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Gyro Observability Research of a Relative Attitude Determination System Based on Stereo Vision

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Abstract

The analysis of the observability of system states is very important in the design of an optimal filter estimation algorithm. A relative attitude estimation algorithm is developed based on a stereo vision system and a gyroscope, and the observability of this algorithm is studied. First, we build the error model of the relative attitude determination system. Second, the observability of every state of the filter is studied. Third, by choosing different variables as the states of the error model, the unobservable subspace of the system is confirmed. Furthermore, the system structural decomposition reveals that this type of relative attitude determination system can only determine the relative attitude between the deputy and the chief and that their gyro drift errors are unobservable. In addition, the structural decomposition also tells us that when the feature points measured by the stereo vision system are greater than two, increasing the number of feature points provides little benefit for improving the observability of the gyro drift errors. Considering the incomplete observability of the original system, the star sensor is added into the system to enable it to be completely observable. The final simulation result indicates that after adding the star sensor, the system, which becomes completely observable, can estimate the body attitude, the relative attitude and the gyro error while providing improved accuracy.

Keywords: observability analysis, stereo vision, relative attitude determination, gyro error

1. Introduction

To form a constellation with small satellites, the relative attitude among each of the satellites must be determined independently. A stereo vision system is one of the vital pieces of equipment used to measure the relative position and the relative attitude. In addition, after adding a gyroscope into the system, the angular velocities can also be obtained. However, currently, increasing numbers of smaller and cheaper sensors are used in small satellites, and the precision of the relative attitude is becoming increasingly lower. But the optimal filter estimation algorithm can evaluate not only the attitude parameters of a small satellite but also the uncertain parameters in the observations. In this way, we can use smaller and cheaper sensors to achieve higher relative attitude determination accuracy.

Relative attitude determination based on stereo vision has attracted much attention and has been used in practice recently (Shay and Pini, 2009; Robert et al., 2000; Zhang et al., 2008). In particular, the optimal filtering algorithm is widely used to determine relative attitude (Kim et al., 2007; He et al., 2007). However, previous research studies did not consider the observability of a relative attitude determination system with stereo vision and gyros. Observability is an important factor in the estimation filter. Only observable states can be evaluated correctly and precisely. Maessen and Gill (2012) presented an investigation of the relative state estimation and observability for two formation flying satellites using two different relative navigation sensor sets. However, the paper only discusses the observability of the relative position between two satellites and not the relative attitude.

A stereoscopic vision system can provide the relative states between the chief spacecraft and the deputy spacecraft autonomously. Integrated with the gyro drift model, the filter can estimate spacecraft's attitude parameters and gyro drift simultaneously. Howerver, because the measurement model of stereoscopic vision system includes time-varying states, the observability analysis becomes rather difficult. Generally, to analyze the observability of a time-varying system, the Gram matrix of the system must be calculated, and then the Gram matrix must be analyzed to determine if it is nonsingular. However, the Gram matrix is obtained by numerical computation, so the character of the system cannot be researched theoretically. In 1992, Meskin and Itzhack presented the theory of PWCS (piece-wise constant system) to solve the inertial navigation in-flight alignment problem, and the theory was successfully implemented in many fields (Drora and Itzhack, 1992). However, when applying the PWCS theory to the relative attitude determination system based on stereo vision system with gyros, the observability of every state in the filter cannot be evaluated clearly.

The purpose of this paper is to study the observability of the relative attitude determination system based on stereo vision. The organization of this paper proceeds as follows. First, the relative coordinate systems are provided. Second, the state equation of the relative attitude errors and the measurement model of the stereoscopic vision system are given. Based on the state equation and measurement model, by choosing different variables as the states of the error model, the unobservable subspace of the system is confirmed. Furthermore, the structural decomposition of the system reveals the relationship between the unobservable states. Under the judgment that the system is incompletely observable, the star sensor is added into the system to enable it to be completely observable. Finally, the simulation results and conclusions are presented.

2. Relative Attitude Error Model

2.1 Coordinate Definition

Determining the deputy's relative attitude involes solving the rotation matrix, which is the deputy's body frame with respect to the chief's body frame. The following coordinate systems are used in this paper.

Orbital Frame: The origin O_o is located at the centroid of the spacecraft. The O_oZ_o axis is in the nadir direction; the O_oY_o axis is in the negative direction of the orbit normal direction; the O_oX_o axis completes the triad in the velocity vector direction for circular orbits. The deputy's orbital frame is expressed as $O_{oc}X_{oc}Y_{oc}Z_{oc}$, and the chief's orbital frame is expressed as $O_{ot}X_{ot}Y_{ot}Z_{ot}$. In the subscript, the letter "c" denotes the variables of the deputy or the chaser, the letter "t" denotes variables of the chief or the target, and the letter "o" denotes the variables in the orbital frame. In this paper, the chief's orbital frame $O_{ot}X_{ot}Y_{ot}Z_{ot}$ is the reference frame.

Body Frame: The origin O_b is located at the centroid of the spacecraft. The three axes are parallel to the three principal axes of the body, and are fixed to the spacecraft. If the spacecraft's attitude is zero, then the body frame coincides with the orbital frame. Generally, O_bX_b is called the roll axis, O_bY_b is called the pitch axis, and O_bZ_b is called the yaw axis. Likewise, the deputy's body frame is represented as $O_{bc}X_{bc}Y_{bc}Z_{bc}$, and the chief's body frame is represented as $O_{bt}X_{bt}Y_{bt}Z_{bt}$. In the subscript, the letter "b" denotes variables in the body frame.

