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# Research of the Robust Stability of Control Systems Using a New Approach to the Lyapunov Functions Construction

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**Abstract**

We investigate a new approach to the construction of vector Lyapunov functions. An approach to the construction of Lyapunov functions as vector functions is developed based on a geometrical interpretation of the second method of Lyapunov. The negative of the gradient is determined from the components of the time derivative of the state vector (i.e., the right-hand side of the state equation). The region of stability of a closed-loop linear, stationary system with uncertain parameters is governed by inequalities in the matrix elements of the closed-loop system. This study developed a method for analysing the robust stability of SISO and MIMO linear systems in canonical forms.

**Keywords:** control systems, robust stability, superstability, Lyapunov's direct method, modelling, simulation

## 1. Introduction

Currently, control problems are characterised by increasingly complex, high-order systems, requirements for high efficiency and stability, numerous uncertainties and incomplete information. Robust stability can be viewed as one of the outstanding issues in control theory, but it is also of a great practical interest. Control system design is one of the main tasks in automation in all branches of industry including manufacturing, energy, electronics, chemicals, medical devices, metals, textiles, transportation, robotics, aviation, space systems, and high-precision military/defence systems. In these systems, uncertainty can occur because of the presence of uncontrolled disturbances acting on the system (Kurzhansky, 1978) or because the true values of the parameters of the system are unknown, either initially or as the system changes over time (Kurzhansky, 1978; Polyak & Shcherbakov, 2002; Bacciotti & Rosier, 2001).

The main goal in control system design is, in some sense, to provide the best protection against uncertainty in the knowledge of the system. The ability of a control system to maintain stability in the presence of parametric or nonparametric uncertainties is known as system robustness. In general, robust stability analysis consists of determining the ranges of values of uncertain parameters for which the closed-loop system remains stable (Polyak & Shcherbakov, 2002). A considerable volume of work has been devoted to the development of robust stability theory.

In this study, we investigate a new approach to the construction of vector Lyapunov functions (Karafyllis & Tsinias, 2003). Vector Lyapunov functions are constructed using a geometrical interpretation of the second method of Lyapunov presented in (Barbashin, 2004; Malkin, 1966). The components of the time derivative of the state vector (i.e., the right-hand side of the state equation) are used to form the negative of the gradient. The robust stability of the system is ensured by choosing the controller parameters so that the scalar product of the gradient vector and the time derivative of the state vector are a negative function (Dorato & Rama, 1990, Antsaklis & Michel, 1997). Stability conditions can be obtained from the positivity of the Lyapunov function in the form of a system of inequalities involving the uncertain parameters of the system (i.e., plant) and the parameters (i.e., gains) of the controller.

We investigate the robust stability of single-input, single-output (SISO) and multi-input, multi-output (MIMO) linear, stationary dynamic systems in canonical form.

In the study of stability, the state equation is defined in terms of perturbations $\Delta x$ about a nominal state; i.e., the state vector $x(t)$ is defined as the difference between the perturbed state $X(t)$ and unperturbed state

$X_S(t)x(t)=\Delta x(t)=X(t)-X_S(t)$

This difference is called a perturbation. Therefore, the origin corresponds to a predetermined condition of the system, the unperturbed state $X_S(t)$. Hence, the right-hand side of the state equation expresses the rates of the perturbations (deviations) of $x(t)$, and we can assume that the vector of perturbation rates for a stable system is directed toward the origin.

Using a geometric interpretation of the second method of Lyapunov, determining stability is reduced (Barbashin, 1967, Karafyllis, 2004, Malkin, 1966, Dorato & Rama, 1990) to the construction of a family of closed surfaces surrounding the origin with the property that the integral curves corresponding to the solutions of the state equation (with respect to perturbations), i.e., the trajectories of the system, cross these surfaces from the exterior to the interior, where the interior contains the origin. The unperturbed condition is stable if it is possible to construct such families of surfaces.

If the total time derivative of the Lyapunov function is negative and the rate vector is directed toward the origin, then each integral curve emanating from a sufficiently small neighbourhood of the origin will necessarily cross each of the surfaces from the exterior to the interior because the Lyapunov function monotonically decreases. In this case, the integral curves approach the origin, so the unperturbed condition is asymptotically stable.

The remainder of this paper is organised as follows. In section 2, we introduce the basic equations of the model and their expanded form and review the Lyapunov function, its geometric interpretation, the gradient vector components and the super-stability condition. In section 3, we consider the existence of stability, robust stability, and super-stability of the nominal system and define the condition of robust stability. In Section 4, we present the results of simulations with a practical example.

## 2. Single-Input, Single-Output Systems

We now consider a system with one input and one output (Beisenbi & Uskenbayeva, 2014 a; Callier & Desoer, 1991; Zhou, Doyle & Clover, 1995; Narendra, Wang & Chen, 2014).

Let the open-loop system be described by the equation

$$\frac{dx}{dt}=Ax+bu, x\in R^n, u\in R^1 \tag{1}$$

where

$$A=\begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & a_1 \end{vmatrix}, \; b=\begin{vmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{vmatrix}, \; x=\begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}.$$

The state feedback control law is given by the scalar function

$$u(t)=-k^T x(t) \tag{2}$$

where $k^T=||k_1, k_2,\dots, k_n||$ (dimensions $1\times n$). Then, system (1) in explicit form can be represented as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -(a_n+k_1)x_1-(a_{n-1}+k_2)x_2-\dots-(a_1+k_n)x_n \end{cases} \tag{3}$$

We apply Lyapunov's direct method (Polyak & Shcherbakov, 2002) to determine the stability of the system in (3): for the system to be asymptotically stable, it is necessary and sufficient that there exists a positive Lyapunov function $V(x)$ such that the derivative with respect to time along the solution of the state equation (3) is negative; i.e., The time derivative of the Lyapunov function in (4) with regard to the state equation (3) is given by the scalar product of the gradient vector $\frac{\partial V(x)}{\partial x}$ and the state rate vector $\frac{dx}{dt}$. To determine the stability of a system (Kurzhansky, 1978; Malkin, 1966), the nominal, or unperturbed, state must be chosen.

