r)}{dr} - \ell(\ell+1)R_3(r) = -k^2r^2R_2(r) & (a_3) \\ \vdots \\ \vdots \\ \delta^p : r^2 \dfrac{d^2R_p(r)}{dr^2} + 2r\dfrac{dR_p(r)}{dr} - \ell(\ell+1)R_p(r) = -k^2r^2R_{(p-1)}(r) & (a_p) \end{cases} \tag{50}$$

where $R_p(0) = R'_p(0) = 0$ for p = 1, 2, 3, ….., Consequently, the solution of Eq. (50-a_o) is

$$R_0(r) = C_1 r^\ell + \frac{C_2}{r^{\ell+1}}$$

(51)

Where C_1 and C_2 are arbitrary constants. Subsisting the initial condition R(r = 0) = 0 and taking into account the condition that R(r) is bounded at r = 0, we get

$$R_0(r) = C_1 r^\ell \tag{52}$$

After setting $R_0(r) = C_1 r^\ell$ in Eq. (50-a_1) we get

$$r^2 \frac{d^2R_1(r)}{dr^2} + 2r\frac{dR_1(r)}{dr} - \ell(\ell+1)R_1(r) = -C_1 k^2 r^{\ell+2} \tag{53}$$

The general solution of Eq. (53) is given by:

$$R_1(r) = A_1 - C_1 \frac{k^2}{4\ell+6} r^{\ell+2} \tag{54}$$

Where the arbitrary constant $A_1 = 0$, because $R_1(r) = 0$ at (r = 0). Therefore, the solution takes the form

$$R_1(r) = \frac{-k^2 C_1}{4\ell+6} r^{\ell+2} \tag{55}$$

The normalized radial dependent solution $R_1(r)$ is illustrated in Fig. (1) at different values of ℓ. Similarly, after setting $R_1(r) = -(k^2C_1/4\ell+6)r^{\ell+2}$ into Eq. (50-a_2) we get

$$r^2 \frac{d^2R_2(r)}{dr^2} + 2r\frac{dR_2(r)}{dr} - \ell(\ell+1)R_2(r) = +C_1 \frac{k^4}{4\ell+6} r^{\ell+4} \tag{56}$$

which has the solution

$$R_2(r) = A_2 + C_1 \frac{k^4}{(4\ell+6)(8\ell+20)} r^{\ell+4} \tag{57}$$

and the arbitrary constant $A_2 = 0$, because $R_2 = 0$ at (r = 0). Therefore,

$$R_2(r) = +C_1 \frac{k^4}{(4\ell+6)(8\ell+20)} r^{\ell+4} \tag{58}$$

The normalized radial dependent solution $R_2(r)$ is shown in Fig. (2) at different values of ℓ.
Equivalently, the solution of Eq. (50-a_3) is given by

$$R_3(r) = -C_1 \frac{k^6}{(4\ell+6)(8\ell+20)(12\ell+42)} r^{\ell+6} \tag{59}$$

In general, the p^{th} term is given by

$$R_p(r) = C_1 \frac{(-1)^p k^{2p} (2\ell+1)!(\ell+p)!}{\ell! p!(2\ell+2p+1)!} r^{\ell+2p} \tag{60}$$

It is worth to mention that the solutions $R_p(r)$ for p = 0, 1, 2, 3, …., satisfy Eq. (42).

As a result, according to Eq. (44), the solution of Eq. (42) can be constructed as

$$R(r) = \sum_{p=0}^{\infty} C_1 \frac{(-1)^p k^{2p} (2\ell+1)!(\ell+p)!}{\ell! p!(2\ell+2p+1)!} r^{\ell+2p} \tag{61}$$

For computing purposes, the solution provided by Eq. (61) as R(r) can be approximated by the p^{th} term $R_p(r)$

$$R_p(r) \approx \sum_{p=0}^{\infty} C_1 \frac{(-1)^p k^{2p} (2\ell+1)!(\ell+p)!}{\ell! p!(2\ell+2p+1)!} r^{\ell+2p} \tag{62}$$

The general solution R(r) up to the third term is illustrated in Fig. (3).

4. Conclusion

As a result, we conclude that the HPM can provides the expected solution and works successfully in handling differential equations directly which produces the solutions in terms of convergent series with easily computable components, and requires less computational work when compared with other methods. A convergent series solution for the radial dependent Schrödinger equation has been obtained, which is realistic and effective.