Earth-Centered Inertial (ECI) Frame: The origin O is located at the center of the earth. The OX axis is in the vernal equinox direction. The OZ axis is the Earth's rotation axis, perpendicular to the equatorial plane. The OY axis is in the equatorial plane and finishes the triad of unit vectors. In the subscript, the letter "i" denotes variables in the ECI frame.

The relationship between each frame is depicted in Figure 1.

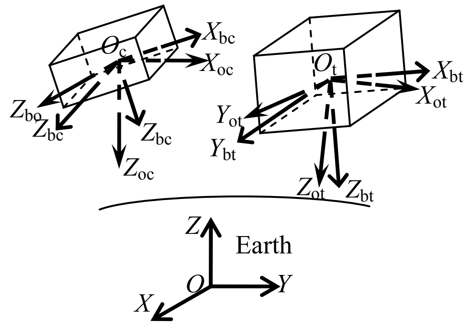

Figure 1. Relationship between the coordinate systems

2.2 Relative Attitude

Define the attitude quaternion $\boldsymbol{q} = [\boldsymbol{q}_{13} \quad q_4]^{\mathrm{T}} = [q_1 \quad q_2 \quad q_3 \quad q_4]^{\mathrm{T}}$, where $\boldsymbol{q}_{13}$ is the vector element, and the scalar q_4 is called the scalar element. This quaternion is considered to denote the true attitude of the spacecraft. However, in the real world, we can only measure this value by instruments, and we cannot determine the true value. As a result, the hat "^" is used to denote the variable that is estimated through the measured value. That is, $\hat{\boldsymbol{q}}$ is the estimated quaternion of $\boldsymbol{q}$. Consequently, there will be an error quaternion $\delta\boldsymbol{q} = [\delta\boldsymbol{\alpha} \quad \delta q_4]^{\mathrm{T}}$ between them. The error quaternion is defined as:

$$\boldsymbol{q} = \hat{\boldsymbol{q}} \otimes \delta\boldsymbol{q} \tag{1}$$

where "$\otimes$" represents the product of two quaternions.

Similarly, we define the relative attitude quaternion $\boldsymbol{q}_{ct}$, the estimated value of $\hat{\boldsymbol{q}}_{ct}$, and the error quaternion between them $\delta\boldsymbol{q}_{ct}$. Then, the relationship between them can be described as follows:

$$\boldsymbol{q}_{ct} = \hat{\boldsymbol{q}}_{ct} \otimes \delta\boldsymbol{q}_{ct} \tag{2}$$

Currently, the rate gyroscope is widely used in satellites. If we fix three rate gyroscopes in the spacecraft, then their input axes are parallel to the three body axes. Consequently, a simple but realistic gyro model can be described as (Farrenkopf, 1978):

$$\boldsymbol{\omega}_{gc}(t) = \boldsymbol{\omega}_{cbi}(t) + \boldsymbol{b}_{cbi}(t) + \boldsymbol{n}_{c1} \tag{3}$$

where $\boldsymbol{\omega}_{gc}(t)$ denotes the output vector of the deputy's three gyros; $\boldsymbol{\omega}_{cbi}(t)$ contains the true values of the three angular velocities along the deputy's body axes; the vector $\boldsymbol{b}_{cbi}(t)$ is the drift-rate bias of the deputy's three gyros; $\boldsymbol{n}_{c1}$ is its drift-rate noise, which is assumed to be a Gaussian white-noise process that obeys $N(0, \sigma_{gci}^2)$, ($i = x$, y, z). The gyro drift-rate bias $\boldsymbol{b}_{cbi}(t)$ is driven by another Gaussian white-noise, that is

$$\dot{\boldsymbol{b}}_{cbi}(t) = \boldsymbol{n}_{c2} \tag{4}$$

The three elements of $\boldsymbol{n}_{c2}$ obey $N(0, \sigma_{bci}^2)$, ($i = x$, y, z).

In Eq. (3), $\boldsymbol{\omega}_{cbi}(t)$ and $\boldsymbol{b}_{cbi}(t)$ are all the values we must determine, but actually, we can only determine $\hat{\boldsymbol{\omega}}_{cbi}(t)$ and $\hat{\boldsymbol{b}}_{cbi}(t)$, which are the estimated values of $\boldsymbol{\omega}_{cbi}(t)$ and $\boldsymbol{b}_{cbi}(t)$. Then, we can define $\Delta\boldsymbol{\omega}_{cbi}(t)$ and $\Delta\boldsymbol{b}_{cbi}(t)$, as the errors of $\boldsymbol{\omega}_{cbi}(t)$ and $\boldsymbol{b}_{cbi}(t)$. The relationship between these auantities can be described as follows:

$$\boldsymbol{\omega}_{cbi} = \hat{\boldsymbol{\omega}}_{cbi} + \Delta\boldsymbol{\omega}_{cbi} \tag{5}$$

$$\boldsymbol{b}_{cbi} = \hat{\boldsymbol{b}}_{cbi} + \Delta\boldsymbol{b}_{cbi} \tag{6}$$