The equations of system (1) or (3) are always formed in terms of deviations Δ from a steady state $X_S(x=\Delta x=X-X_S)$.

Applying a geometric interpretation of Lyapunov's theorem (Malkin, 1966; Dorato & Rama, 1990), we define negative gradients of the candidate Lyapunov function as

$$\begin{cases} -\dfrac{dx_1}{dt} = \dfrac{\partial V_1(x)}{\partial x_2} = x_2 \\ -\dfrac{dx_2}{dt} = \dfrac{\partial V_2(x)}{\partial x_3} = x_3 \\ \dots \\ -\dfrac{dx_{n-1}}{dt} = \dfrac{\partial V_{n-1}(x)}{\partial x_n} = x_n \\ -\dfrac{dx_n}{dt} = \dfrac{\partial V_n(x)}{\partial x_1} + \dfrac{\partial V_n(x)}{\partial x_2} + \dfrac{\partial V_n(x)}{\partial x_3} + \dots + \dfrac{\partial V_n(x)}{\partial x_n} = \\ = -[(a_n + k_1)x_1 + (a_{n-1} + k_2)x_2 + \dots + (a_1 + k_n)x_n]^2 \end{cases} \tag{4}$$

Then, we obtain the complete time derivative of the candidate vector Lyapunov function as

$$\begin{cases} \dfrac{dV_1(x)}{dt} = -x_2^2 \\ \dfrac{dV_2(x)}{dt} = -x_3^2 \\ \dots \\ \dfrac{dV_2(x)}{dt} = -x_n^2 \\ \dfrac{dV_n(x)}{dt} = -[(a_n + k_1)x_1 + (a_{n-1} + k_2)x_2 + \dots + (a_1 + k_n)x_n]^2 \end{cases} \tag{5}$$

From (5), it follows that the complete time derivative of a candidate vector Lyapunov function will always be a negative function.

The complete time derivative of the Lyapunov function $V(x) = V_1(x) + V_2(x) + \dots + V_n(x)$ can be expressed in scalar form as

$$\frac{dV(x)}{dt} = -x_2^2 - x_3^2 - \dots - [(a_n + k_1)x_1 + (a_{n-1} + k_2)x_2 + \dots + (a_1 + k_n)x_n]^2 \tag{6}$$

From (4), we can obtain a candidate vector Lyapunov function (Beisenbi & Uskenbayeva, 2014 b):

$$V_1(x) = (0, -\frac{1}{2}x_2^2, 0, \dots, 0)$$

$$V_2(x) = (0, 0, -\frac{1}{2}x_3^2, \dots, 0)$$

$$\dots$$

$$V_{n-1}(x) = (0, 0, 0, \dots, -\frac{1}{2}x_n^2)$$

$$V_n(x) = (\frac{1}{2}(a_n + k_1)x_1^2, \frac{1}{2}(a_{n-1} + k_2)x_2^2, \dots, \frac{1}{2}(a_1 + k_n)x_n^2)$$

The entries of the candidate vector Lyapunov function $V_i (i = 1, \dots, n)$ are constructed from the gradient vector. The Lyapunov function can be expressed in scalar form as

$$V(x) = \frac{1}{2}(a_n + k_1)x_1^2 + \frac{1}{2}(a_{n-1} + k_2 - 1)x_2^2 + \frac{1}{2}(a_{n-2} + k_3 - 1)x_3^2 + \dots + \frac{1}{2}(a_1 + k_n - 1)x_n^2 \tag{7}$$

Given that the function in (7) must be positive and the quadratic forms in (5) are negative, we obtain the following conditions for the stability of the system in (3):

$$\begin{cases} a_n + k_1 > 0 \\ a_{n-1} + k_2 - 1 > 0 \\ a_{n-2} + k_3 - 1 > 0 \\ \dots \\ a_1 + k_n - 1 > 0 \end{cases} \quad (8)$$

In control systems, a precise mathematical formulation is often inaccessible. In reality, systems inevitably contain uncertainty. For a system to satisfy the constraints (8) in the presence of uncertainties in the parameters, we can determine a robust stability radius

$$G = ((g_{ij})),\ g_{ij} = g_{ij}^0 + \Delta_{ij},\ \left|\Delta_{ij}\right| < \gamma m_{ij},\ i = 1,\dots,n$$

where the nominal system matrix $G_0 = g_{ij}^0$ is super-stable, $g_{ij} = a_{ij} - b_j k_j$ are the entries of the closed-loop system matrix, $G_0 = ((g_{ij}^0))$ is the nominal system matrix (1), $\Delta = ((\Delta_{ij})), |\Delta| < m_{ij}$ is the matrix of uncertainties, the matrix $m = ((m_{ij}))$ scales changes in the entries $g_{ij}$ of matrix $G$, and $\gamma > 0$ is the uncertainty range.

