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A Skewed Truncated Cauchy Uniform Distribution and Its Moments

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Abstract

Although usually normal distribution is considered for statistical analysis, however in many practical situations, distribution of data is asymmetric and using the normal distribution is not appropriate for modeling the data. Base on this fact, skew symmetric distributions have been introduced. In this article, between skew distributions, we consider the skew Cauchy symmetric distributions because this family of distributions doesn't have finite moments of all orders. We focus on skew Cauchy uniform distribution and generate the skew probability distribution function of the form $2f(u)G(\lambda u)$, where f is truncated Cauchy distribution and G is the distribution function of uniform distribution. The finite moments of all orders and distribution function for this new density function are provided. At the end, we illustrate this model using exchange rate data and show, according to the maximum likelihood method, this model is a better model than skew Cauchy distribution. Also the range of skewness and kurtosis for $0 \leq \lambda \leq 10$ and the graphical illustrations are provided.

Keywords: skew symmetric distribution, truncated Cauchy distribution, uniform distribution

1. Introduction

Skew symmetric distributions have been an attentive topic for many statistical researchers in the past few years. Many researchers have worked on this topic and introduced many new functions in this area. As a matter of fact, the story of skew symmetric distribution started by a paper published by Azzalini (1985). He introduced skew normal distribution with the following structure:

$$2f(x)G(\lambda x) \qquad -\infty < x < \infty, \lambda \in \mathbb{R}$$

where f and G respectively are the density and distribution function of the normal distribution. According to the lemma in the same paper, f should be a symmetric distribution around 0, G should be a continuous distribution function so that G' is symmetric around 0, and λ is a real constant. After Azzalini (1985), Mukhopadhyay and Vidakovic (1995) suggested to use the way which one can take f and G to belong to various families of probability distribution function. Therefore, other researchers could introduce many different skew symmetric distribution functions for using in various situations. For example, Nadarajah and Kotz (2003-2009) presented skewed distributions generated by normal, student's t, logistic, Cauchy, Laplace and uniform kernel. As an example for skew distribution with the uniform kernel, they took f to be the density function of uniform distribution and replaced G with normal, student's t, Laplace, Cauchy, logistic and uniform distribution functions. For all of these distributions were identified some properties such as finite moments of all orders and characteristic functions. But for skew distributions with Cauchy kernel only characteristic functions were provided. Gupta et al. (2002) introduced new models of skew symmetric distribution, where f and G are replaced with the pdf and cdf of normal, student's t, logistic, Cauchy, Laplace and uniform distribution. They also introduced some of their properties like characteristic function and their moments. They also could not find the finite moments for skew Cauchy distribution.

Some statistical researchers have worked on skewed distributions generated by the Cauchy kernel in order to provide more properties and solve some of the problems related to finite moments. In fact, skew Cauchy distribution is one of the most important distributions because of its ability in illustrating different phenomenas in a wide range of fields from physics to economics where researchers often have to deal with asymmetric data with heavy tails. In such situation, normal distributions are not suitable for modeling the data and skew Cauchy distribution is a better choice for fitting the data. Nadarajah and Kotz (2005) introduced skew distribution generated by the Cauchy kernel and presented their characteristic functions. However, they were not able to introduce moments of all orders because of the lack of finite moments. As a result, they tried to solve the problem in later stages of their research. In their first step, Nadarajah and Kotz (2006) introduced truncated Cauchy distribution with the following structure:

$$f(x) = \frac{1}{\sigma}\left\{\arctan\left(\frac{B-\mu}{\sigma}\right) - \arctan\left(\frac{A-\mu}{\sigma}\right)\right\}^{-1} \left\{1 + \left(\frac{x-\mu}{\sigma}\right)^2\right\}^{-1} \tag{1}$$

where $-\infty < A \le x \le B < \infty$, $\sigma > 0$ and $\mu \in \mathbb{R}$. Actually, Johnson and Kotz (1970) introduced this distribution however Nadarajah and Kotz (2006) found the finite moments of all orders for truncated Cauchy distribution. Finally, Nadarajah and Kotz (2007) introduced skewed truncated Cauchy distribution as follows:

$$f(x) = \frac{1}{2\arctan(h)(1+x^2)}\left\{1 + \frac{\arctan(\lambda x)}{\arctan(h)}\right\} \qquad -h \le x \le h, h > 0$$

