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The Random of Lacunary Statistical on Γ^3 Over P-Metric Spaces Defined by Musielak Orlicz Functions

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Abstract

In this paper, we define and study the notion of lacunary statistical convergence and lacunary of statistical Cauchy sequences in random on Γ^3 over p-metric spaces defined by Musielak-Orlicz functions.

Keywords: analytic sequence, triple sequences, Γ^3 space, Musielak-Orlicz function random p-metric space, Lacunary sequence, Statistical convergence

1. Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geo-graphic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{r,s,t\to\infty} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \Gamma^3 E(mnk) = 0$$

Throughout ω, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively.

We write ω^3 for the set of all complex sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, ω^3 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \left(m, n, k = 1, 2, 3, ...\right)$$

A sequence $x = (S_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} \left|x_{mnk}\right|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$\left|x_{mnk}\right|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y)=\sup_{m,n,k}\left\{\left|x_{mnk}-y_{mnk}\right|^{\frac{1}{m+n+k}}: m,n,k:1,2,3,...\right\},$$

for all x = $\{x_{mnk}\}$ and y = $\{y_{mnk}\}$ in Γ^3. Let ϕ = {finite sequences}.

Consider a triple sequence x = (x_{mnk}). The (m, n, k)th section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]}$ = $\sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all m, n, k $\in \mathbb{N}$, where δ_{mnk} is a three dimensional matrix with 1 in the (m, n, k)th position and zero otherwise.

Consider a triple sequence x = (x_{mnk}). The (m, n, k)th section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]}$ = $\sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all m, n, k $\in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a 1 in the (i, j, k)th place for each i, j, q, $\in \mathbb{N}$.

An Orlicz function is a function M : $[0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with M(0) = 0, M(x) > 0, for x > 0 and M(x) $\to \infty$ as x $\to \infty$. If convexity of Orlicz function M is replaced by M(x + y) ≤ M (x) + M (y), then this function is called Orlicz function. An Orlicz function f is said to satisfy Δ2 – condition for all values u, if there exists K > 0 such that M (2u) ≤ Kf (u), u ≥ 0.

1.1 Lemma

Let M be an Orlicz function which satisfies Δ_2 – condition and let 0 < δ < 1. Then for each t ≥ δ, we have M (t) < $K\delta^{-1}$ f (2) for some constant K > 0.

A sequence M = (M_{mnk}) of Orlicz function is called a Musielak-Orlicz function. A sequence g = (g_{mnk}) defined by

$$g_{mnk}(\nu)=\sup\left\{|\nu|u-(M_{mnk})(u): u\geq 0\right\},\quad m, n, k = 1,2,...$$

is called the complementary function of a sequence of Musielak-Orlicz M. For a given sequence of Musielak-Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows.

$$t_M=\left\{x\in\omega^3 : I_M\left(\left|x_{mnk}\right|\right)^{1/m+n+k}\to 0 \text{ as } m,n,k\to\infty\right\},$$

Where IM is a convex modular defined by

$$I_M(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}M_{mnk}\left(\left|x_{mnk}\right|\right)^{1/m+n+k}, x=(x_{mnk})\in t_M.$$

2. Definition and Preliminaries

Let n $\in \mathbb{N}$ and X be a real vector space of dimension ω, where n ≤ ω. A real valued function $d_p(x_1,...,x_n)=\left\|(d_1(x_1),...,d_n(x_n))\right\|_p$ on X satisfying the following four conditions:

(i) $\left\|(d_1(x_1),...,d_n(x_n))\right\|_p=0$ if and only if $d_1(x_1),...,d_n(x_n)$ are linearly dependent,

(ii) $\left\|(d_1(x_1),...,d_n(x_n))\right\|_p$ is invariant under permutation,

(iii) $\left\|(\alpha d_1(x_1),...,d_n(x_n))\right\|_p=|\alpha|\left\|(d_1(x_1),...,d_n(x_n))\right\|_{p,}$ $\alpha\in\mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2,y_n)) = (dx(x_1, x_2, ... x_n)p + d_Y (y_1,y_2,...,y_n)^p)^{1/p}$ for $1 \le p < \infty$;(or)

(v) $d((x_1,y_1),(x_2,y_2),...,(x_n,y_n)):=\sup\left\{d_X(x_1,x_2,...,x_n),d_Y(y_1,y_2,...,y_n)\right\}$,

for x_1 x_2, ... $x_n \in$ X, y_1, y_2, ... $y_n \in$ Y is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n sub-space.