We define the system using the negative of the gradient of a candidate function, i.e., $\dot{x} = \Delta_x V$, which was obtained previously in the form of a Lyapunov function:

$$\begin{cases} \dot{x}_1 = -(a_n + k_1)x_1 \\ \dot{x}_2 = -(a_{n-1} + k_2 - 1)x_2 \\ \dot{x}_3 = -(a_{n-2} + k_3 - 1)x_3 \\ \dots \\ \dot{x}_n = -(a_1 + k_n - 1)x_n \end{cases} \quad (9)$$

super-stability of nominal system (9) is defined using (4).

$$\delta(G_0) = \min(-g_{ij}^0 - \sum_{j \neq 1} g_{ij}^0) = \min((a_n + k_1), \min(a_{n-i} + k_i - 1)) \geq 0 \quad i = 2,\dots,n \quad (10)$$

Suppose that the condition of super-stability is preserved for all matrices of the family

$$-\left(g_{ii}^0 + \Delta_{ii}\right) - \sum_{j \neq i} \left|g_{ij}^0 + \Delta_{ij}\right| \geq 0, i = 1,\dots,n$$

This inequality will be satisfied for all admissible $\Delta_{ii}$ if and only if

$$a_n^0 + k_1^0 - \gamma m_{11} > 0$$

$$a_{n-1}^0 + k_2^0 - 1 - \gamma m_{22} > 0$$

$$\dots$$

$$a_1^0 + k_n^0 - 1 - \gamma m_{nn} > 0$$

i.e.,

$$\gamma < \gamma^* = \min(\frac{a_n^0 + k_1^0}{m_{11}}, \min \frac{a_{n-1}^0 + k_i^0 - 1}{m_{ii}}), i = 1,..,n-1$$

Thus, we can explicitly find the radius of robust stability for the family of systems.

*2.1 Example*

As an example, we analyse a third-order system, the block diagram of which is shown in Fig. 1.

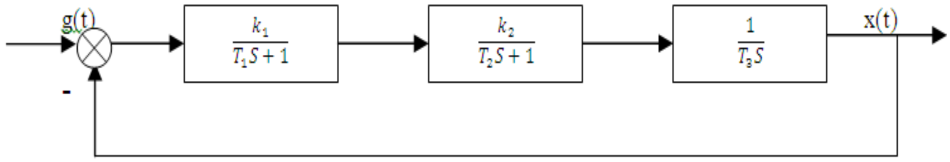


Figure 1. System block diagram for example 1

The transfer function for the open-loop system has the form

$$W(s) = \frac{k_1 k_2}{(T_1 s+1)(T_2 s+1)T_3 s}$$

where $T_1$, $T_2$ and $T_3$ are time constants and $k_1$ and $k_2$ are gains.

Letting $k = k_1 k_2$, we obtain the transfer function for the closed-loop system

$$H(s) = \frac{k}{(T_1 s+1)(T_2 s+1)T_3 s+k}$$

The secular equation of the closed-loop system has the form

$$G(s) = (T_1 s+1)(T_2 s+1)T_3 s + k = 0$$

which can be expressed as

$$G(s) = b_0 s^3 + b_1 s^2 + b_2 s + b_3 = 0$$

Where

$$b_0 = T_1, T_2, T_3, b_1 = (T_1 + T_2)T_3, b_2 = T_3, b_3 = k$$

Dividing through by $b_0$, we obtain

$$G(s) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where

$$G = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{Vmatrix}$$

$$a_1 = \frac{b_1}{b_0} = \frac{(T_1+T_2)T_3}{T_1 T_2 T_3};$$

$$a_2 = \frac{b_2}{b_0} = \frac{T_3}{T_1 T_2 T_3}; \quad a_3 = \frac{b_3}{b_0} = \frac{k}{T_1 T_2 T_3};$$

The state equation for the closed-loop system can be written as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots \\ \dot{x}_3 &= -\dfrac{k}{T_1T_2T_3}x_1 - \dfrac{T_3}{T_1T_2T_3}x_2 - \dfrac{(T_1+T_2)T_3}{T_1T_2T_3}x_3 \end{cases}$$

Using the method described, we construct a Lyapunov function for which the complete time derivative is

$$\frac{dV(x)}{dt} = -x_2^2 - x_3^2 - (\frac{k}{T_1T_2T_3}x_1 + \frac{T_3}{T_1T_2T_3}x_2 + \frac{(T_1+T_2)T_3}{T_1T_2T_3}x_3)^2$$

The Lyapunov function is therefore

$$V(x) = \frac{1}{2}\frac{k}{T_1T_2T_3}x_1^2 + \frac{1}{2}\frac{T_3(1-T_1T_2)}{T_1T_2T_3}x_2^2 + \frac{1}{2}\frac{(T_1+T_2)T_3 - T_1T_2T_3}{T_1T_2T_3}x_3^2$$

The conditions for system stability are reduced to the form

$$(\frac{1}{T_1} + \frac{1}{T_2}) - 1 > 0;$$

$$\frac{1}{T_1T_2} - 1 > 0;$$

$$\frac{k}{T_1T_2T_3} > 0$$

2.1.1 Stability Limits

We can define the following stability limits:

1. The aperiodic stability limit (zero root s=0), which is given by

$$\frac{k}{T_1T_2T_3} = 0, k = 0;$$

2. The vibrational stability limit, which is given by

$$(\frac{1}{T_1} + \frac{1}{T_2}) = 1 \text{ and } \frac{1}{T_1T_2} = 1$$

3. The stability limit corresponding to an infinite root $(s = \infty)$, which is given by

$$T_1T_2T_3 = 0$$

**3. Multi-Input, Multi-Output Systems**

We will investigate a method for determining the robust stability of linear systems with *m* inputs and *n* outputs based on a vector Lyapunov function, and we will obtain the conditions for robust stability (Antsaklis & Michel, 1997; Ma, Lu, Chen D. & Chen Y., 2014; Terra, Cerri & Ishihara, 2014).