Considering $\delta\boldsymbol{q}_{ct} = [\delta\boldsymbol{\alpha}_{ct} \quad \delta q_{ct4}]^{\mathrm{T}}$, under the small angle assumption, $\delta\boldsymbol{\alpha}_{ct}$ is nearly zero, δq_{ct4} is nearly 1. We can directly use the result of Kim *et al.* (2007), regarding the dynamics of $\delta\boldsymbol{\alpha}_{ct}$, which is described as follows:

$$\delta\dot{\boldsymbol{\alpha}}_{ct} = -\hat{\boldsymbol{\omega}}_{cbi} \times \delta\boldsymbol{\alpha}_{ct} - 0.5\Delta\boldsymbol{b}_{cbi} + 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})\Delta\boldsymbol{b}_{tbi} - 0.5\boldsymbol{n}_{c1} + 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})\boldsymbol{n}_{t1} \tag{7}$$

First, assuming that the deputy's and chief's drift-rate bias errors are all known variables, when only choosing $\delta\boldsymbol{\alpha}_{ct}$ as a state variable, the state equation can be modeled as follows:

$$\delta\dot{\boldsymbol{\alpha}}_{ct} = -[\hat{\omega}_{cbi} \times]\delta\boldsymbol{\alpha}_{ct} + [0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})^{\mathrm{T}} \quad -0.5\boldsymbol{I}_{3\times3} \quad 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})^{\mathrm{T}} \quad -0.5\boldsymbol{I}_{3\times3}] \begin{bmatrix} \Delta\boldsymbol{b}_{tbi} \\ \Delta\boldsymbol{b}_{cbi} \\ \boldsymbol{n}_{t1} \\ \boldsymbol{n}_{c1} \end{bmatrix} \tag{8}$$

Rewriting Eq. (8) with matrix symbols,

$$\delta\dot{\boldsymbol{\alpha}}_{ct} = \boldsymbol{F}_1\delta\boldsymbol{\alpha}_{ct} + \boldsymbol{G}_1\boldsymbol{n}_b \tag{9}$$

Actually, errors exist in every sensor, so we cannot determine the gyro's drift precisely. Thus, the deputy's and the chief's drift-rate bias errors should be estimated. Choosing $\delta\boldsymbol{\alpha}_{ct}$, $\Delta\boldsymbol{b}_{cbi}(t)$ and $\Delta\boldsymbol{b}_{tbi}(t)$ as state variables, the state equation can be modeled as follows:

$$\begin{bmatrix} \delta\dot{\boldsymbol{\alpha}}_{ct} \\ \Delta\dot{\boldsymbol{b}}_{tbi} \\ \Delta\dot{\boldsymbol{b}}_{cbi} \end{bmatrix} = \begin{bmatrix} -[\hat{\boldsymbol{\omega}}_{cbi}\times] & 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct}) & -0.5\boldsymbol{I}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\alpha}_{ct} \\ \Delta\boldsymbol{b}_{tbi} \\ \Delta\boldsymbol{b}_{cbi} \end{bmatrix} + \begin{bmatrix} 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct}) & -0.5\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_{t1} \\ \boldsymbol{n}_{c1} \\ \boldsymbol{n}_{t2} \\ \boldsymbol{n}_{c2} \end{bmatrix} \quad (10)$$

Rewriting Eq. (10) with matrix symbols,

$$\Delta\dot{\boldsymbol{X}} = \boldsymbol{F}\Delta\boldsymbol{X} + \boldsymbol{G}\boldsymbol{n} \quad (11)$$

2.3 Stereo Vision Measurement Equation

The stereo vision system provides the relative states between the chief and the deputy through the following three stages. First, obtain an image of the chief spacecraft, and then identify and match the pre-defined feature points in the image. Second, according to the parameters between the stereo vision system and the deputy, calculate the coordinate values of the feature points in the deputy's body frame. Finally, based on this information, the relative states can be estimated.

Under the assumption that the former two stages are accomplished by a stereo vision system, this paper only considers the situation in which the coordinate values of the feature points are given by the stereo vision system. Suppose $\boldsymbol{Z}_L(t_k)$ is the L-th feature point coordinate value in the deputy's body frame at the time step t_k. $\boldsymbol{R}_L$ is the corresponding points' coordinate values in the chief body frame; this value can be obtained by analyzing the chief's shape before-hand. $\boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))$ represents the attitude rotation matrix from the chief's body frame to the deputy's body frame at the time step t_k. $\boldsymbol{\rho}=[x \quad y \quad z]^{\mathrm{T}}$ is the position of the deputy in the chief's orbital frame. The relationship between the feature points' coordinates values in the deputy's body frame and the relative states can be modeled as,

$$\boldsymbol{Z}_L(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))[\boldsymbol{R}_L - \boldsymbol{\rho}(t_k)] \quad (12)$$

If we want to solve the relative position $\boldsymbol{\rho}$ and the relative attitude $\boldsymbol{q}_{ct}(t_k)$ from this equation, at least six equations are required. Namely, at least two feature points must be measured. Here, assuming L ($L \geqslant 2$) feature points is measured, and the coordinate values in the deputy body frame is defined as $\boldsymbol{Z}_1(t_k)$, $\boldsymbol{Z}_2(t_k)$, $\cdots$, $\boldsymbol{Z}_L(t_k)$. The corresponding points' coordinate values in the chief's body frame are known beforehand to be $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\cdots$, $\boldsymbol{R}_L$. For $\forall\ l$, $m \in N$, let $l \neq m$, the vector pair $\boldsymbol{Z}_m(t_k)$, $\boldsymbol{Z}_l(t_k)$ and $\boldsymbol{R}_m$, $\boldsymbol{R}_l$ are related by the following equations,

$$\boldsymbol{Z}_m(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))[\boldsymbol{R}_m - \boldsymbol{\rho}(t_k)] \quad (13)$$

$$\boldsymbol{Z}_l(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))[\boldsymbol{R}_l - \boldsymbol{\rho}(t_k)] \quad (14)$$