by using the density and distribution function of truncated Cauchy distribution with $\mu = 0$ and $\sigma = 1$. In this paper, we consider skew Cauchy uniform model in paper of Nadarajah and Kotz (2005) and try to find moments of all orders of this model. In accordance to the paper of Nadarajah and Kotz (2007) and to the lemma introduced by Azzalini (1985), we take f as a truncated Cauchy distribution with the following structure

$$f(x) = \frac{1}{2\,\sigma\arctan\left(\frac{h}{\sigma}\right)} \left\{1 + \left(\frac{x}{\sigma}\right)^2\right\}^{-1} \qquad -h \le x \le h, \sigma > 0$$

and G as a cdf of the uniform distribution on $[-h, h]$. We define the pdf of skew truncated Cauchy uniform distribution as follows:

$$f_X(x) = \frac{1}{\sigma\ \arctan\left(\frac{h}{\sigma}\right)\left(1 + \frac{x^2}{\sigma^2}\right)} \frac{\lambda x + h}{2h} \tag{2}$$

where $-h \le x \le h$. We consider $\lambda \ge 0$, becauce there are the same properties for $\lambda < 0$ by using the fact that $G(\lambda x) = 1 - G(-\lambda x)$. When $\lambda = 0$, skew truncated Cauchy uniform pdf reduces to truncated Cauchy pdf.

The rest of this article is delivered as follows: In section 2 we introduce the basic of skew truncated Cauchy uniform distribution and its cumulative distribution function. In section 3 we provide finite moments of all orders when n is odd and even. Finally, in section 4, the application of this function in the economics, based on the maximum likelihood method, is illustrated. In this section, we use exchange rate data of Pound to Dollar from 1800 to 2003. Furthermore, the range of skewness and kurtosis for $0 \le \lambda \le 10$ and graphical illustration for different values of λ are provided.

For performing the calculations, we use the following lemma:

Lemma:

(Equation (3.194.5), Gradshteyn & Ryzhik, 2000): For$\mu > 0$,

$$\int_0^u \frac{x^{\mu-1}}{1+\beta x}dx = \frac{u^\mu}{\mu} \quad {}_2F_1(1, \mu; 1+\mu; -\beta u)$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}.$$

2. The Basic of Skew Truncated Cauchy Uniform Distribution

The density function of Cauchy distribution $C(\mu, \sigma)$ is

$$f_X(x) = \frac{1}{\pi\sigma\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)} \qquad -\infty < x < \infty$$

where $\mu \epsilon \mathbb{R}$ and $\sigma > 0$. In spite of its name, the first person who analyzed the properties of Cauchy distribution was a French mathematician (Poisson) in 1824. This distribution function is a symmetric distribution about μ and the spread of the distribution is related to σ. It means when the spread of distribution increases, the value of σ increases as well. It is possible to use Cauchy distribution in a number of different situation and areas. For example, the Cauchy distribution is utilized in explaining the distribution of the point of intersection P, which is fixed on a straight line with another variable straight line that randomly oriented according to the fixed point A. The intersection of vertical line from A to the fixed line is called O. The distance OP which is the distance of the point of intersection P from the point O has a Cauchy distribution with $\theta = 0$. According to this type of definition, the Cauchy distribution is used for describing the distribution of the points of particles impact from a point-source (A) with a fixed straight line. The Cauchy distribution also is used in physics to calculate the distribution of the energy of an unstable state in quantum mechanics with the name of Lorentzian distribution. Use of this function is becoming as common as normal distribution because the pdf of this distribution function is more peaked in the middle and has the fatter tails than the normal distribution. As a result, this function can be utilized in different areas including extreme risk analysis as well as financial applications. This is because of the functions tails. They are more realistic in the real world applications. The Cauchy distribution doesn't have any moments. For example to find the expectation value of standard Cauchy distribution we have:

$$E(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx \qquad -\infty < x < \infty.$$

It can be clearly seen that this integral is not completely convergence. Therefore, because of the main weakness of the Cauchy distribution which is the fact that it does not have any moments, the application of this distribution remains fairly limited.

