Modern Applied Science
www.ccsenet.org/mas.html
Vol. 1, No. 2
July 2007

Eventually Strong Wrpp Semigroups
Whose Idempotents Satisfy Permutation Identities

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Abatract

The aim of this paper is to study eventually strong wrpp semigroups whose idempotents satisfy permutation identities, that is, so-called PI-strong wrpp semigroups. After some properties are obtained, the structure of such semigroups are investigated. In particular, the structure of special cases are established.

Keywords: eventually strong wrpp, normal band, eventually PI-strong wrpp, spined product

1. Introduction and preliminaries

Let S be a semigroup, A a subset of S and let

$$\sigma = \begin{pmatrix} 1, & 2, & \ldots, & n \\ \sigma(1), & \sigma(2), & \ldots, & \sigma(n) \end{pmatrix} \qquad (*)$$

A non-identity permutation on n objects. Then A is said to satisfy the permutation identity determined by σ (in short, to satisfy a permutation identity if there is no ambiguity) if

$$(\forall x_1, x_2, \ldots, x_n \in A) \quad x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)},$$

Where $x_1 x_2 \ldots x_n$ is the product of $x_1, x_2, \ldots$ and x_n in S. If $A = S$, then S is called a PI-semigroup.

In connection with this, regular semigroup whose idempotents satisfy permutation identities were investigated by Yamada (1967, P.371). Strong wrpp semigroup whose idempotents satisfy permutation identities were investigated by Guo(1996, P.1947) Eventually strong wrpp semigroup whose idempotents satisfy permutation identities were studied by Du et al.(2001,P.424).

Du(2001, P.5)introduced the relation $L^{(**)}$ which generalizes the relation L^{**}, and the concept of eventually wrpp semigroup was introduced.

Let $a, b \in S$. Then $aL^{(**)}b \Leftrightarrow axRay$ if and only if $bxRby$ for all $x, y \in S$ .we write as $L_a^{(**)}$ with respect to $L^{(**)}$- class containing a for any $a \in S$. Clearly, $L^{**} \subseteq L^{(**)}$. In particular, we have $L^{**} = L^{(**)}$ when $S = S^1$.

We denote

$I_a = \{e \in E(S) \mid (\forall x \in S) eax = ax, \text{ and } xae = xa\}$

for all $a \in E(S)$.

A semigroup is called eventually stong wrpp semigroup if each $L^{(**)}$-class of S contains an idempotent, and $|L_a^{(**)} \cap I_a| = 1$ for all $a \in S$. Here, denotes unique idempotent by a^+.

A eventually strong wrpp semigroup S is called eventually PI-strong wrpp semigroup if idempotents of S satisfy a permutation identity.

Throughout this paper, the terminologies and notations are not defined can be found in Howie(1976).

Lemma1.1 (Yammada, 1967, PP.371-392) let B be a band. Then the following conditions are equivalent:

(1) B Satisfy a permutation identity;

(2) B is a normal band;

(3) B is a strong semilattice of rectangular bands.

Lemma 1.2(Tang, 1997, PP.1499-1504) Let Y be a semilattice, and $S=[Y;S_\alpha,\Phi_{\alpha,\beta}]$ a strong semilattice of semigroup S_α. If for any $a\in S_\alpha$, $b\in S_\beta,(a,b)\in R$, then $\alpha=\beta$.

Definition 1.3 A semigroup S is called R-cancellative monoids,if for any $a,b,c\in S$, $(ca,ab)\in R\Rightarrow(a,b)\in R$ and $(ac,bc)\in R\Rightarrow(a,b)\in R$.

2. Some lemma

In what follows S is always a eventually strong semigroup whose idempotents satisfy permutation identity(*). Let

$$k=min\{i\mid\sigma(i)\neq i\},\ m=\sigma^{-1}(k),$$

then $\sigma(k)>k$. For any $e\in E(S)$, we let

$$S_e=\{a\in S\mid a^+=e\}.$$

Lemma 2.1 The following conditions hold:

(1) A subsemigroups of S satisfy formula (*);

(2) $E(S)$ is a normal band;

(3) $L^+(S)|_T\subseteq L^+(T)$ for any a subsemigroup T of S;

(4) $ab=aa^+b=ab^+b=aba^+b^+=a^+b^+ab$ For any $a,b\in S$.

Proof. Proof of (1) and (2) refer to Yammada(1967,PP.371-392). (3) is trivial. We only show that (4). According to $xaa^+=xa,a^+ax=ax$ and $a^+a=aa^+$,we have $ab=aa^+b=ab^+b$.