Subtracting the above two equations, we have

$$\boldsymbol{Z}_m(t_k) - \boldsymbol{Z}_l(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))[\boldsymbol{R}_m - \boldsymbol{R}_l] \quad (15)$$

Substituting the vector ($\boldsymbol{Z}_m(t_k) - \boldsymbol{Z}_l(t_k)$) and ($\boldsymbol{R}_m - \boldsymbol{R}_l$) into $\boldsymbol{M}_j(t_k)$ and $\boldsymbol{S}_l$, where j=1, …, N, and then the relative attitude measurement equation is modeled as,

$$\boldsymbol{M}_j(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))\boldsymbol{S}_j \quad (16)$$

Considering the factor that errors exist in every sensor, suppose $\boldsymbol{n}_{RL}(t_k)$ is the measurement noise of the stereo vision system and the three elements of $\boldsymbol{n}_{RL}(t_k)$ obey $N(0, \sigma_R^2)$. Then, the measurement model of stereo vision system can be described as,

$$\boldsymbol{M}_j(t_k) = \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))\boldsymbol{S}_j + \boldsymbol{n}_{RL}(t_k) \quad (17)$$

Rewriting the above equation with simple matrix symbols, its discrete form is as follows,

$$\boldsymbol{Z}_{Mk} = \boldsymbol{h}(\boldsymbol{q}_{ct,k}) + \boldsymbol{n}_{RLk} \quad (18)$$

This equation is a nonlinear equation for $\Delta\boldsymbol{X}$ and it must be linearized. Based on the theory of the Kalman filter, the partial derivative of $\boldsymbol{h}(\Delta\boldsymbol{X}_k)$ with respect to $\Delta\boldsymbol{X}$ can be computed as follows,

$$\boldsymbol{H}_k = \begin{bmatrix} 2[\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_1\times] & \boldsymbol{0}_{3\times6} \\ \vdots & \vdots \\ 2[\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_N\times] & \boldsymbol{0}_{3\times6} \end{bmatrix} \quad (19)$$

With the state equation and the measurement model, the relative states can be estimated using the filter algorithm.

3. Observability Analysis

3.1 Observability of the Relative Attitude

Considering a linear stochastic system that is composed of state equation (8) and measurement model (17), state matrix $\boldsymbol{F}_1$ can be supposed as a constant matrix because the deputy rotates around the earth at a constant angular velocity. This linear system is same as the one in Li *et al*. (1996). Farrenkopf (1978) concluded that if a few star (at least two) measurements around an orbit are available, then the system is completely observable. Based on this result, the three-order system expressed by Eq. (8) and Eq. (17) is completely observable. Thus, the relative attitude can be estimated by the stereo vision system without considering the sensor's error.

3.2 Observability of the Gyro Drift

Considering the fact that errors exist in every sensor, the actual filter uses a linear stochastic system expressed by Eq. (10) and Eq. (17). Make the following assumption,

$$\begin{cases} \boldsymbol{V}_{1,k} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix}^{\mathrm{T}} = 2\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_1 \\ \vdots \\ \boldsymbol{V}_{N,k} = \begin{bmatrix} v_{N1} & v_{N2} & v_{N3} \end{bmatrix}^{\mathrm{T}} = 2\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_N \end{cases} \tag{20}$$

Based on assumption of Eq. (20), the measurement matrix can be rewritten as,

$$\boldsymbol{H}_k = \begin{bmatrix} [\boldsymbol{V}_{1,k}\times] & \boldsymbol{0}_{3\times 6} \\ \vdots & \vdots \\ [\boldsymbol{V}_{L,k}\times] & \boldsymbol{0}_{3\times 6} \end{bmatrix} \tag{21}$$

Considering the n th power of $\boldsymbol{F}$ in Eq. (11), where $n \geq 1$.

$$\boldsymbol{F}^n = \begin{bmatrix} \boldsymbol{F}_{11}^n & \boldsymbol{F}_{12}^n & \boldsymbol{F}_{13}^n \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \end{bmatrix} \tag{22}$$

where,

$$\begin{aligned} &\boldsymbol{F}_{11}^n = (-1)^n \left[\hat{\boldsymbol{\omega}}_{cbi}\times\right]^n, \boldsymbol{F}_{13}^n = (-1)^n 0.5\left[\hat{\boldsymbol{\omega}}_{cbi}\times\right]^{n-1} \\ &\boldsymbol{F}_{12}^n = (-1)^{n-1} 0.5\left[\hat{\boldsymbol{\omega}}_{cbi}\times\right]^{n-1} \boldsymbol{A}(\hat{\boldsymbol{q}}_{ct}) \end{aligned} \tag{23}$$

Considering the state-space model, which is composed of $\boldsymbol{F}$ and $\boldsymbol{H}$, the observability matrix can be computed as,

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{a1}^{\mathrm{T}} & \cdots & \boldsymbol{Q}_{an}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{24}$$

where,

$$\boldsymbol{Q}_{a1} = \begin{bmatrix} [\boldsymbol{V}_{1,k}\times] & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \\ \vdots & \vdots & \vdots \\ [\boldsymbol{V}_{L,k}\times] & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \end{bmatrix} \tag{25}$$

$$\boldsymbol{Q}_{an} = \begin{bmatrix} [\boldsymbol{V}_{1,k}\times]\boldsymbol{F}_{11}^n & [\boldsymbol{V}_{1,k}\times]\boldsymbol{F}_{12}^n & [\boldsymbol{V}_{1,k}\times]\boldsymbol{F}_{13}^n \\ \vdots & \vdots & \vdots \\ [\boldsymbol{V}_{L,k}\times]\boldsymbol{F}_{11}^n & [\boldsymbol{V}_{L,k}\times]\boldsymbol{F}_{12}^n & [\boldsymbol{V}_{L,k}\times]\boldsymbol{F}_{13}^n \end{bmatrix} \tag{26}$$