Johnson and Kotz (1970) introduced the truncated Cauchy distribution (1) for solving the problems of the Cauchy distribution. They provided its cumulative distribution function as follows:

$$F(x) = \frac{\arctan\left(\frac{x-\mu}{\sigma}\right) - \arctan\left(\frac{A-\mu}{\sigma}\right)}{\arctan\left(\frac{B-\mu}{\sigma}\right) - \arctan\left(\frac{A-\mu}{\sigma}\right)}$$

for $-\infty < A \leq x \leq B < \infty$, $\mu = 0$ and $\sigma \geq 0$. They discussed estimation issues for symmetric standard case when $A = -B, \mu = 0$ and $\sigma = 1$. Moreover, Rohatgi (1976) provided the first two moments for truncated Cauchy distribution when $\mu = 0$ and $\theta = 1$. The choice of the limits, A and B is based on the historical records. Nadarajah and Kotz (2006) provided the moments of all orders for truncated Cauchy distribution. Therefore, truncated Cauchy distribution can be more useful in many different areas. A good example for efficiency of truncated Cauchy distribution is its use in characterizing employment productivity distribution. The main problem when analyzing employment productivity distribution is how to find the reasonable measure of minimal and maximal productivity of employees. One is likely to detect the mass of measurement errors according to downright faulty data or time accumulation problems. According to the Swedish data, Forsund and Londh (2004) choose mean of pay costs as the measure of minimal supportable productivity. They also detected that the empirical employment distribution between these two productivity values was well defined by a truncated Cauchy distribution. Truncated Cauchy distribution is also common prior density function to Bayesian models especially for analyzing economic data; A good example regarding to what was mentioned is provided by Bauwens et al. (1999).

In fact, truncated distributions can be used in many industrial settings. Final productions are topics for inspections of experiments before being sent to the client. The usual action is as follows: if a productions' implementation lies within certain tolerance limits, it is confirmed and sent to the client otherwise the product is rejected and, therefore, discarded for redoing. Therefore, the real distribution for the client is truncated. Another example can be detected in the multistage production process, where the inspection is implemented at each production stage. If only confirming products are sent to the next stage, the real distribution is truncated distribution. Another example that can be mentioned here is accelerated life testing with samples censored. In point of fact, the meaning of a truncated distribution plays a significant role in analyzing a variety of production processes, process optimization, and quality improvement.

Nadarajah and Kotz (2005) introduced skew distribution with Cauchy kernel as follows:

$$f(x)=\frac{2}{\pi\sigma}\left\{1+\left(\frac{x}{\sigma}\right)^2\right\}^{-1}G(\lambda x) \qquad -\infty<x<\infty$$

where $\lambda\in\mathbb{R}$ and $\sigma>0$ and G was replaced with distribution function of normal, Cauchy, Laplace, logistic, student's t and uniform distribution. However they faced with the same problems that exist in Cauchy distribution. Actually, there were not finite moments of all orders for skew distribution with Cauchy kernel. They tried to solve the distribution problems and make it more applicable in different areas. Therefore, Nadarajah and Kotz (2007) introduced skew truncated Cauchy distribution. They replaced f with pdf of truncated Cauchy distribution with $\mu=0$ and $\sigma=1$ and took G to be the cumulative distribution function of truncated Cauchy distribution and found finite moments of all orders. In this article, we focus on skew Cauchy uniform distribution and try to find finite moments of all orders. Hence, we introduce skew truncated Cauchy uniform distribution. According to the lemma in Azzalini (1985), we take f to be the truncated Cauchy distribution with $\mu=0$ and replace G with the distribution function of uniform distribution when $-h\le x\le h$ with the following structure:

$$G(x)=\begin{cases}0 & -h<x\\ \dfrac{x+h}{2h} & -h\le x<h\\ 1 & x\ge h.\end{cases}$$

The cdf of skew truncated Cauchy uniform is as follows:

$$F(x)=\begin{cases}0 & x<-\delta\\ \dfrac{\lambda\sigma}{4h\arctan\left(\frac{h}{\sigma}\right)}\left(\ln(\sigma^2+x^2)-\ln(\sigma^2+\delta^2)\right)+\dfrac{\arctan\left(\frac{x}{\sigma}\right)+\arctan\left(\frac{\delta}{\sigma}\right)}{2\arctan\left(\frac{h}{\sigma}\right)} & -\delta\le x<\delta\\ \dfrac{\arctan\left(\frac{x}{\sigma}\right)}{\arctan\left(\frac{h}{\sigma}\right)} & \delta\le x<h\\ 1 & x\ge h\end{cases} \tag{3}$$

where $\delta=\min\left\{h,\frac{h}{\lambda}\right\}$.