A trivial example of p product metric space is the p norm space is $X=\mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\left\|\left(d_1(x_1),\ldots,d_n(x_n)\right)\right\|_E = \sup\left(\left|\det\left(d_{mn}(x_{mn})\right)\right|\right) =$$

$$\sup\left(\left|\begin{matrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{12}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{mn}(x_{nm}) \end{matrix}\right|\right)$$

Where $x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n$ for each i = 1, 2, 3, ... n.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p-metric. Any complete p-metric space is said to be p-Banach metric space.

Let X be a linear metric space. A function $\omega: X \to \mathbb{R}$ is called paranorm, if

(1) $\omega(x) \geq 0$, for all $x \in X$;

(2) $\omega(-x) = \omega(x)$, for all $x \in X$;

(3) $\omega(x+y) \leq \omega(x) + \omega(y)$, for all $x, y \in X$;

(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $\omega(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\omega(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm ω for which $\omega(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, ω) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, p.183).

By the convergence of a triple sequence we mean the convergence on the Pringsheim sense that is, a triple sequence $x = (x_{mnk})$ has Prinsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{mnk} - L| < \epsilon$ whenever m, n, k > N. We shall write more briefly as P – convergent.

The triple sequence $\theta_{i,l,j} = \{(m_i, n_l, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0,\ h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h}_\ell = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h}_j = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \left\{(m, nk) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n < n_\ell \text{ and } k_{j-1} < k \leq k_j\right\},$$

$$q_k = \frac{m_k}{m_{k-1}}, \overline{q}_\ell = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \overline{\varphi}_s, \theta_{rs}$ is determined by

$$I_{rs} = \left\{(m, n) : m_{r-1} < m \leq m_r \text{ and } n_{s-1} < n \leq n_s\right\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \overline{q}_s.$$

The notion of λ-triple gai and triple analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{000} < \lambda_{111} < \ldots \text{and } \lambda_{mnk} \to \infty \text{ as } m, n, k \to \infty$$

and said that a sequence $x = (x_{mnk}) \in \omega^3$ is λ - convergent to 0, called a the λ - limit of x, if $\mu_{mnk}(x) \to 0$ as m, n, $k \to \infty$, where

$$\mu_{mnk}(x) = \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}}$$

$$\left(\Delta_\upsilon^{m-1} x_{mn} - \Delta_\upsilon^{m-1} x_{mn+1} - \Delta_\upsilon^{m-1} x_{mn+2} - \Delta_\upsilon^{m-1} x_{m+1n} - \Delta_\upsilon^{m-1} x_{m+1n+1} - \Delta_\upsilon^{m-1} x_{m+1n+2} - \Delta_\upsilon^{m-1} x_{m+2n}\right)$$

The sequence $x = (x_{mnk}) \in \omega^3$ is λ - triple analytic if $\sup_{u\upsilon w} |\mu_{mnk}(x)| < \infty$. If $\lim_{mnk} x_{mnk} = 0$ in the ordinary sense of convergence, then $\lim_{mnk}$

$$\left(\frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \left(\Delta_\upsilon^{m-1} x_{mn} - \Delta_\upsilon^{m-1} x_{mn+1} - \Delta_\upsilon^{m-1} x_{mn+2} - \Delta_\upsilon^{m-1} x_{m+1n} - \Delta_\upsilon^{m-1} x_{m+1n+1}\right)\right). = 0$$

This implies that

$$\lim_{mn} |\mu_{mnk}(x) - 0| = \lim_{mn}$$

$$\left|\left(\frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \left(\Delta_\upsilon^{m-1} x_{mn} - \Delta_\upsilon^{m-1} x_{mn+1} - \Delta_\upsilon^{m-1} x_{mn+2} - \Delta_\upsilon^{m-1} x_{m+1n} - \Delta_\upsilon^{m-1} x_{m+1n+1}\right)\right)\right| = 0.$$

Which yields that $\lim_{uvw} \mu_{mnk}(x) = 0$ and hence $x = (x_{mnk}) \in \omega^3$ is λ-convergent to 0.