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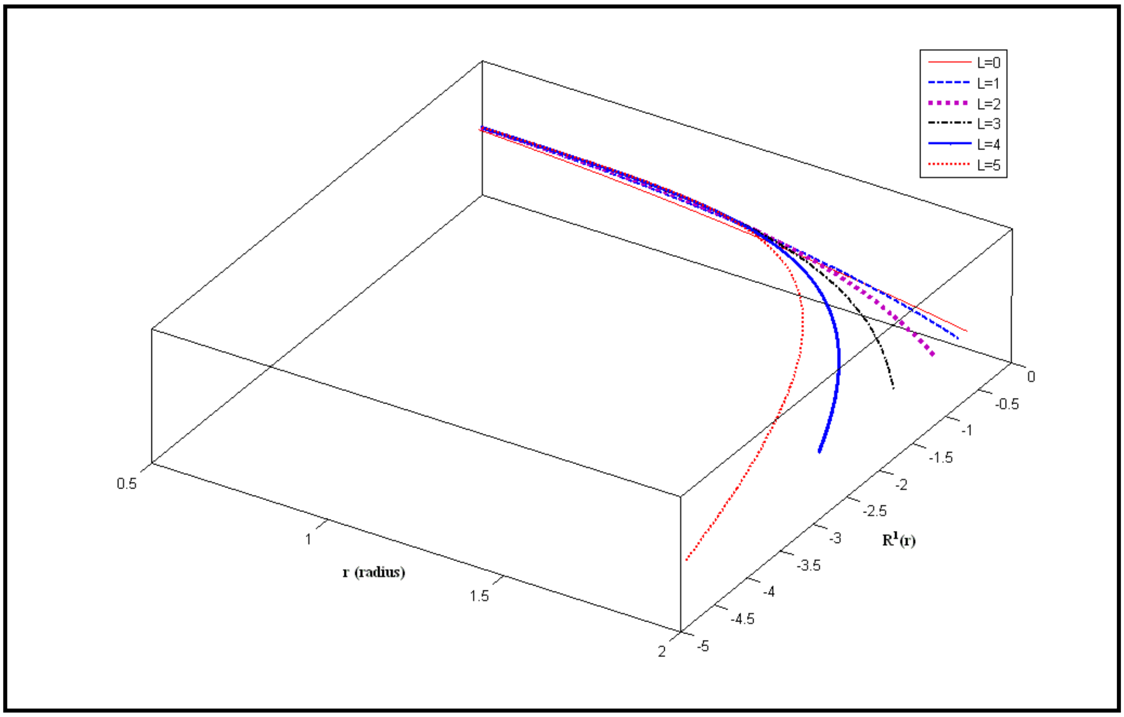

Figure 1. The radial solution $R_1(r)$ of Radial dependent Schrödinger equation (spherical Bessel equation) at different values of ℓ.