Assume a linear system given by

$$\dot{x} = Ax + Bu, x \in R^n, u \in R^m$$

$$y = cx \quad y \in R^{\ell} \tag{11}$$

and a state feedback controller

$$u = -Kx \tag{12}$$

where

$$A = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}, B = \begin{Vmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{Vmatrix},$$

$$C = \begin{Vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{22} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{l1} & c_{l2} & \dots & c_{1\mathrm{n}} \end{Vmatrix}, x = \begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix}, y = \begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{Vmatrix}$$

$$K = \begin{Vmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ k_{m1} & k_{m2} & \dots & k_{mn} \end{Vmatrix}, u = \begin{Vmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Vmatrix},$$

$$u_i = -k_{i1}x_1 - k_{i2}x_2 - \dots - k_{in}x_n, i = 1,2,\dots,n$$

Equation (11) can be expanded as

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m \\ \dots\dots\dots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m \end{cases} \tag{13}$$

Let the matrix $G = A\text{-}BK$ represent the closed-loop system. Expressing system (13) in matrix-vector form, we can write

$$\dot{x} = Gx, x \in R^n$$

where

$$G = \begin{Vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{Vmatrix}$$

$$g_{ij} = a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj}$$

Hence, (13) can be written as

$$\begin{cases} \dot{x}_1 = (a_{11} - \sum_{k=1}^{m} b_{1k}k_{k1})x_1 + (a_{12} - \sum_{k=1}^{m} b_{1k}k_{k2})x_2 + \ldots + (a_{1n} - \sum_{k=1}^{m} b_{1k}k_{kn})x_n \\ \dot{x}_2 = (a_{21} - \sum_{k=1}^{m} b_{2k}k_{k1})x_1 + (a_{22} - \sum_{k=1}^{m} b_{2k}k_{k2})x_2 + \ldots + (a_{2n} - \sum_{k=1}^{m} b_{2k}k_{kn})x_n \\ \ldots\ldots \\ \dot{x}_n = (a_{n1} - \sum_{k=1}^{m} b_{nk}k_{k1})x_1 + (a_{n2} - \sum_{k=1}^{m} b_{nk}k_{k2})x_2 + \ldots + (a_{nn} - \sum_{k=1}^{m} b_{nk}k_{kn})x_n \end{cases} \tag{14}$$

A Lyapunov function $V(x)$ is defined as a vector $V(V_1(x), V_2(x), \ldots, V_n(x))$, and the gradient of the vector Lyapunov function can be written as

$$\begin{cases} \frac{\partial V_1(x)}{\partial x_1} = -\left(a_{11} - \sum_{k=1}^{m} b_{1k}k_{k1}\right)x_1, \\ \frac{\partial V_1(x)}{\partial x_2} = -\left(a_{12} - \sum_{k=1}^{m} b_{1k}k_{k2}\right)x_2, \ldots, \frac{\partial V_1(x)}{\partial x_n} = -\left(a_{1n} - \sum_{k=1}^{m} b_{1k}k_{kn}\right)x_n \\ \frac{\partial V_2(x)}{\partial x_1} = -\left(a_{21} - \sum_{k=1}^{m} b_{2k}k_{k1}\right)x_1, \\ \frac{\partial V_2(x)}{\partial x_2} = -\left(a_{22} - \sum_{k=1}^{m} b_{2k}k_{k2}\right)x_2, \ldots, \frac{\partial V_2(x)}{\partial x_n} = -\left(a_{2n} - \sum_{k=1}^{m} b_{2k}k_{kn}\right)x_n \\ \ldots \\ \frac{\partial V_n(x)}{\partial x_1} = -\left(a_{n1} - \sum_{k=1}^{m} b_{nk}k_{k1}\right)x_1, \\ \frac{\partial V_n(x)}{\partial x_2} = -\left(a_{n2} - \sum_{k=1}^{m} b_{nk}k_{k2}\right)x_2, \ldots, \frac{\partial V_n(x)}{\partial x_n} = -\left(a_{nn} - \sum_{k=1}^{m} b_{nk}k_{kn}\right)x_n \end{cases} \tag{15}$$

The time derivatives of the components of the vector Lyapunov function can be obtained from the state equation (13) or (14) using the scalar product of the components of the gradient of the vector Lyapunov function and the components of the state rate vector $\frac{dx_i}{dt}$, i.e.,

$$\frac{dV_i(x)}{dt} = -\left[\left(a_{i1} - \sum_{k=1}^{m} b_{ik}k_{k1}\right)x_1 + \left(a_{i2} - \sum_{k=1}^{m} b_{ik}k_{k2}\right)x_2 + \ldots + \left(a_{in} - \sum_{k=1}^{m} b_{ik}k_{kn}\right)x_n\right]^2 \quad i = 1,2,\ldots,n \tag{16}$$

The time derivatives of the elements of the vector Lyapunov function $V_i(x)$ are given in (16). From the geometrical interpretation of Lyapunov's theorem, these functions will be negative; i.e., the conditions for asymptotic stability of system (14) will always be satisfied.

Using the components of the gradient vector, we construct the elements of the vector Lyapunov function:

$$\begin{cases} V_1(x_1, x_2, ..., x_n) = -\left( a_{11} - \sum_{k=1}^{m} b_{1k} k_{k1} \right) x_1^2 - \\ -\left( a_{12} - \sum_{k=1}^{m} b_{1k} k_{k2} \right) x_2^2 -, ..., -\left( a_{1n} - \sum_{k=1}^{m} b_{1k} k_{kn} \right) x_n^2 \\ V_2(x_1, x_2, ..., x_n) = -\left( a_{21} - \sum_{k=1}^{m} b_{2k} k_{k1} \right) x_1^2 - \\ -\left( a_{22} - \sum_{k=1}^{m} b_{2k} k_{k2} \right) x_2^2 -, ..., -\left( a_{2n} - \sum_{k=1}^{m} b_{2k} k_{kn} \right) x_n^2 \\ \ldots\ldots \\ V_n(x_1, x_2, ..., x_n) = -\left( a_{n1} - \sum_{k=1}^{m} b_{nk} k_{k1} \right) x_1^2 - \\ -\left( a_{n2} - \sum_{k=1}^{m} b_{nk} k_{k2} \right) x_2^2 -, ..., -\left( a_{nn} - \sum_{k=1}^{m} b_{nk} k_{kn} \right) x_n^2 \end{cases} \tag{17}$$

