Strong Cconvergence Theorems for Strictly Pseudocontractive Mappings by Viscosity Approximation Methods

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Abstract

In this paper, we introduce a modified Mann iterative process for strictly pseudocontractive mappings and obtain a strong convergence theorem in the framework of Hilbert spaces. Our results improve and extend the recent ones announced by many others.

Keywords: Strong convergence, Strictly pseudocontractive mappings, Viscosity approximation methods

1. Introduction and Preliminaries

Let H be a real Hilbert space, C a subset of H. Recall that A mapping $T : C \to C$ is said to be a strict pseudo-contraction (Browder & Petryshn, 1967, 197-228) if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \quad \text{For all } x, y \in C. \tag{1.1}$$

(If (1.1) holds, we also say that T is a k-strict pseudocontraction.) Strict pseudocontractions in Hilbert spaces were introduced by Browder and Petryshyn (1967, 197-228), which are extension of extensions of nonexpansive mappings which satisfy the inequality (1.1) with $k = 0$. That is, $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

Mann's iteration process (Mann, 1953, 506-510) which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \ge 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$ is often used to approximate a fixed point of a nonexpansive mapping. However, Mann's iteration process has no strong convergence for nonexpansive maps even in Hilbert spaces (Genel & Lindenstrass, 1975, 81-86). If T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.2) converges weakly to a fixed point of T. This is also valid in a uniformly convex Banach space with a Frechet differentiable norm (Reich, 1979, 274-276). Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Recently, Kim and Xu (2005, 51-60) modified Mann iterative process to get a strong convergence theorem for nonexpansive mappings. In (Moudafi, 2000, 46-55), Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces.

Recently Xu (2004, 279-291)

Studied the viscosity approximation methods proposed by Moudafi (2000, 46-55) for nonexpansive mappings in a uniformly smooth Banach space. More precisely, he proved following theorems.

Theorem 1.1 (Xu, 2004, 279-291). Let E be a Hilbert space, C a closed convex subset of E and $T : C \to C$ a nonexpansive mapping with $F(T) \ne \phi$ and $f \in \Pi_C$. Then the sequence $\{x_n\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$,

$t \in (0,1)$ converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \rightarrow F(T)$ by $Q(f) = \lim_{t \rightarrow 0} x_t$ the $Q(f)$ solves the variational inequality $\langle (I - f)Q(f), Q(f) - x \rangle \leq 0$, $f \in \Pi_C$, $x \in F(T)$.

In this paper, we try to modified Mann iterative scheme (1.2) for strictly pseudocontractive mappings to have strong convergence theorem by using viscosity approximation methods in the framework of Hilbert spaces. More precisely, we introduce the composite iteration process as follows:

$$\begin{cases} y_n = (1 - \beta_n) x_n + \beta_n T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n. \end{cases} \quad (1.3)$$

We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (1.3) converges to some fixed point of T, which solves some variational inequality.

It is our purpose in this paper is to introduce this composite iteration scheme for approximating some fixed point of strictly pseudocontractive mappings by using viscosity methods in the framework of Hilbert spaces. We establish the strong convergence of the sequence $\{x_n\}$ defined by (1.3). Our results improve and extend the ones announced by Kim and Xu (2005, 51-60), Xu (2004, 279-291) and some others.

We need the following lemmas for the proof of our main results.

Lemma 1.1 (Xu, 2002, 240-256). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property $\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \gamma_n \sigma_n$, $n \geq 0$, where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

2. Main Results

Theorem 2.1 Let C be a closed convex subset of a Hilbert space E and let $T : C \rightarrow C$ be a strictly pseudocontractive mapping with a fixed point in C. The initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfying the following conditions:

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < a \leq \beta_n < \gamma$ for some $a \in (0, \gamma]$ and $\gamma = \min\{1, 2k\}$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Let $\{x_n\}_{n=0}^{\infty}$ be the composite process defined by (1.3). Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to some fixed point $p \in F(T)$ which solves the variational inequality

$$\langle (I - f)Q(f), Q(f) - p \rangle \leq 0, \ f \in \Pi_C, \ p \in F(T). \quad (2.1)$$

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, taking a fixed point p of $F(T)$ and using (1.1), we have

that $\left\| y_n - p \right\|^2 = \left\| (x_n - p) + \beta_n(Tx_n - x_n) \right\|^2 = \left\| x_n - p \right\|^2 - 2\beta_n \left\langle x_n - Tx_n, x_n - p \right\rangle + \beta_n^2 \left\| x_n - Tx_n \right\|^2$

$$\leq \left\| x_n - p \right\|^2 - \beta_n(2k - \beta_n)\left\| x_n - Tx_n \right\|^2 \leq \left\| x_n - p \right\|^2. \tag{2.2}$$