By S satisfying equation (*), we have

$$ab=aa^+bb^+=a(a^+)^k a^+(a^+)^{(m-k)}bb^+(b^+)^{(n-m)}b^+=a(a^+)^k bb^+a^+b^+=aba^+b^+.$$

Dually, we have $ab=a^+b^+ab$.

Lemma2.2 $(ab)^+=a^+b^+$ holds for any $a,b\in S$.

Proof. Let all $x,y\in S$. Then

$$\begin{aligned}abxRaby &\Rightarrow a^+bb^+Ra^+bb^+ \quad (aL^{(**)}a^+, a^+b=a^+bb^+)\\
&\Rightarrow b^+a^+bb^+xRb^+a^+bb^+y \quad (R \text{ is a left congruence})\\
&\Rightarrow (b^+)^k a^+(b^+)^{(m-k)}b(b^+)^{(n-m+1)}xR(b^+)^k a^+(b^+)^{(m-k)}b(b^+)^{(n-m+1)}y\\
&\Rightarrow ba^+b^+x=b^+ba^+b^+xRb^+ba^+b^+y=ba^+b^+y \quad (S \text{ satisfies } (*))\\
&\Rightarrow b^+a^+b^+xRb^+a^+b^+y \quad (bL^{(**)}b^+)\\
&\Rightarrow a^+b^+xRa^+b^+y\\
&\Rightarrow abxRaby \quad (aba^+b^+=ab).\end{aligned}$$

Consequently, $a^+b^+\in L^+_{ab}\cap E(S)$. By lemma 2.1(4), we know that $(ab)^+=a^+b^+$.

Lemma 2.3 Let $e\in E(S)$. Then S_e is an inflation of commutative R-cancellative monoids.

Proof. According to lemma 2.2, we know that S_e is a subsemigroup of S. Noticed that

$$eS_e=S_ee\subseteq S_e^2=\{ab\mid a,b\in S_e\}=\{abe\mid a,b\in S_e\}\subseteq S_ee.$$

Thus $eS_e=S_ee=S_e^2$, so S_e^2 contains a identity element e, and satisfies (*). For any $a,b\in S_e^2$, we have

$$ab=e^kae^{m-k}be^{n-m+1}=ebae=ba,$$

That is, S_e^2 is commutative. For any $c\in S_e^2$ such that $caRcb$, by cL^+e,we have $a=eaRe b=b$. Thus S_e^2 is $R-$left cancellative, so S_e^2 is a R-cancellative monoid. The mapping ϕ_e defined by the following rule:

$$\phi_e:\ S_e\to S_e^2,\ x\mapsto ex,$$

Then for any $x,y\in S_e$, we have $xy=exy=exey=x\phi_e y\phi_e$. So S_e is an inflation of commutative R-cancellative monoids S_e^2.

Lemma 2.4 Let B be a normal band, and η the least semilattice congruence on B. Then B is a eventually PI- strong wrpp semigroup and $L^+(B)=\eta$.

Proof. Let $[Y;E_\alpha,\xi_{\alpha,\beta}]$ be a strong semilattice decomposition of B with structure homomorphism $\xi_{\alpha,\beta}$, where Y is a semilattice, and $E_\alpha(\alpha\in Y)$ is a rectangular band. For any $a\in B$, we have $a\in L_a^{(**)}\cap I_a$. If $e\in L_a^{(**)}\cap I_a$, then we have $eax=ax\ xae=xa$ and $ea=ae$ for any $x\in B$. By $eL^{(**)}(B)a$ and $ae=ea=eaa=aa=a$, we have $e=eeRea=a$. Therefore, $e=ae=a$, so $a=a^+$. According to lemma 1.1, we know that B is a eventually PI- strong wrpp semigroup.

Let $a\in E_{\alpha},b\in E_{\beta}(\alpha,\beta\in Y)$, and $aL^{(**)}b$ .By $ab=a\square b\xi_{\beta,\alpha\beta}$,we Have $b=bbRb\square b\xi_{\beta,\alpha\beta}$. According to lemma1.2, $\beta=\alpha\beta$, similarly, $\alpha=\alpha\beta$. Therefore $L^{(**)}(B)\subseteq\eta$.