Let I^3 – be an admissible ideal of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}, \theta_{rst}$ be a triple lacunary sequence, $M = (M_{mnk})$ be a Musielak-Orlicz function and $\left(X, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p\right)$ be a p – metric space, $q = (q_{mnk})$ be triple analytic sequence of strictly positive real numbers. By ω^3 (p-X) we denote the space of all sequences defined over $\left(X, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p\right)$. The following inequality will be used through out the paper. If $0 \le q_{mnk} \le \sup_{mnk} = H$, $K = \max(1, 2^{H-1})$ then

$$|a_{mnk} + b_{mnk}|^{q_{mnk}} \le K\left\{|a_{mnk}|^{qmnk} + |b_{mnk}|^{q_{mnk}}\right\}$$

for all m, n, k and $a_{mnk}, b_{mnk} \in \mathbb{C}$. Also $|a|^{q_{mnk}} \le \max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$. In the present paper we define the following sequence spaces.

$$\left[\Gamma_{M\mu}^{3q}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p^\varphi\right]_{\theta_{rst}}^{I^3} =$$

$$\left\{r, s, t \in I_{rst} : \left[M_{mnk}\left(\left\|\mu_{mnk}(x), (d(x_1), d(x_2), ..., d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}} \ge \epsilon\right\} \in I^3,$$

$$\left[\Lambda_{M\mu}^{3q}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p^\varphi\right]_{\theta_{rst}}^{I^3} =$$

$$\left\{r, s, t \in I_{rst} : \left[M_{mn}\left(\left\|\mu_{mnk}(x), (d(x_1), d(x_2), ..., d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}} \ge K\right\} \in I^3,$$

If we take $M_{mnk}(x) = x$, we get

$$\left[\Gamma_{M\mu}^{3q}, \|(d(x_1), d(x_2), ..., d(x_{n-1}))\|_p^\varphi\right]_{\theta_{rst}}^{I^3} =$$

$$\left\{r,s,t\in I_{rst}:\left[\left(\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\geq\in\right\}\in I^3,$$

$$\left[\Lambda^{3q}_{M\mu},\left\|\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}=$$

$$\left\{r,s,t\in I_{rst}:\left[\left(\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\geq K\right\}\in I^3,$$

If we take $q = (q_{mnk}) = 1$, we get

$$\left[\Gamma^{3}_{M\mu},\left\|\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}=$$

$$\left\{r,s,t\in I_{rst}:\left[M_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p\right)\right]\geq\in\right\}\in I^3,$$

$$\left[\Lambda^{3}_{M_\mu},\left\|\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}=$$

$$\left\{r,s,t\in I_{rst}:\left[M_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p\right)\right]\geq K\right\}\in I^3,$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\left[\Gamma^{3q}_{M\mu},\left\|\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ and $\left[\Lambda^{3q}_{M\mu},\left\|\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ which we shall discuss in this paper.

3. Main Results

3.1 Theorem

Let M = (M_{mnk}) be a Musielak-Orlicz functions, q = (q_{mnk}) be a triple analytic sequence of strictly positive real numbers, the sequence spaces $\left[\Gamma^{3q}_{M_\mu}, \left\|\left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ and $\left[\Lambda^{3q}_{M_\mu}, \left\|\left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ are linear spaces.

Proof

The proof can be established using standard technique.

3.2 Theorem

Let M = (M_{mnk}) be a Musielak-Orlicz functions, q = (q_{mnk}) be a triple analytic sequence of strictly positive real numbers, the sequence space $\left[\Gamma^{3q}_{M\mu}, \left\|\left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ is a paranormed space with respect to the paranorm defined by $g(x) = \inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}} \le 1\right\}$.

Proof

Clearly g (x) ≥ 0 for x = (x_{mnk}) $\in \left[\Gamma^{3q}_{M\mu}, \left\|\left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$

Since M_{mnk} (0) = 0, we get g (0) = 0.