Proof:

When $-\delta\le x\le 0$

$$F(x)=\int_{-\delta}^{x}\frac{\sigma(\lambda t+h)}{2h\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt=\frac{\lambda\sigma}{2h\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{x}\frac{t}{\sigma^2+t^2}dt+\frac{\sigma}{2\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{x}\frac{1}{\sigma^2+t^2}dt$$

$$=\frac{\lambda\sigma}{4h\arctan(h)}\left(\ln(\sigma^2+x^2)-\ln(\sigma^2+\delta^2)\right)+\frac{\arctan\left(\frac{x}{\sigma}\right)+\arctan\left(\frac{\delta}{\sigma}\right)}{2\arctan\left(\frac{h}{\sigma}\right)}.$$

When $0\le x\le\delta$

$$F(x)=\int_{-\delta}^{0}\frac{\sigma(\lambda t+h)}{2h\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt+\int_{0}^{\delta}\frac{\sigma(\lambda t+h)}{2h\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt$$

$$=\frac{\lambda\sigma}{4h\arctan(h)}\left(\ln(\sigma^2+x^2)-\ln(\sigma^2+\delta^2)\right)+\frac{\arctan\left(\frac{x}{\sigma}\right)+\arctan\left(\frac{\delta}{\sigma}\right)}{2\arctan\left(\frac{h}{\sigma}\right)}.$$

When $\delta<x\le h$

$$F(x)=\int_{-\delta}^{\delta}\frac{\sigma\lambda t}{2h\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt+\int_{-\delta}^{\delta}\frac{\sigma}{2\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt+\int_{\delta}^{x}\frac{\sigma}{\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt$$

$$=\frac{\lambda\sigma}{2h\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{\delta}\frac{t}{\sigma^2+t^2}dt+\frac{\sigma}{2\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{\delta}\frac{1}{\sigma^2+t^2}dt+\int_{\delta}^{x}\frac{\sigma}{\arctan\left(\frac{h}{\sigma}\right)(\sigma^2+t^2)}dt$$

$$=\frac{\arctan\left(\frac{\delta}{\sigma}\right)}{\arctan\left(\frac{h}{\sigma}\right)}+\frac{\arctan\left(\frac{x}{\sigma}\right)-\arctan\left(\frac{\delta}{\sigma}\right)}{\arctan\left(\frac{h}{\sigma}\right)}=\frac{\arctan\left(\frac{x}{\sigma}\right)}{\arctan\left(\frac{h}{\sigma}\right)}.$$

3. Moment

Theorem1: If X has the pdf (2) then

$$E(X^n)=\frac{h^{n+1}}{(n+1)\sigma\arctan\left(\frac{h}{\sigma}\right)}\ {}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};-\frac{h^2}{\sigma^2}\right) \tag{4}$$

for n even.

Proof:

$$E(X^n)=\int_{-\delta}^{\delta}\frac{x^n}{\sigma\arctan\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})}\frac{\lambda x+h}{2h}dx+\int_{\delta}^{h}\frac{x^n}{\arctan\left(\frac{h}{\sigma}\right)\left(1+\frac{x^2}{\sigma^2}\right)}dx$$

$$=\int_{-\delta}^{\delta}\frac{\lambda x^{n+1}}{\sigma\arctan\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})}\frac{1}{2h}dx+\int_{-\delta}^{\delta}\frac{hx^n}{\sigma\arctan\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})}\frac{1}{2h}dx+\int_{\delta}^{h}\frac{x^n}{\sigma\arctan\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})}dx$$

$$=\frac{\lambda}{2h\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{\delta}\frac{x^{n+1}}{(1+\frac{x^2}{\sigma^2})}dx+\frac{1}{2\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{-\delta}^{\delta}\frac{x^n}{(1+\frac{x^2}{\sigma^2})}dx+\frac{1}{\arctan\left(\frac{h}{\sigma}\right)}\int_{\delta}^{h}\frac{x^n}{(1+\frac{x^2}{\sigma^2})}dx$$

$$=\frac{1}{\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{0}^{\delta}\frac{x^n}{1+\frac{x^2}{\sigma^2}}dx+\frac{1}{\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{0}^{h}\frac{x^n}{1+\frac{x^2}{\sigma^2}}dx-\frac{1}{\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{0}^{\delta}\frac{x^n}{1+\frac{x^2}{\sigma^2}}dx$$

$$=\frac{1}{\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{0}^{h}\frac{x^n}{1+\frac{x^2}{\sigma^2}}dx=\frac{1}{2\sigma\arctan\left(\frac{h}{\sigma}\right)}\int_{0}^{h^2}\frac{z^{\frac{n-1}{2}}}{1+\frac{z}{\sigma^2}}dz$$