The positiveness of the vector Lyapunov function can be expressed as

$$\begin{cases} -\left( a_{11} - \sum_{k=1}^{m} b_{1k} k_{k1} \right) > 0, -\left( a_{12} - \sum_{k=1}^{m} b_{1k} k_{k2} \right) > 0, ..., -\left( a_{1n} - \sum_{k=1}^{m} b_{1k} k_{kn} \right) > 0 \\ -\left( a_{21} - \sum_{k=1}^{m} b_{2k} k_{k1} \right) > 0, -\left( a_{22} - \sum_{k=1}^{m} b_{2k} k_{k2} \right) > 0, ..., -\left( a_{2n} - \sum_{k=1}^{m} b_{2k} k_{kn} \right) > 0 \\ \ldots\ldots \\ -\left( a_{n1} - \sum_{k=1}^{m} b_{nk} k_{k1} \right) > 0, -\left( a_{n2} - \sum_{k=1}^{m} b_{nk} k_{k2} \right) > 0, ..., -\left( a_{nn} - \sum_{k=1}^{m} b_{nk} k_{kn} \right) > 0 \end{cases} \tag{18}$$

We will consider the radius of robust stability of the vector Lyapunov function components. For this purpose, we can address parametric families of coefficients of the vector Lyapunov function components in the form (Pupkova & Egunova, 2004)

$$d_{ij} = d_{ij}^0 + \Delta_{ij}, \left|\Delta_{ij}\right| \le \gamma m_{ij}, i, j = 1,2,...,n,$$

where the coefficients $d_{ij}^0 = -\left( a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{ij}^0 \right)$ of nominal matrix $D_0$ correspond to a strictly positive Lyapunov function, i.e.,

$$\sigma(D_0) = \min_i \min_j -\left( a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) > 0$$

We will require that the coefficients be strictly positive for all functions in the family

$$-\left( a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) + \Delta_{ij} > 0, i = 1,2,...,n; j = 1,2,...,n$$

This inequality holds for all admissible $\Delta_{ij}$ if and only if

$$-\left( a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) + \gamma m_{ij} > 0, i = 1,2,...,n; j = 1,2,...,n$$

i.e.

$$\gamma < \gamma^* = \min_i \min_j \frac{-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0\right)}{m_{ij}}$$

In particular, if $m_{ij} = 1$ (in which case the scales of all of the coefficients of the components of the Lyapunov function are identical), then

$$\gamma^* = \sigma(D_0)$$

Thus, the stability radius of the family of positive functions is equal to the smallest value among the coefficients of the vector Lyapunov function.

**4. System in Block-Diagonal Form**

Assume a system described by the equations (Zhou, Doyle & Clover, 1995; Ma, Lu, Chen W. & Chen Y., 2014).

$$\dot{x} = Ax + Bu, x \in R^n, u \in R^m \tag{19}$$

$$y = cx \quad y \in R^{\ell}$$

with a feedback control law

$$u = -Kx \tag{20}$$

where

$$A = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}, B = \begin{Vmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{Vmatrix},$$

$$C = \begin{Vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{22} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{l1} & c_{l2} & \dots & c_{1\mathrm{n}} \end{Vmatrix}, x = \begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix}, y = \begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{Vmatrix}$$

$$K = \begin{Vmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ k_{m1} & k_{m2} & \dots & k_{mn} \end{Vmatrix}, u = \begin{Vmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Vmatrix},$$

$$u_i = -k_{i1}x_1 - k_{i2}x_2 - \ldots - k_{in}x_n, i = 1,2,\ldots,n$$

The dynamics matrix $A$ can be transformed with a matrix $P$, whose columns are functions of $A$, to the block-diagonal form (Callier & Desoer 1991)

$$\widetilde{A} = P^{-1}AP = diag\{\Lambda, J_1, \ldots, J_m, J_1', \ldots, J_k'\} \tag{21}$$

where the blocks are of the form

$$\Lambda = diag\{\lambda_1, \ldots, \lambda_l\} \tag{22}$$

$$J_j = \begin{Vmatrix} \lambda_j & 1 & ... & 0 & 0 \\ 0 & \lambda_j & ... & 0 & 0 \\ ... & . & ... & ... & ... \\ 0 & 0 & ... & \lambda_j & 1 \\ 0 & 0 & ... & 0 & \lambda_j \end{Vmatrix}, N_j \times N_j\, j = 1,...,m, \tag{23}$$

$$J_j^{'} = \begin{Vmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{Vmatrix}, j = 1,...,k. \tag{24}$$

and where $\lambda_1,...,\lambda_l$ are real, distinct roots, $\lambda_j$ are real, repeated roots ($N_j$ roots with the same value), $\lambda_j = \alpha_j \pm j\beta_j$ are complex roots of matrix $A$, and it follows that $l + N_1 + ... + N_m + 2k = n.$

We show that the designated structure (21) allows for the analysis of separate control terms through the representation of the system, with (22), (23) and (24) corresponding to the block diagonal matrix $\widetilde{A}$

For this purpose, we express (19) as

$$\widetilde{x} = \widetilde{A}\widetilde{x} + \widetilde{B}u = \begin{Vmatrix} \Lambda & & 0 \\ & J & \\ 0 & & J^{'} \end{Vmatrix} \widetilde{x} + \begin{Vmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{B}_3 \end{Vmatrix} \widetilde{u} \tag{25}$$

$$u = -\widetilde{k}\widetilde{x} \tag{26}$$

where

$$\widetilde{x} = P^{-1}x,\ \widetilde{A} = P^{-1}AP,\ \widetilde{B} = P^{-1}B, \widetilde{k} = kP$$