From Eq. (23), we can see $\boldsymbol{F}_{13}$ right multiplied by $\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})$ is $\boldsymbol{F}_{12}$, and as an attitude matrix, $\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct})$ is a nonsingular matrix. This means $\boldsymbol{F}_{13}$ can be changed into $\boldsymbol{F}_{12}$ by a finite number of elementary column operations, so a column of $\boldsymbol{F}_{13}$ and a column of $\boldsymbol{F}_{12}$ are linearly correlated. Additionally, vectors $\boldsymbol{V}_{1,k}$, …,

$\boldsymbol{V}_{L,k}$ are linearly independent, so the rank of the observability matrix $\boldsymbol{Q}$ is not greater than 6. The dimension of the state equation is 9, i.e., the system is incomplete observable. Compared with the linear system expressed by Eq. (8) and Eq. (17), this system only adds the states of the deputy's and the chief's drift-rate bias errors; i.e., these errors consist of unobservable states. We will analyze the unobservable states by system structural decomposition.

3.3 Unobservable States Analysis

Choose the 1st, 2nd, 4th, 13th, 14th and 16th rows from the observability matrix $\boldsymbol{Q}$, and choose another three row vectors that are linearly independent to the former six row vectors. Then, all nine of the row vectors form the transformation matrix $\boldsymbol{P}$.

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{P}_1^{\mathrm{T}} & \boldsymbol{P}_2^{\mathrm{T}} & \boldsymbol{P}_3^{\mathrm{T}} & \boldsymbol{P}_4^{\mathrm{T}} & \boldsymbol{P}_5^{\mathrm{T}} & \boldsymbol{P}_6^{\mathrm{T}} & \boldsymbol{P}_7^{\mathrm{T}} & \boldsymbol{P}_8^{\mathrm{T}} & \boldsymbol{P}_9^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \tag{27}$$

where, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$, $\boldsymbol{P}_4$, $\boldsymbol{P}_5$, $\boldsymbol{P}_6$ are the 1st, 2nd, 4th, 13th, 14th and 16th rows of the observability matrix $\boldsymbol{Q}$, respectively, and the remaining $\boldsymbol{P}_7$, $\boldsymbol{P}_8$ and $\boldsymbol{P}_9$ form the matrix as follows,

$$\begin{bmatrix}\boldsymbol{P}_7\\ \boldsymbol{P}_8\\ \boldsymbol{P}_9\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}_{3\times6} & \boldsymbol{I}_{3\times3}\end{bmatrix} \tag{28}$$

Performing the system structural decomposition by the linearly nonsingular transformation $\Delta\boldsymbol{Y}=\boldsymbol{P}\Delta\boldsymbol{X}$, we have,

$$\Delta\dot{\boldsymbol{Y}}=\boldsymbol{PFP}^{-1}\Delta\boldsymbol{Y}+\boldsymbol{PGn}=\boldsymbol{F}_{o\bar{o}}\Delta\boldsymbol{Y}+\boldsymbol{G}_{o\bar{o}}\boldsymbol{n} \tag{29}$$

$$\boldsymbol{Z}_k=\boldsymbol{H}_k\boldsymbol{P}^{-1}\Delta\boldsymbol{Y}+\boldsymbol{n}_{RLk}=\boldsymbol{H}_{ko\bar{o}}\Delta\boldsymbol{Y}+\boldsymbol{n}_{RLk} \tag{30}$$

where, $\boldsymbol{F}_{o\bar{o}}$, $\boldsymbol{G}_{o\bar{o}}$ and $\boldsymbol{H}_{ko\bar{o}}$ are computed as follows,

$$\boldsymbol{F}_{o\bar{o}}=\left[\begin{array}{cc|c}\boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3}\\ \boldsymbol{0}_{3\times3} & \boldsymbol{F}_{33} & \boldsymbol{0}_{3\times3}\\ \hline \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3}\end{array}\right],\quad \boldsymbol{G}_{o\bar{o}}=\left[\begin{array}{c}\boldsymbol{G}_o\\ \hline \boldsymbol{G}_{\bar{o}}\end{array}\right] \tag{31}$$

$$\boldsymbol{H}_{ko\bar{o}}=\left[\begin{array}{c|c}\boldsymbol{H}_{ko} & \boldsymbol{0}_{3L\times3}\end{array}\right] \tag{32}$$

The non-zero elements in the above equation can be computed by symbolic computation software, so we do not show them in details here. From the newly transformed linear system, the last three states of $\Delta\boldsymbol{Y}$ are clearly unobservable and the relationship between $\Delta\boldsymbol{Y}$ and $\Delta\boldsymbol{X}$ is given by Eq. (33)~ Eq. (35).