Conversely, suppose that g (x) = 0, then

$$\inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}} \le 1\right\}$$

Suppose that $\mu_{mnk}(x) \ne 0$ for each m, n, k ∈ $\mathbb{N}$. Then $\left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p} \to \infty$. It follows that

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \to \infty.$$

which is a contradiction. Therefore μ_{mnk} (x) = 0. Let

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \le 1$$

and

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(y), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \le 1$$

Then by using Minkowski's inequality, we have

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(x+y), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \le$$

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H}+$$

$$\left(\left[M_{mnk}\left(\left\|\mu_{mnk}(y),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\right)^{1/H}.$$

So we have

$$g(x+y)=\inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(x+y),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\le 1\right\}\le$$

$$\inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\le 1\right\}+$$

$$\inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(y),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\le 1\right\}$$

Therefore,

$$g(x+y)\le g(x)+g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=\inf\left\{\left[M_{mnk}\left(\left\|\mu_{mnk}(\lambda x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\le 1\right\}.$$

Then

$$g(\lambda x)=\inf\left\{\left((|\lambda|t)^{q_{mnk}/H}:\left[M_{mnk}\left(\left\|\mu_{mnk}(\lambda x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\right)\le 1\right\}$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mnk}}\le \max\left(1,|\lambda|^{\sup p_{mnk}}\right)$, we have

$$g(\lambda x)\le \max\left(1,|\lambda|^{\sup p_{mnk}}\right)$$

$$\inf\left\{t^{q_{mnk}/H}:\left[M_{mnk}\left(\left\|\mu_{mnk}(\lambda x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}}\le 1\right\}$$

This completes the proof.

3.3 Theorem

(i) If the sequence (M_{mnk}) satissfies Δ_2 – condition, then

$$\left[\Gamma^{3q}_{M\mu},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right]^{I^{3\alpha}}_{\theta_{rst}}=\left[\Gamma^{3q\mu}_{g},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right]^{I^{3}}_{\theta_{rst}}.$$

(ii) If the sequence (g_{mnk}) satisfies Δ_2 – condition, then

$$\left[\Gamma^{3q\mu}_{g},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right]^{I^{3\alpha}}_{\theta_{rst}}=\left[\Gamma^{3q}_{M\mu},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right]^{I^{3}}_{\theta_{rst}}$$

Proof

Let the sequence (M_{mnk}) satisfies Δ_2 – condition, we get

$$\left[\Gamma^{3q\mu}_{g},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right]^{I^{3}}_{\theta_{rst}}\subset\left[\Gamma^{3q}_{M\mu},\left\|\mu_{mnk}(x),(d(x_1),d(x_2),...,d(x_{n-1}))\right\|^{\varphi}_p\right] \quad (3.1)$$

To prove the inclusion

$$\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3\alpha}}_{\theta_{rst}} \subset \left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}},$$

let

$$a \in \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3\alpha}}_{\theta_{rst}}.$$

Then for all $\{x_{mnk}\}$ with

$$(x_{mnk}) \in \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}}$$

we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|x_{mnk}a_{mnk}\right| < \infty. \tag{3.2}$$

Since the sequence (M_{mnk}) satisfies Δ_2 – condition then

$$(y_{mnk}) \in \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}},$$

we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|\frac{\varphi_{rst}y_{mnk}a_{mnk}}{\Delta^{m}\lambda_{mnk}}\right| < \infty.$$

by (3.2). Thus

$$(\varphi_{rst}a_{mnk}) \in \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}} =$$
$$\left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}}$$

and hence

$$(a_{mnk}) \in \left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}}.$$

This gives that

$$\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3\alpha}}_{\theta_{rst}} \subset \left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}} \tag{3.3}$$

we are granted with (3.1) and (3.3)

$$\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3\alpha}}_{\theta_{rst}} = \left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}}$$

(ii) Similarly, one can prove that

$$\left[\Gamma^{3q\mu}_{g}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3\alpha}}_{\theta_{rst}} \subset \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^{3}}_{\theta_{rst}}$$

if the sequence (g_{mnk}) satisfies Δ_2 – condition.

3.4 Proposition

If $0 < q_{mnk} < p_{mnk} < \infty$ for each m, n and k then

$$\left[\Lambda^{3q}_{f\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \subseteq \left[\Lambda^{3q}_{f\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

Proof

The proof can be established using standard technique.

3.5 Proposition

(i) If $0 < \inf q_{mnk} \le q_{mnk} < 1$ then

$$\left[\Lambda^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

(ii) If $I \le q_{mnk} \le \sup q_{mnk} < \infty$, then

$$\left[\Lambda^{3}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

Proof

The proof can be established using standard technique.

3.6 Proposition

Let $M' = \left(M'_{mnk}\right)$ and $M'' = \left(M''_{mn}\right)$ are sequences of Musielak – Orlicz functions, we have

$$\left[\Lambda^{3q}_{M'\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(d_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \cap$$

$$\left[\Lambda^{3q}_{M''\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \subseteq \left[\Lambda^{3q}_{M'+M''\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

Proof

The proof can be established using standard technique.