Using equation (3.194.5), Gradshteyn & Ryzhik (2000)

$$E(X^n)=\frac{h^{n+1}}{(n+1)\sigma\arctan\left(\frac{h}{\sigma}\right)}\ {}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};-\frac{h^2}{\sigma^2}\right).$$

Theorem2: If X has the pdf (2) then:

$$E(X^n) = \frac{\lambda\delta^{n+2}}{(n+2)\,\mathrm{h\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+2}{2}; \frac{n+4}{2}; -\frac{\delta^2}{\sigma^2}\right)$$

$$+ \frac{h^{n+1}}{(n+1)\,\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{h^2}{\sigma^2}\right)$$

$$- \frac{\delta^{n+1}}{(n+1)\,\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{\delta^2}{\sigma^2}\right) \qquad (5)$$

for n odd.

Proof:

$$E(X^n) = \int_{-\delta}^{\delta} \frac{\lambda x^n}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})} \frac{\lambda x + h}{2h} dx + \int_{\delta}^{h} \frac{x^n}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})} dx$$

$$= \frac{\lambda}{2\sigma h \arctan\left(\frac{h}{\sigma}\right)} \int_{-\delta}^{\delta} \frac{x^{n+1}}{(1+\frac{x^2}{\sigma^2})} dx + \frac{1}{2\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{-\delta}^{\delta} \frac{x^n}{(1+\frac{x^2}{\sigma^2})} dx + \frac{1}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{\delta}^{h} \frac{x^n}{(1+\frac{x^2}{\sigma^2})} dx$$

$$= \frac{\lambda}{h\,\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{0}^{\delta} \frac{x^{n+1}}{(1+\frac{x^2}{\sigma^2})} dx + \int_{\delta}^{h} \frac{x^n}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)(1+\frac{x^2}{\sigma^2})} dx$$

$$= \frac{\lambda}{h\ \mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{0}^{\delta} \frac{x^{n+1}}{1+\frac{x^2}{\sigma^2}} dx + \frac{1}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{0}^{h} \frac{x^n}{1+\frac{x^2}{\sigma^2}} dx - \frac{1}{\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \int_{0}^{\delta} \frac{x^n}{1+\frac{x^2}{\sigma^2}} dx$$

Using equation (3.194.5), Gradshteyn & Ryzhik (2000)

$$E(X^n) = \frac{\lambda\delta^{n+2}}{(n+2)\,\mathrm{h\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+2}{2}; \frac{n+4}{2}; -\frac{\delta^2}{\sigma^2}\right)$$

$$+ \frac{h^{n+1}}{(n+1)\,\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{h^2}{\sigma^2}\right)$$

$$- \frac{\delta^{n+1}}{(n+1)\,\mathrm{\sigma arctan}\left(\frac{h}{\sigma}\right)} \quad {}_2F_1\left(1, \frac{n+1}{2}; \frac{n+3}{2}; -\frac{\delta^2}{\sigma^2}\right).$$

According to these two theorems, we can find the moments for all orders of this function. For example, we can find mean and variance of this model when $\sigma = 1$as follows:

$$E(X) = \frac{\lambda\delta^3}{3\,\mathrm{harctan}(h)} \quad {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; -\delta^2\right) + \frac{h^2}{2\,arctan(h)} \quad {}_2F_1(1,1;2;-h^2)$$

$$- \frac{\delta^2}{2\,arctan(h)} \quad {}_2F_1(1,1;2;-\delta^2)$$

$$= \frac{\lambda(\delta - \arctan(\delta))}{h\arctan(h)} + \frac{\ln(1+h^2) - \ln(1+\delta^2)}{2\arctan(h)}$$

and

$$V(X) = E(X^2) - \left(E(X)\right)^2.$$

On the other hand

$$E(X^2) = \frac{h^3}{3\arctan(h)} \quad {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; -h^2\right) = \frac{h - \arctan(h)}{\arctan(h)}$$

hence

$$V(X) = \frac{h - \arctan(h)}{\arctan(h)} - \left(\frac{\lambda(\delta - \arctan(\delta))}{\text{harctan}(h)} + \frac{\ln(1 + h^2) - \ln(1 + \delta^2)}{2\arctan(h)}\right)^2.$$