Conversely, let $a\eta b$. Then $\alpha=\beta$, and for any $x\in E_{\gamma}, y\in E_{\lambda}$, we have

$$\begin{aligned} axRay &\Leftrightarrow a\xi_{\alpha,\alpha\gamma}x\xi_{\gamma,\alpha\gamma}Ra\xi_{\alpha,\alpha\lambda}y\xi_{\lambda,\alpha\lambda} \\ &\Leftrightarrow \alpha\gamma=\alpha\lambda \\ &\Leftrightarrow b\xi_{\beta,\beta\gamma}x\xi_{\gamma,\beta\gamma}Rb\xi_{\beta,\beta\lambda}y\xi_{\lambda,\beta\lambda} \\ &\Leftrightarrow bxRby. \end{aligned}$$

Thus $aL^{+}b$, so $\eta\subseteq L^{(**)}(B)$.

Lemma 2.5 the following conditions are equivalent:

(1) $E(S)$ is a rectangular band;

(2) S is $L^{(**)}$-single;

(3) S is an inflation of the semidirect product of a commutative R-cancellative monoids and a rectangular band.

Proof. (1) $\Rightarrow$(2). Let $E(S)$ be a rectangular band. For any $a,b\in S, x,y\in S$,

We have

$$\begin{aligned} axRay &\Leftrightarrow a^{+}xRa^{+}y \quad (aL^{(**)}a^{+}) \\ &\Leftrightarrow b^{+}a^{+}xRb^{+}a^{+}y \quad (R \text{ is a left congruence}) \\ &\Leftrightarrow b^{+}a^{+}x^{+}xRb^{+}a^{+}y^{+}y \quad (\text{by lemma 2.1(4)}) \\ &\Leftrightarrow b^{+}x^{+}xRb^{+}y^{+}y \quad (E(S) \text{ is a rectangular band}) \\ &\Leftrightarrow b^{+}xRb^{+}y \quad (\text{by lemma 2.1(4)}) \\ &\Leftrightarrow bxRby \quad (bL^{(**)}b^{+}). \end{aligned}$$

Therefore, $aL^{(**)}b$.

(2)$\Rightarrow$(3). Let S is $L^{(**)}$-single. By lemma 2.1 (2)and (3), we know that $E(S)$ is a normal band, and $L^{(**)}|_{E(S)}=\omega|_{E(S)}$. So $E(S)$ is a rectangular band since $L^{(**)}|_{E(S)}$ is the least semilattice congruence on $E(S)$. According to lemma 2.1(4) and lemma 2.2, we have $S^{2}=\bigcup_{e\in E(S)}S_{e}e$. The mapping defined by the following rule:

$$\varphi:\ S\to S^{2},\quad x\mapsto xx^{+},$$

By lamma 2.3 and its proof, we know that definition of φ is good. For any $a,b\in S$, we have $ab=aa^{+}bb^{+}=a\varphi\, b\varphi$. Thus S is an inflation of S^{2}. We select a fixed $e\in E(S)$, The mapping defined by the following rule:

$$\Psi:\ S^{2}\to S_{e}^{2}\times E(S),\ \ x\mapsto(exe,x^{+}).$$

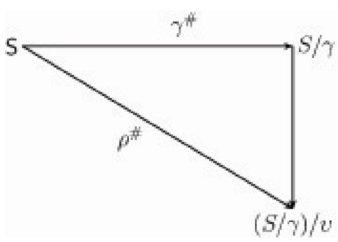
Let $x,y\in S^{2}$ such that $x\Psi=y\Psi$. Then $x^{+}=y^{+}$, and $x=x^{+}ex^{+}xx^{+}ex^{+}$ $=x^{+}exex^{+}=y^{+}eyey^{+}=y$. Therefore Ψ is injective. For any $(x,f)\in S_{e}^{2}\times E(S)$, We have $(fxf)\Psi=(x,f)\Psi$. Thus Ψ is surjective. On the other hand, for any $x,y\in S^{2}$, we have

$$\begin{aligned} (xy)\Psi &= (exye,(xy)^{+})=(exx^{+}ey^{+}ye,x^{+}y^{+}) \\ &= (exx^{+}eey^{+}ye,x^{+}y^{+}) \\ &= (exe\,eye,x^{+}y^{+}) \\ &= x\Psi\ y\Psi. \end{aligned}$$

Therefore, Ψ is a isomorphic mapping.

(3)$\Rightarrow$(1). It is trivial.