$$\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{bmatrix}=\begin{bmatrix}0 & -v_{13} & v_{12}\\ v_{13} & 0 & -v_{11}\\ 0 & -v_{23} & v_{22}\end{bmatrix}\delta\boldsymbol{\alpha}_{ct} \tag{33}$$

$$\begin{bmatrix}\Delta y_7\\ \Delta y_8\\ \Delta y_9\end{bmatrix}=\Delta\boldsymbol{b}_{cbi} \tag{34}$$

$$\begin{bmatrix}\Delta y_4\\ \Delta y_5\\ \Delta y_6\end{bmatrix}=\begin{bmatrix}p_{41} & p_{42} & p_{43}\\ p_{51} & p_{52} & p_{53}\\ p_{61} & p_{62} & p_{63}\end{bmatrix}\delta\boldsymbol{\alpha}_{ct}+\begin{bmatrix}p_{44} & p_{45} & p_{46}\\ p_{54} & p_{55} & p_{56}\\ p_{64} & p_{65} & p_{66}\end{bmatrix}\Delta\boldsymbol{b}_{tbi}+\frac{1}{2}\begin{bmatrix}0 & v_{13} & -v_{12}\\ -v_{13} & 0 & v_{11}\\ 0 & v_{23} & -v_{22}\end{bmatrix}\Delta\boldsymbol{b}_{cbi} \tag{35}$$

From the above equations, we can obtain the following two results:

(1) $\Delta\boldsymbol{b}_{cbi}$ and $\Delta\boldsymbol{b}_{tbi}$ are unobservable states, and $\delta\boldsymbol{\alpha}_{ct}$ can be computed from $[\Delta y_1 \;\; \Delta y_2 \;\; \Delta y_3]^{\mathrm{T}}$, so $\delta\boldsymbol{\alpha}_{ct}$ are observable states.

(2) When $L \geq 2$, the observability of this system has no relationship with the number of feature points measured by the stereo vision system.

According to the above results, the drift-rate bias errors $\Delta\boldsymbol{b}_{cbi}$ and $\Delta\boldsymbol{b}_{tbi}$ cannot be evaluated by the stereo vision system alone. One way to improve the observability of the original linear system is to add external sensors that measure deputy's attitude, and then the drift-rate bias errors $\Delta\boldsymbol{b}_{cbi}$ and $\Delta\boldsymbol{b}_{tbi}$ can be estimated at the same time. This method will be described in detail in the following section.

4. Modified Relative Attitude Filter

4.1 Deputy's Attitude

Directly using the results of Lefferts et al. (1982), choose the deputy's drift-rate bias errors $\Delta\boldsymbol{b}_{cbi}$ and $\delta\boldsymbol{q}_{13}$ as state variables, where $\delta\boldsymbol{q}_{13}$ is the first three elements of $\delta\boldsymbol{q}_c$, which is the deputy's attitude error quaternion. Then, the state equation, which is used in the attitude determination filter, is described as follows:

$$\begin{bmatrix} \delta\dot{\boldsymbol{q}}_{13} \\ \Delta\dot{\boldsymbol{b}}_{cbi} \end{bmatrix} = \begin{bmatrix} -[\hat{\boldsymbol{\omega}}_{cbo}\times] & -0.5\boldsymbol{I}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q}_{13} \\ \Delta\boldsymbol{b}_{cbi} \end{bmatrix} + \begin{bmatrix} -0.5\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_{c1} \\ \boldsymbol{n}_{c2} \end{bmatrix} \tag{36}$$

Here, we adopt a star sensor as the deputy's attitude measurement sensor. At the time step t_k, suppose the sensor coordinate system is parallel to the body frame, then the measurement model of the star sensor is

$$\boldsymbol{Z}_j(t_k) = \boldsymbol{A}(\boldsymbol{q}_c(t_k))\boldsymbol{A}_{oi}(t_k)\boldsymbol{L}_j + \boldsymbol{n}_{sj}(t_k) \tag{37}$$

where, $\boldsymbol{Z}_j(t_k)$ is the j th measurement vector of the star sensor in the deputy's body frame. $\boldsymbol{A}(\boldsymbol{q}_c(t_k))$ represents the attitude rotation matrix from the deputy's orbital frame to the deputy's body frame at time step t_k. $\boldsymbol{A}_{oi}(t_k)$ is the attitude rotation matrix from the ECI frame to the orbital frame at the time step t_k. $\boldsymbol{L}_j$ is the j th starlight vector described in the ECI frame, which is already known in the ephemeris. $\boldsymbol{n}_{sj}(t_k)$ represents measurement noise, and the three elements of $\boldsymbol{n}_{sj}(t_k)$ obey $N(0,\sigma_S^2)$.

4.2 Attitude Fusion Filter

Adding $\delta\boldsymbol{q}_{13}$ to the state equation (10), we can redefine the state vector as $\Delta\boldsymbol{X}_{re}$ = $[\delta\boldsymbol{\alpha}_{ct} \quad \Delta\boldsymbol{b}_{tbi} \quad \Delta\boldsymbol{b}_{cbi} \quad \delta\boldsymbol{q}_{13}]^{\mathrm{T}}$, and then form the fusion filter,

$$\Delta\dot{\boldsymbol{X}}_{re} = \boldsymbol{F}_{re}\Delta\boldsymbol{X}_{re} + \boldsymbol{G}_{re}\boldsymbol{n} \tag{38}$$

The undefined variables are given as follows,

$$\boldsymbol{F}_{re} = \begin{bmatrix} -[\hat{\boldsymbol{\omega}}_{cbi}\times] & 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct}) & -0.5\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & -0.5\boldsymbol{I}_{3\times3} & -[\hat{\boldsymbol{\omega}}_{cbo}\times] \end{bmatrix} \tag{39}$$

$$\boldsymbol{G}_{re} = \begin{bmatrix} 0.5\boldsymbol{A}(\hat{\boldsymbol{q}}_{ct}) & -0.5\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \boldsymbol{0}_{3\times3} & -0.5\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \tag{40}$$