4. Discussion

The Cauchy distribution has been applied in many fields including but not limited to biological analysis, physics, survival analyzing, economics and reliability. In all of these fields, there has been no evidence found that supports the theory in which the empirical moments of any orders should be infinite. Therefore, the selection of the Cauchy distribution or skewed Cauchy symmetric distributions as a model is unreasonable. It is mainly because there are no finite moments of all orders. In this paper, we introduced skew truncated Cauchy uniform distribution and calculated finite moments of all of the orders. By doing so we have provided a model that is more useful in practical situations. As a very good example, we can point to the application of the skew truncated Cauchy uniform distribution to exchange rate (ER) data of United Kingdom Pound to the United State Dollar from the years 1800 to 2003. Data is obtained from the official website of Global Financial Data organization accessible in http://www.globalfinancialdata.com/. Global Financial Data (GFD) organization specializes in providing financial and economy data that extends from the 1200s to present. Since the Raw data of the chosen case study is not proper to use, we need to transform data to obtain logical fits. We transform data using logarithms and relative change from one year to the next year $\left(\frac{(old-new)}{0.5(old+new)}\right)$. The advantage of using relative change is that the data consists of pure numbers and it is independent of the units of measurement. We fitted both skew Cauchy distribution and skew truncated Cauchy uniform distribution by using maximum likelihood method. The maximum likelihood method is used to estimate the parameters of a model, test hypothesis about parameters and finally compare two models of the same data. A set of data and a mathematical model are essential elements in the maximum likelihood method. A mathematical model will have special unknown quantities which called parameters. The least square method finds the estimates of parameters of the model according to the minimum sum of square prediction error while the maximum likelihood method estimates the parameters base on maximizing the probability of a model fitting the data. For comparing different models in maximum likelihood method, the likelihood ratio test is used. Actually, for fitting and comparing different models, there are two other different tests which are called Wald test and Score test. For large samples, three of them converge but for small samples, most researchers prefer to use likelihood ratio test.

We considered $h = 1.5$ and $\sigma = 1$. A quasi-Newton algorithm nlm in R software was used to solve the likelihood equation. The two models are not nested but have the equal numbers of parameters. Final results for skew Cauchy distribution and skew truncated Cauchy uniform distribution are as follows, respectively:

$$\hat{\lambda} = -0.4809, \quad -\log L = 236.2354$$

and

$$\hat{\lambda} = -0.4179, \quad -\log L = 140.5781.$$

According to the standard likelihood ratio test, the skew truncated Cauchy uniform distribution is a much better model for exchange rate data.

On the other hand, we know the main feature of skew symmetric distribution is that the new parameter control skewness and kurtosis and provide a more flexible model which presents the data as properly as possible. According to the manner which was suggested in Azzalini (1986), we calculate the skewness and kurtosis of the standard truncated Cauchy distribution on $[-1,1]$. The skewness and kurtosis are 0 and 2.024 respectively. The skewness and kurtosis of standard skew truncated Cauchy uniform also is calculated on $[-1,1]$ for λ from 0 to 10. The range of possible values of skewness and kurtosis are $(-0.364, 0.208)$ and $(1.954, 2.331)$ respectively. It can be clearly seen that the new model exhibit the both positive and negative skewness and higher degree of peakness.

Figure 1 presents shapes of skew truncated Cauchy uniform distribution for different values of λ. It is obvious that it presents a variety of shapes.

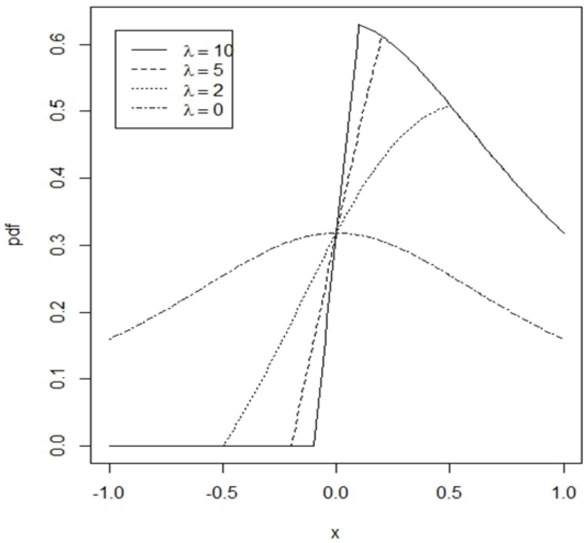

Figure 1. Examples of skew truncated Cauchy uniform distribution for $\lambda = 0,2,5,10, h = 1$ and $\sigma = 1$

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