Lemma 2.6 Let $\rho=\{(a,b)\in S\times S\mid a^{+}\eta b^{+}\}$. Then ρ is a semilattice congruence on S ,and each $\rho-$class of S is a


S
$\gamma^{\#}$
S/γ
$\rho^{\#}$
$(S/\gamma)/v$

Noticed that for any $a \in S_\alpha, b \in S_\beta, \alpha, \beta \in Y$, we have

$$ab = aa^+b^+a^+b^+bb^+a^+b^+ = aa^+b^+a^+bb^+a^+b^+$$
$$= aa^+\xi_{\alpha,\alpha\beta}b^+\xi_{\beta,\alpha\beta}a^+\xi_{\alpha,\alpha\beta}bb^+\xi_{\beta,\alpha\beta}a^+\xi_{\alpha,\alpha\beta}b^+\xi_{\beta,\alpha\beta}$$
$$= (aa^+\xi_{\alpha,\alpha\beta})(bb^+\xi_{\beta,\alpha\beta})$$
$$= a\Phi_{\alpha,\alpha\beta}\, b\Phi_{\beta,\alpha\beta}.$$

Thus $\Phi_{\alpha,\beta}$ is a structure homomorphism, where $\alpha, \beta \in Y$, and $\alpha \geq \beta$.

(2)$\Rightarrow$(3). Let $S = [Y; S_\alpha, \Phi_{\alpha,\beta}]$ is a strong semilattice of S_α, where $\alpha, \beta \in Y$, S_α is an inflation of the direct product of a commutative $R-$cancellative monoid M_α and a rectangular band E_α with respect to the mapping φ_α. Then $S_\alpha^2 = M_\alpha \times E_\alpha$, and for $\beta \leq \alpha$, we have

$$(M_\alpha \times E_\alpha)\Phi_{\alpha,\beta} = S_\alpha^2\Phi_{\alpha,\beta} \subseteq M_\beta \times E_\beta.$$

Therefore, $M = \bigcup_{\alpha\in Y} M_\alpha \times E_\alpha$ forms a subsemigroup of S, and

$$M = [Y; M_\alpha, \Phi_{\alpha,\beta}|_{M_\alpha \times E_\alpha}] \quad (\alpha, \beta \in Y).$$

Let $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ denote the mapping which M_α onto M_β elicited by $\Phi_{\alpha,\beta}|_{M_\alpha \times E_\alpha}$ and the mapping E_α onto E_β, respectively, identity element of M_α write as e_α, and $M' = [Y; M_\alpha, \phi_{\alpha,\beta}], B = [Y; E_\alpha, \psi_{\alpha,\beta}]$. According to lemma 9, $M = M' \times B$. For any $x \in S_\alpha (\alpha \in Y)$ such that $x\varphi_\alpha = (a,e) \in M_\alpha \times E_\alpha$. Then let $x^* = (e_\alpha, e)$. For any $\gamma \in Y$, and $\gamma < \alpha$, then $(x\Phi_{\alpha,\gamma})^* = x^*\Phi_{\alpha,\gamma}$. If not, then

$$(x\Phi_{\alpha,\gamma})^* = (e_\gamma, f) \neq (e_\alpha\phi_{\alpha,\gamma}, e\psi_{\alpha,\gamma}) = (e_\gamma, e\psi_{\alpha,\gamma}) = x^*\Phi_{\alpha,\gamma}.$$

So

$$((xx^*)\Phi_{\alpha,\gamma})^* = (x\Phi_{\alpha,\gamma}x^*\Phi_{\alpha,\gamma})^* = (x\Phi_{\alpha,\gamma})^* x^*\Phi_{\alpha,\gamma} = (e_\gamma, fe\psi_{\alpha,\gamma})$$
$$\neq (e_\gamma, e\psi_{\alpha,\gamma} f) = x^*\Phi_{\alpha,\gamma}(x\Phi_{\alpha,\gamma}) = ((x^*x)\Phi_{\alpha,\gamma})^*.$$

This is contrary to $x^*x = x^*x\varphi_\alpha = x\varphi_\alpha = x\varphi_\alpha x^* = xx^*$. For any $y \in S_\beta (\beta \in Y)$,

We have

$$xy = x\Phi_{\alpha,\alpha\beta}y\Phi_{\beta,\alpha\beta} = x\Phi_{\alpha,\alpha\beta}(x\Phi_{\alpha,\alpha\beta})^* y\Phi_{\beta,\alpha\beta}(y\Phi_{\beta,\alpha\beta})^* = (xx^*)\Phi_{\alpha,\alpha\beta}(yy^*)\Phi_{\beta,\alpha\beta}.$$

Therefore, S is an inflation of M with respect to the mapping $\theta: S \to M, x \mapsto xx^*$.