Suppose that the stereo vision system identifies the chief's three vertices, $\boldsymbol{R}_1$=$[-0.5\ 0.5\ 0.5]^{\mathrm{T}}$, $\boldsymbol{R}_2$=$[-0.5\ 0.5\ -0.5]^{\mathrm{T}}$, $\boldsymbol{R}_3$=$[-0.5\ -0.5\ 0.5]^{\mathrm{T}}$. Meanwhile, suppose that the star sensor identifies two stars. Then, Eq. (17) and Eq. (37) are combinde to form the measurement model,

$$\boldsymbol{Z}_{RSk} = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))\boldsymbol{S}_1 \\ \boldsymbol{A}(\boldsymbol{q}_{ct}(t_k))\boldsymbol{S}_2 \\ \boldsymbol{A}(\boldsymbol{q}_c(t_k))\boldsymbol{A}_{oi}(t_k)\boldsymbol{L}_1 \\ \boldsymbol{A}(\boldsymbol{q}_c(t_k))\boldsymbol{A}_{oi}(t_k)\boldsymbol{L}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{n}_{R1}(t_k) \\ \boldsymbol{n}_{R2}(t_k) \\ \boldsymbol{n}_{s1}(t_k) \\ \boldsymbol{n}_{s2}(t_k) \end{bmatrix} \tag{41}$$

Rewriting the above equation with simple matrix symbols, its discrete form is as follows,

$$\boldsymbol{Z}_{RSk} = \boldsymbol{h}_{RS}(\boldsymbol{q}_{c,k}, \boldsymbol{q}_{ct,k}) + \boldsymbol{n}_{s\,Rk} \tag{42}$$

The partial derivative of $\boldsymbol{h}_{RS}(\Delta\boldsymbol{X}_{re})$ with respect to $\Delta\boldsymbol{X}_{re}$ can be computed as follows,

$$\boldsymbol{H}_{RS,k} = 2\begin{bmatrix} [A(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_1\times] & \boldsymbol{0}_{3\times6} & \boldsymbol{0}_{3\times3} \\ [A(\hat{\boldsymbol{q}}_{ct,k})\boldsymbol{S}_2\times] & \boldsymbol{0}_{3\times6} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times6} & [A(\hat{\boldsymbol{q}}_{c,k})\boldsymbol{A}_{oi,k}\boldsymbol{L}_1\times] \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times6} & [A(\hat{\boldsymbol{q}}_{c,k})\boldsymbol{A}_{oi,k}\boldsymbol{L}_2\times] \end{bmatrix} \tag{43}$$

Considering the linear system expressed by Eq. (38) and Eq. (42), due to the addition of star sensor, the rank of measurement matrix $\boldsymbol{H}_{RS,k}$ can reach 6. Then, we can prove that if the measurement vectors are linearly independent vectors, the modified system will be a completely observable system.

5. Modified Relative Attitude Filter

According to the preceding analysis, the drift-rate bias errors cannot be estimated by the stereo vision system alone. In addition, after adding the star sensor, the filter can estimate the deputy's attitude, relative attitude and gyro error simultaneously. To validate this result, filters with a star sensor and without a star sensor are all simulated under same parameters at the same time. The parameters used in the simulation are shown in the following table.

Table 1. Simulation parameters

Gyros	Sensors	Others
Standard deviation of the drift-rate noise σ_{gi}: 0.05 °/h	Star sensor σ_s : 50 arcsec	Orbit angular velocity ω_o: 10^{-3} rad/s
Standard deviation of the drift-rate bias driven noise σ_{bi} : 0.03 °/h	Stereo vision system σ_R : 0.05 m	Sampling time T : 1 s
Drift-rate bias b: 150 °/h (Deputy), 100 °/h (Chief)		

The initial condition is shown as follows: $\Delta\hat{\boldsymbol{X}}_{re0} = \boldsymbol{0}_{15\times3}$, $\Delta\hat{\boldsymbol{b}}_{cbi0} = \Delta\hat{\boldsymbol{b}}_{tbi0} = \boldsymbol{0}_{3\times1}$, $\hat{\boldsymbol{q}}_{c0} = \hat{\boldsymbol{q}}_{ct0}$ =[0 0 0 1]$^{\mathrm{T}}$, $\boldsymbol{P}_0$ = diag[10^{-4} $\boldsymbol{I}_{3\times3}$ 0.005^2 $\boldsymbol{I}_{3\times3}$ 0.005^2 $\boldsymbol{I}_{3\times3}$ 10^{-4} $\boldsymbol{I}_{3\times3}$]. Under the same measurement data, the estimation results of the chief's and the deputy's gyro drift-rate bias are shown in the following figures.