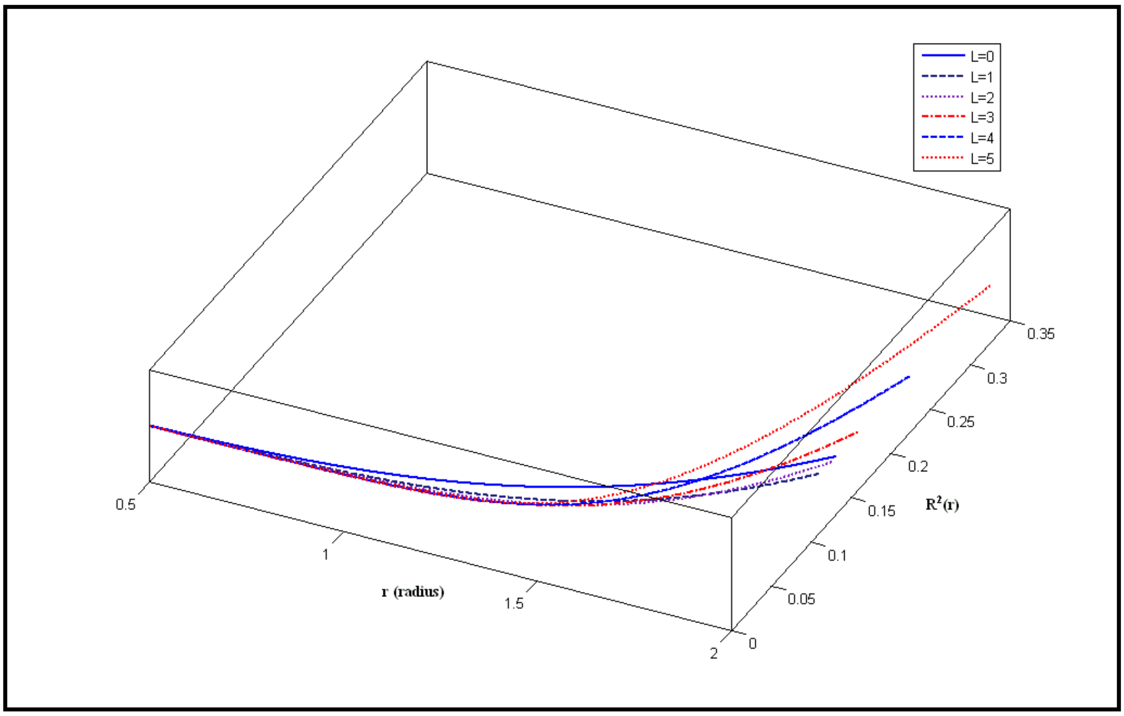

Figure 2. The radial solution $R_2(r)$ of the radial dependent Schrödinger equation (spherical Bessel equation) at different values of ℓ.

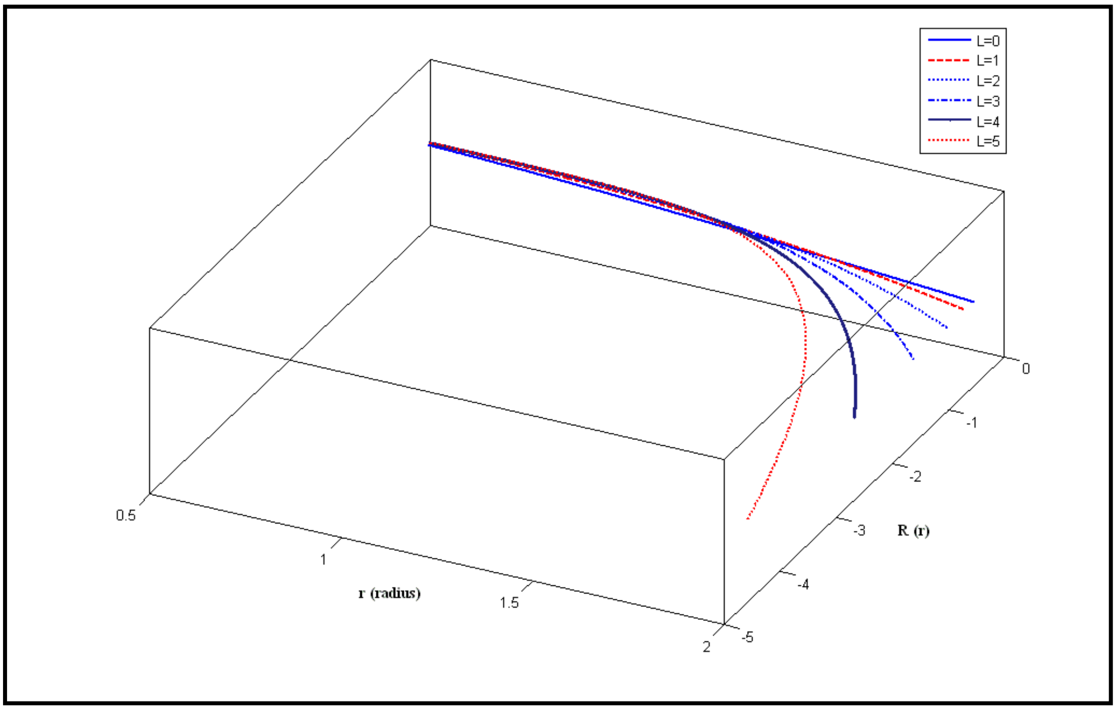

Figure 3. The general solution R(r) of the radial dependent Schrödinger equation (spherical Bessel equation) up to the third term of Eq. (61), at different values of ℓ.