(4)$\Rightarrow$(5). Let S is an inflation of spined product of a strong semilattice of a commutative $R-$cancelative monoid M and a normal band B with respect to it's the common greatest homomorphism image under the mapping φ. Then S is also an inflation of a strong semilattice M' of commutative $R-$cancelative monoids and a normal band B with respect to the mapping φ. Let the greatest semilattice decomposition of M and B be

$$M = [Y; M_\alpha, \phi_{\alpha,\beta}], B = [Y; E_\alpha, \psi_{\alpha,\beta}].$$

Let $a, b, x, y \in S$, and such that $a\varphi = (k,e) \in M_\alpha \times E_\alpha, x\varphi = (l,f) \in M_\beta \times E_\beta, b\varphi = (m,g) \in M_\gamma \times E_Y, y\varphi = (n.h) \in M_\delta \times E_\delta, e_\alpha$ is an identity element of M_α. Then

$$axRay \Leftrightarrow (kl, ef)R(kn, eh)$$
$$\Leftrightarrow (ke, ef)R(ke_\alpha n, eh)$$
$$\Leftrightarrow (e_\alpha l, ef)R(e_\alpha n, eh), \alpha\beta = \alpha\gamma \quad (M_\alpha \text{ is a } R\text{-cancellative moniod})$$
$$\Leftrightarrow (e_\alpha, e)xR(e_\alpha, e)y.$$

Thus $aL^+(e_\alpha, e)$. Clearly, $xa(e_\alpha, e) = xa,\ (e_\alpha, e)ax = ax$. Let $(e_\lambda, u) \in M_\lambda \times E_\lambda (\lambda \in Y)$

Such that $(e_\lambda, u)L^+a$, and $xa(e_\lambda, u) = xa;\ (e_\lambda, u)ax = ax$. Then we imply that $\beta\alpha\lambda = \beta\alpha$ by $xa(e_\lambda, u) = xa$. Let $\beta = \alpha$. Then $\alpha\lambda = \alpha$. So $\lambda \geq \alpha$. Again $(e_\lambda, u)\Phi_{\lambda,\alpha}a\varphi = (e_\lambda, u)a = a(e_\lambda, u) = a\varphi(e_\lambda, u)\Phi_{\lambda,\alpha}$ u), where $\Phi_{\alpha,\beta}$ is a structure homomorphism of a strong semilattice M' of a commutative $R-$commutative monoid and a rectangular band, so $(e_\lambda, u)\Phi_{\lambda,\alpha}$ is an identity element of $a\varphi$ in $M_\alpha \times E_\alpha$. Thus $(e_\lambda, u)\Phi_{\lambda,\alpha} = (e_\alpha, e)$. Noticed that $(e_\alpha, e) = (e_\alpha, e)(e_\alpha, e) = (e_\lambda, u)(e_\alpha, e)$,

so $(e_\alpha, e) = (e_\lambda, u)(e_\alpha, e) = (e_\lambda, u)(e_\lambda, u) = (e_\lambda, u)$, consequently, $(e_\alpha, e) = a^+$, S is an eventually strong wrpp semigroup. According to M is a commutative

$R-$cancellative monoid and B is a normal band, we obtain

$$axyb = (klnm, efgh) = (knlm, egfh) = ayxb \cdot$$

Therefore, (5) holds.

Definition 3.2 A wrpp semigroup S is called to satisfy conditions (L).if there exist a unique $e \in L^{**} \cap E(S)$ such that $eae = a$ for all $a \in S$.

Corollary3.3 an eventually PI-strong semigroup S is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*)

If and only if $S^2 = S$.

Proof. Let S be an eventually PI-strong wrpp semigroup. By theorem 3.1, we know that S is an inflation of the spined product of a commutative $R-$cancellative monoid M and a normal band B with respect to its common greatest semilattcice homomorphism image Y. If $S^2 = S$, then $S = M \times_Y B$. According to the proof of theorem 3.1, for any $(k,e) \in M_\alpha \times E_\alpha \subseteq S, (k,e)^+ = (e_\alpha, e)$ and $(k,e)^+(k,e)(k,e)^+ = (k,e)$, we can imply that S is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*). Conversely, if S is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*), then we have $a^+a = a$ for any $a \in S$. Therefore $S^2 = S$.

Corollary3.4 the following conditions are equivalent:

(1) S is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*);

(2) S is an eventually PI-strong wrpp semigroup, and $S^2 = S$;

(3) S is a strong semilattice of the direct product of a commutative $R-$cancellative monoid and a rectangular band;

(4) S is a spined product of a strong semilattice of commutative $R-$cancellative monoid and a normal band;

(5) S is a semigroup with satisfying the conditions (L) and satisfying a permutation identity $x_1x_2x_3x_4 = x_1x_3x_2x_4$.

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