Thus, the dimensions of the matrices $\widetilde{B}_1, \widetilde{B}_2 and\ \widetilde{B}_3$ and the vector of inputs $u$ correspond to the dimensions of the square matrices $\Lambda, J, J^{'}$.. Based on (25), having defined $\widetilde{B}_2 = 0,\ \widetilde{B}_3 = 0$, it is straightforward to show that we can manipulate the coordinates of the system corresponding to the matrix $\Lambda$ while maintaining the invariant coordinates of the system defined by matrices $J$ and $J'$ when $\widetilde{B}_1 = 0\ \ and\ \ \widetilde{B}_3 = 0$ or $\widetilde{B}_1 = 0$ and $\widetilde{B}_2 = 0$. Thus, the task is reduced to analysing the robust stability of the subsystems

$$\dot{x}_1 = \Lambda\widetilde{x}_1 + \widetilde{B}_1 u \tag{27}$$

$$\dot{x}_2 = \Lambda\widetilde{x}_2 + \widetilde{B}_2 u \tag{28}$$

$$\dot{x}_3 = J^{'}\widetilde{x}_3 + \widetilde{B}_3 u \tag{29}$$

where

$$\widetilde{x}_1 = \begin{Vmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ x_l \end{Vmatrix}, \widetilde{x}_2 = \begin{Vmatrix} \widetilde{x}_{l+1} \\ \widetilde{x}_{l+2} \\ \vdots \\ x_{l+L} \end{Vmatrix}, L = N_1 + \ldots + N_{m,} \widetilde{x}_3 = \begin{Vmatrix} \widetilde{x}_{l+L+1} \\ x_{l+L+2} \\ \vdots \\ \widetilde{x}_n \end{Vmatrix},$$

with the matrices defined in (22) – (24). We now consider the robust stability of the systems in (27) and (28) separately using the proposed method of constructing Lyapunov functions.

For simplicity, we assume that

$$B = \begin{Vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Vmatrix}, u \in R,\ u = -k^T x,\quad k = \begin{Vmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{Vmatrix}$$

As in (25), we express the system as

$$\widetilde{x} = \widetilde{A}\widetilde{x} \quad + \widetilde{b}\, u = \begin{Vmatrix} \Lambda & & 0 \\ & J & \\ 0 & & J^{'} \end{Vmatrix} \widetilde{x} + \begin{Vmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \\ \widetilde{b}_3 \end{Vmatrix} \widetilde{u}\ , \tag{30}$$

$$\widetilde{u} = -\widetilde{k}^T \widetilde{x} = -\begin{Vmatrix} \widetilde{k}_1^T \widetilde{k}_2^T \widetilde{k}_3^T \end{Vmatrix} \widetilde{x}, \tag{31}$$

where

$$\widetilde{x} = P^{-1}x,\ \widetilde{A} = P^{-1}A\,P,\quad b = P^{-1}b,\quad \widetilde{k}^T = k^T P, \tag{32}$$

Thus, the dimensions of column vectors $\widetilde{b}_1, \widetilde{b}_2, \widetilde{b}_3$ and the row vectors $\widetilde{k}_1^T, \widetilde{k}_2^T, \widetilde{k}_3^T$ correspond to the dimensions of the square matrices $\Lambda, J, J^{'}$. . Based on (30), (31), and (32), having defined $\widetilde{k}_2^T = 0,\quad \widetilde{k}_3^T = 0,$ it is straightforward to obtain the characteristic determinant of the closed-loop system

$$\left|\lambda I - (\widetilde{A} - \widetilde{b}\widetilde{k}^T)\right| = \left|\lambda I_1 - (\Lambda - \widetilde{b}_1\widetilde{k}_1^T)\right| \left|\lambda I_2 - J\right| \left|\lambda I_3 - J^{'}\right|$$

which illustrates that by properly choosing the entries of the gain vector $\widetilde{k}_1^T$ , it is possible to set the values of the matrix $Q_1 = (\Lambda - \widetilde{B}_1\widetilde{k}_1^T)$ without affecting the values of the matrices $J$ or $J^{'}$, . Choosing either $\widetilde{k}_1^T = 0, \widetilde{k}_3^T = 0$ or $\widetilde{k}_1^T = 0, \widetilde{k}_2^T = 0$ allows the subsystems in (27), (28), and (29) to be analysed separately.

$$\dot{\widetilde{x}} = \Lambda\widetilde{x} + \widetilde{b}_1 u \tag{33}$$

$$\dot{\widetilde{x}} = J\widetilde{x} + \widetilde{b}_2 u \tag{34}$$

$$\dot{\widetilde{x}} = J^{'}\widetilde{x} + \widetilde{b}_3 u \tag{35}$$

*4.1 Analysis of the First Subsystem*

The system in (33) can be expanded as

$$\begin{cases} \dot{\tilde{x}}_1 = (s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1 \\ \dot{\tilde{x}}_2 = (s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2 \\ \ldots \\ \dot{\tilde{x}}_l = (s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l \end{cases}$$

For the gradient of the candidate Lyapunov function $V(x_1,...,x_l)$, we obtain

$$\frac{\partial V(\tilde{x})}{\partial \tilde{x}_1} = -(s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1,$$

$$\frac{\partial V(\tilde{x})}{\partial \tilde{x}_2} = -(s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2 \quad ,..., \frac{\partial V(\tilde{x})}{\partial \tilde{x}_l} = -(s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l$$