3.7 Proposition

For any sequence of Musielak-Orlicz functions $M = (M_{mnk})$ and $q = (q_{mnk})$ be triple analytic sequence of strictly positive real numbers. Then

$$\left[\Gamma^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

Proof

The proof can be established using standard technique.

3.8 Proposition

The sequence space $\left[\Gamma^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ is solid

Proof

Let

$$x = (x_{mnk}) \in \left[\Lambda^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}},$$

$$\sup_{mnk} \left[\Lambda^{3q}_{M\mu}, \left\| \mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} < \infty.$$

Let (α_{mnk}) be triple sequence of scalars such that $\left|\alpha_{mnk}\right| \le 1$ for all m, n, k $\in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then we get.

$$\sup_{mnk}\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(\alpha x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}} \leq$$
$$\sup_{mnk}\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}.$$

This completes the proof.

3.9 Proposition

The sequence space $\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$ is monotone

Proof

The proof follows from Proposition 3.8.

3.10 Proposition

If $M = (M_{mnk})$ be any Musielak-Orlicz functions. Then

$$\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^{**}}\right]^{I^3}_{\theta_{rst}}$$

if and only if $\sup_{r,s,t\geq 1}\frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty.$

Proof

Let $x \in \left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1)d(x_2),..,d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{I^3}_{\theta_{rst}}$ and $N = \sup_{r,s,t\geq 1}\frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty.$

Then we get

$$\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^{**}_{rst}}\right]^{I^3}_{\theta_{rst}} = N\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^*_{rst}}\right]^{I^3}_{\theta_{rst}} = 0.$$

Thus

$$x \in \left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^{**}}\right]^{I^3}_{\theta_{rst}}.$$

Conversely, suppose that

$$\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{\varphi^*}_{N_\theta} \subset \left[\Lambda^{3qu}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^{**}}\right]^{I^3}_{\theta_{rst}}$$

and

$$x \in \left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{I^3}_{\theta_{rst}}.$$

Then

$$\left[\Lambda^{3q}_{M\mu},\left\|\mu_{mnk}(x),\left(d(x_1),d(x_2),...,d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{I^3}_{\theta_{rst}} < \epsilon, \text{ for every } \epsilon > 0.$$

Suppose that $\sup_{r,s,t\geq 1}\frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} = \infty,$ then there exists a sequence of members (rst_{ijk}) such that $\lim_{i,j,k\to\infty}\frac{\varphi^*_{ijk}}{\varphi^{**}_{ijk}} = \infty.$

Hence, we have

$$\left[\Lambda^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi^*_{rs}}\right]^{I^3}_{\theta_{rst}} = \infty.$$

Therefore, $x \notin \left[\Lambda^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi^{**}}\right]^{I^3}_{\theta_{rst}}$, which is a contradiction. This completes the proof.

3.11 Proposition

If M = (M_{mnk}) be any Musielak-Orlicz functions. Then

$$\left[\Lambda^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi^*}\right]^{I^3}_{\theta_{rst}} = \left[\Lambda^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi^{**}}\right]^{I^3}_{\theta_{rst}}$$

if and only if $\sup_{r,s,t\geq 1} \frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty, \sup_{r,s,t\geq 1} \frac{\varphi^{**}_{rst}}{\varphi^*_{rst}} > \infty.$

Proof

The proof can be established using standard technique.

3.12 Proposition

The sequence space $\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$ is not solid

Proof

The result follows from the following example.

Example

Consider

$$x = (x_{mnk}) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & \ldots & 1 \end{pmatrix} \in \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), \ldots, d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}.$$

Let

$$\alpha_{mnk} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \end{pmatrix}, \text{for all } m, n, k \in \mathbb{N}.$$

Then $\alpha_{mnk} x_{mnk} \notin \left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$. Hence

$$\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$

is not solid.

3.13 Proposition

The sequence space $\left[\Gamma^{3q}_{M\mu}, \left\|\mu_{mnk}(x), \left(d(x_1), d(x_2), ..., d(x_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ is not monotone

Proof

The proof follows from Proposition 3.12.

Competing Interests

Authors have declared that no competing interests exist.

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