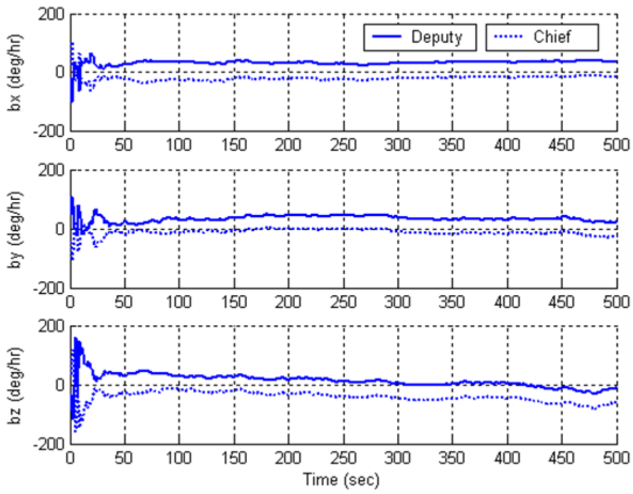

Figure 2. Gyro rate bias estimation of the deputy and of the chief by the stereo vision system alone

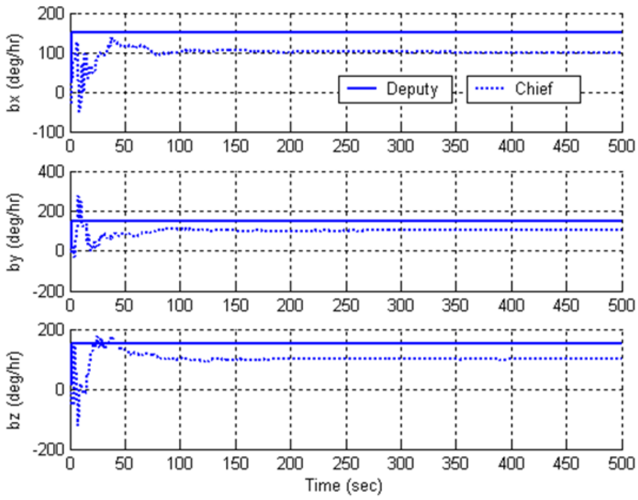

Figure 3. Gyro rate bias estimation of the deputy and of the chief after adding the star sensor

From figure 2, we can see that the gyro bias of the chief and the deputy are not correctly estimated by the stereo vision system alone, although they converge to certain values. This figure validates the result that $\Delta \boldsymbol{b}_{cbi}$ and $\Delta \boldsymbol{b}_{tbi}$ are unobservable states using a numerical approach. In addition, figure 3 shows that gyro bias values of the chief and of the deputy converge to 100º/h and 150º/h, respectively. These values are the same as the ones predefined in Table 1. Due to the addition of the star sensor in the deputy, figure 3 also shows that the gyro bias values of the chief and of the deputy are not converged in the meantime. This result shows that the gyro drift-rate bias errors are corrected by the star sensor, and the modified system is a completely observable system.

6. Conclusions

This paper analyses the gyro drift observability of the relative attitude determination filter based on the stereo vision system. By choosing different variables as states of the error model, we conclude that the gyro drift-rate bias errors are unobservable states. Further analysis of the system structural decomposition reveals that increasing the number of feature points does little to improve the observability of the gyro drift error when the feature points measured by the stereo vision system are greater than two. Based on the above results, a star sensor is added in the original system to form a modified system. The simulation results indicated that the modified system becomes completely observable. The analysis presented here represents a beneficial reference for designing a stable relative attitude filter.

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References

Drora, G. M., & Itzhack, Y. B. I. (1992). Observability Analysis of Piece-wise Constant Systems. *IEEE Transaction on Aerospace and Electronic Systems, 28*(4), 1056-1075. http://dx.doi.org/10.1109/7.165367

Farrenkopf, R. L. (1978). Analytic Steady-state Accuracy Solution for Two Common Spacecraft Attitude Estimators. *Journal of Guidance Control and Dynamics, 1*(4), 282-284. http://dx.doi.org/10.2514/3.55779

He, D. L., Cao, X. B., & Miao, Y. (2007). Relative States Autonomous Determination of Double Satellites Formation Based on Square Root UKF Filter. *Systems Engineering and Electronic, 29*(1),139–142. (in Chinese).

Kim, S. G., Crassidis, J. L., & Yang, C., *et al*. (2007). Kalman Filtering for Relative Spacecraft Attitude and Position Estimation. *Journal of Guidance Control and Dynamics, 30*(1), 133-143. http://dx.doi.org/10.2514/1.22377

Lefferts, R., J., Markley, F. L., & Shuster, M., D. (1982). Analytic Steady-state Accuracy Solution for Two Common Spacecraft Attitude Estimators. *Journal of Guidance Control and Dynamics, 5*(5), 417-429. http://dx.doi.org/10.2514/3.56190

Li, X. P., Jeffrey, B. B., & Arthur J. T. (1996). Stability Analysis on Earth Observing System AM-1 Spacecraft Attitude Determination. AIAA Guidance, Navigation, and Control Conference, San Diego, USA, p.1-6. http://dx.doi.org/10.2514/3.21651

Maessen, D., C., & Gill, E. (2012). Relative State Estimation and Observability Analysis for Formation Flying Satellites. *Journal of Guidance Control and Dynamics*, *35*(1), 321-326. http://dx.doi.org/10.2514/1.55125

Robert, A., Crassidis, J. L., & Junkins, J. L. (2000). Vision-based Relative Navigation for Formation Flying of Spacecraft. AIAA Guidance, Navigation, and Control Conference, Denver, USA, p.1-11. http://dx.doi.org/10.2514/6.2000-4439

Shay, S., & Pini, G. (2009). Stereoscopic Vision-based Spacecraft Relative State Estimation. AIAA Guidance, Navigation, and Control Conference, Chicago, USA, p.1-23. http://dx.doi.org/10.2514/6.2009-6094

Zhang, Q. J., Hu, X. L., & Ye, B. *et al*. (2008). Binocular Vision-based Relative Position and Attitude Determination Between Spacecrafts. *Journal of Astronautics, 29*(1), 156-161.

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