The time derivative of the Lyapunov function is

$$\frac{dV(\tilde{x})}{dt} = \sum_{i=1}^{l} \frac{\partial V(\tilde{x})}{\partial \tilde{x}_i}\frac{d\tilde{x}_i}{dt} = \sum_{i=1}^{l} (s_i - \tilde{b}_i k_i)^2 \tilde{x}_i^2$$

which will be negative. The Lyapunov function has the form

$$V(\tilde{x}) = -(s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1^2 - (s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2^2 -,...,-(s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l^2$$

The Lyapunov function will be positive if the following inequalities are satisfied:

$$s_1 - \tilde{b}_1\tilde{k}_1 < 0,\ s_2 - \tilde{b}_2\tilde{k}_2 < 0 ,..., s_l - \tilde{b}_l\tilde{k}_l < 0 \tag{36}$$

where $s_i - \tilde{b}_i\tilde{k}_i = \mu_i, i = 1,...,l$ are eigenvalues of the dynamics matrix of the closed-loop system, and for stability, we have the condition

*4.2 Analysis of the Second Subsystem (One Jordan Block)*

The subsystem in (34) can be expressed as

$$\begin{cases} \dot{\tilde{x}}_i == s_i\tilde{x}_i + \tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i \\ \dot{\tilde{x}}_{i+1} == s_i\tilde{x}_{i+1} + \tilde{x}_{i+2} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1} \\ \ldots \\ \dot{\tilde{x}}_{i+N_i} == s_i\tilde{x}_{i+N_i} - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\tilde{x}_{i+N_i} \end{cases} \quad i = 1,...,m;$$

The gradient of the candidate vector Lyapunov function, in accordance with the suggested approach, is

$$\frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(s_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i; \qquad \frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\tilde{x}_{i+1}$$

$$\frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(s_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}; \qquad \frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+2}} = -\tilde{x}_{i+2}$$

$$\ldots$$

$$\frac{\partial V_{i+N_i}(\tilde{x})}{\partial \tilde{x}_{i+N_i}} = -(s_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i})\tilde{x}_{i+N_i}$$

The time derivatives of the vector Lyapunov function components have the form

$$\frac{dV_i(\tilde{x})}{dt} = -(s_i\tilde{x}_i + \tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i)^2$$

$$\frac{dV_{i+1}(\tilde{x})}{dt} = -(s_i\tilde{x}_{i+1} + \tilde{x}_{i+2} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1})^2$$

$$\cdots$$

$$\frac{dV_{i+N_i}(\tilde{x})}{dt} = -(s_i\tilde{x}_{i+N_i} - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\tilde{x}_{i+N_i})^2 .$$

The time derivatives are all negative and meet the condition for asymptotic stability. The candidate vector Lyapunov function is

$$V_i(\tilde{x}) = -(s_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i^2 - \tilde{x}_{i+1}^2$$

$$V_{i+1}(\tilde{x}) = -(s_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}{}^2 - \tilde{x}_{i+2}^2$$

$$\cdots$$

$$V_{i+N_i-1}(\tilde{x}) = -(s_i - \tilde{b}_{i+N_i-1}\tilde{k}_{i+N_i-1})\tilde{x}_{i+N_i-1}{}^2 - \tilde{x}_{i+N_i}^2 \quad V_{i+N_i}(\tilde{x}) = -(s_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i})\tilde{x}_{i+N_i}{}^2 .$$

For system (33), the conditions for the Lyapunov function to be positive are

$$s_i - \tilde{b}_i\tilde{k}_i < 0,\ \ s_i + 1 - \tilde{b}_{i+1}\tilde{k}_{i+1} < 0, \ldots,\ s_i + 1 - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i} < 0\ \ i = 1, \ldots, m. \tag{37}$$

The set of inequalities in (37) also ensures that the real-valued roots of the secular equation of the closed-loop system will be negative.

*4.3 Analysis of the Third Subsystem*

The subsystem in (35) can be expanded. For one block, we have

$$\begin{cases} \dot{\tilde{x}}_i = \alpha_i\tilde{x}_i + \beta_i\tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i & i = 1,..,k \\ \dot{\tilde{x}}_{i+1} = -\beta_i\tilde{x}_i + \alpha_i\tilde{x}_{i+1} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1} & \end{cases}$$

If we construct the Lyapunov function with candidates $V_i(\tilde{x})$ and $V_{i+1}(\tilde{x})$, we obtain the following gradient vector of the candidate Lyapunov function components:

$$\frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_i} = -(\alpha_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i,\ \frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\beta_i\tilde{x}_{i+1},$$

$$\frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_i} = \beta_i\tilde{x}_i,\ \frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}$$

The time derivatives of the components of the candidate vector Lyapunov function are

$$\frac{dV_i(\tilde{x})}{dt} = -(\alpha_i\tilde{x}_i + \beta_i\tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i)^2,$$

$$\frac{dV_{i+1}(\tilde{x})}{dt} = -(-\beta_i\tilde{x}_i + \alpha_i\tilde{x}_{i+1} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1})^2$$

These functions are negative and meet the conditions for asymptotic stability. The Lyapunov function in scalar form is given by

$$V_i(\tilde{x}) = -(\alpha_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i^2 - (\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}^2, \quad i = 1,..,k$$

$$\tilde{b}_i\tilde{k}_i = \tilde{b}_{i+1}\tilde{k}_{i+1}, \qquad \tilde{x}_i = \tilde{x}_{i+1}$$

The conditions for the Lyapunov function to be positive are

$$\alpha_i - \widetilde{b}_i \widetilde{k}_i < 0 \text{ , } i = 1,..,k \tag{38}$$

The conditions in (38) also ensure that the roots $\mu_i = \alpha_i - \widetilde{b}_i \widetilde{k}_i < 0,\ i = 1,..., k$ of the closed-loop system matrix are negative. It is possible to determine the radius of robust stability if necessary.

For the linear closed-loop system to be stable, the eigenvalues of the system matrix must have negative real parts. The preceding derivation guarantees the stability of the system (Pupkova & Egunova, 2004).

**5. Conclusion**

Research in recent years has shown that the method of Lyapunov functions can be successfully used to analyse the robust stability of linear and nonlinear control systems. Widespread application of this method is constrained by the lack of a general method for selecting or constructing Lyapunov functions and difficulties with their algorithmic representation. An inappropriate choice of a Lyapunov function or the inability to construct one does not indicate instability of the system, only that a proper Lyapunov function has not been found.

An analysis of the robust stability of systems is provided by the new approach, which is derived from a geometric interpretation of the asymptotic stability theorem of Lyapunov. A Lyapunov function is constructed in the form of a vector, and the negative of the gradient is found using the components of the time derivative of the state vector (the right-hand side of the state equation). In this case, the time derivative of the Lyapunov function, which is given by the scalar product of the gradient vector and the time derivative of the state vector, is always a negative function. The region of robust stability of the closed-loop system is defined by the conditions for which the constructed Lyapunov function is positive.

The proposed approach to the construction of Lyapunov functions allows for an evaluation of the region of robust stability in the form of simple inequalities in the uncertain parameters of the controlled system. This study developed a method for analysing the robust stability of SISO and MIMO linear systems in canonical forms. The method ensures the stability of the system; i.e., the real parts of the eigenvalues of the closed-loop system are all negative. The efficiency and applicability of the proposed approach are evident.

**References**

Antsaklis, A. J., & Michel, A. N. (1997). *Linear Systems.* New York: McGraw-Hill.

Bacciotti, A., & Rosier, L. (2005). *Lyapunov functions and stability in control theory Lecture Notes in Control and Information Sciences*, London: Springer-Verlag. Retrieved from http://www.springer.com/us/book/9783540213321

Barbashin, E. A. (1967). *Introduction to the theory of stability of motion.* Moscow: Nauka.

Beisenbi, M., & Uskenbayeva, G. (2014a). *The New Approach of Design Robust Stability for Linear Control System.* Proceeding of the International Conference on Advances in Electronics and Electrical Technology (AEET'14). (pp. 11-18). Bangkok, Thailand: Institute of Research Engineers and Doctors. ISBN: 978-981-07-8859-9. http://dx.doi.org/10.3850/ 978-981-07-8859-9_56

Beisenbi, M., & Uskenbayeva, G. (2014b). The New Approach of Design Robust Stability for Linear Control System *of A.M. Lyapunov functions.* Proceedings of the International Conference on New Trends in Information and Communication Technology. (p. 110-118). Almaty: Kazakhstan: Kazakh British technical university.

Besekersky, V. A., & Nebylov, A. V. (1983). *Robust automatic control system.* Moscow.: Nauka.

Callier, F. M., & Desoer, C. A. (1991). *Linear system theory.* London: Springer-Verlag. Retrieved from http://link.springer.com/book/10.1007/978-1-4612-0957-7

Dorato, P., & Rama, K. Y. (1990). *Recent Advances in the Robust Control.* New York: IEEEpress 3.

Gantmacher, F. R. (1967). *The Theory of Matrices.* Moscow: Nauka.

Jia, L., & Sheng, Z. (2014). Robust stabilization of linear switching systems with both input and communication delays. Abstract and Applied Analysis, 2014(2014), 9. http://dx.doi.org/10.1155/2014/906917

Karafyllis, I. (2004). The non-uniform in time small-gain theorem for a wide class of control systems with outputs. *European Journal of Control, 10*(4), 307–323. http://dx.doi.org/10.3166/ejc.10.307-323

Kurzhansky, A. B. (1978). *Control and observation in the conditions of uncertainty*. Moscow: Nauka. Lancaster

P. *The theory of matrices.* Moscow: Nauka, (1978).

Ma, Y. D., Lu, J. G., Chen, W. D., & Chen, Y. Q. (2014, March). Robust stability bounds of uncertain fractional-order systems. *Fractional Calculus and Applied Analysis, 17*(1), 136-153. http://dx.doi.org/10.2478/s13540-014-0159-3

Malkin, I. G. (1966). *The theory of the motion stability.* Moscow: Nauka.

Narendra, K. S., Wang, Y., & Chen, W. (2014). *Stability, robustness, and performance issues in second level adaptation.* Proceedings of the American Control Conference. (pp. 2377-2382). Portland, OR; United States. Article number 6859503. Retrieved from http://scl.hanyang.ac.kr/scl/database/papers/ACC/2014/media/files/1476.pdf

Neumark, Y. N. (1992). *Automation and Remote Control.* Robust stability and D – partition.

Polyak, B. T., & Shcherbakov, P. S. (2002). *Robust stability and control*. Moscow: Nauka.

Pupkova, K. A., & Egunova, N. D. (2004). *Methods of classical and modern theory of automatic control. Arithmetic models, dynamic behaviour and analysis of automatic control systems.* Moscow: Publishing office of Bauman University.

Terra, M. H., Cerri, J. P., & Ishihara, J. Y. (2014, September). Optimal robust linear quadratic regulator for systems subject to uncertainties. *IEEE Transactions on Automatic Control, 59*(9), 2586-2591. Article number 6754186. http://dx.doi.org/10.1109/TAC.2014.2309282

Voronov, A. A., & Matrosov, V. M. (1987). *Method of Lyapunov's vector functions in the stability theory.* Moscow: Nauka.

Zhou, K., Doyle, J. C., & Clover, K. (1995). *Robust and optimal control.* Upper Saddle River. NJ: Prentice Hall.