It follows from (2.2) that

$$\left\| x_{n+1} - p \right\| \leq \alpha_n \left\| f(x_n) - p \right\| + (1 - \alpha_n)\left\| y_n - p \right\| \leq \max\left\{ \frac{1}{1-\alpha}\left\| f(p) - p \right\|, \left\| x_n - p \right\| \right\}.$$

Now, an induction yields

$$\left\| x_n - p \right\| \leq \max\left\{ \frac{1}{1-\alpha}\left\| f(p) - p \right\|, \left\| x_0 - p \right\| \right\}, n \geq 0, \tag{2.3}$$

Which implies that $\{x_n\}$ is bounded, so is $\{y_n\}$.

Since condition (i), we obtain $\left\| x_{n+1} - y_n \right\| = \alpha_n \left\| f(x_n) - y_n \right\| \to 0$, as $n \to \infty$. $\tag{2.4}$

Next, we claim that $\left\| x_{n+1} - x_n \right\| \to 0$, as $n \to \infty$. $\tag{2.5}$

On the other hand, we have

$$\left\| x_{n+1} - x_n \right\| \leq (1 - \alpha_n)\left\| y_n - y_{n-1} \right\| + \left| \alpha_{n-1} - \alpha_n \right|\left\| y_{n-1} - f(x_{n-1}) \right\| + \alpha\alpha_n \left\| x_n - x_{n-1} \right\|. \tag{2.6}$$

Next, we define $A_n = \beta_n T + (1 - \beta_n)I$. We have A_n is nonexpansive for all n. Indeed, It follows from conditions (ii) and (iii) that

$$\left\| A_n x - A_n y \right\|^2 = \left\| x - y - \beta_n[x - y - (Tx - Ty)] \right\|^2$$

$$\leq \left\| x - y \right\|^2 - \beta_n(2k - \beta_n)\left\| x - Tx - y + Ty \right\|^2 \leq \left\| x - y \right\|^2.$$

Next, we show $F(T) = F(A_n)$. Notice that $p \in F(A_n) \Leftrightarrow p = Tp \Leftrightarrow p \in F(T)$. That is, $F(T) = F(A_n)$.

Observing $y_n = A_n x_n$ and $y_{n-1} = A_{n-1}x_{n-1}$, we obtain

$$\left\| y_n - y_{n-1} \right\| \leq \left\| A_n x_n - A_n x_{n-1} \right\| + \left\| A_n x_{n-1} - A_{n-1}x_{n-1} \right\| \leq \left\| x_n - x_{n-1} \right\| + M_1\left| \beta_n - \beta_{n-1} \right|. \tag{2.7}$$

where $M_1 \geq 0$ is a constant such that $M_1 = \sup_{n \geq 0}\left\{ \left\| x_n \right\| + \left\| Tx_n \right\| \right\}$.

Substitute (2.7) into (2.6) yields that

$$\left\| x_{n+1} - x_n \right\| \leq [1 - (1 - \alpha)\alpha_n]\left\| x_n - x_{n-1} \right\| + M_2(\left| \alpha_{n-1} - \alpha_n \right| + \left| \beta_{n-1} - \beta_n \right|), \tag{2.8}$$

where $M_2 \geq 0$ is a constant such that $M_2 \geq \max\left\{ \left\| y_{n-1} - f(x_{n-1}) \right\|, M_1 \right\}$, for all n. By assumptions (i)-(iii) and Lemma 1.1, we obtain (2.5) holds. Observe that

$$\beta_n \left\| Tx_n - x_n \right\| = \left\| y_n - x_n \right\|. \tag{2.9}$$

On the other hand, we have $\left\| y_n - x_n \right\| \leq \left\| y_n - x_{n+1} \right\| + \left\| x_{n+1} - x_n \right\|$. $\tag{2.10}$

It follows from (2.4) and (2.5) that $\lim_{n \to \infty}\left\| y_n - x_n \right\| = 0$, $\tag{2.11}$

which combines with the condition (ii) and (2.9) yields that $\lim_{n \to \infty}\left\| Tx_n - x_n \right\| = 0$. $\tag{2.12}$

Next, we claim that $\limsup\limits_{n\to\infty}\langle f(q)-q,x_n-q\rangle\le 0$, (2.13)

where $q=Q(f)=s-\lim\limits_{t\to 0}x_t$ with x_t being the fixed point of the contraction $x\mapsto tf(x)+(1-t)Ax$, where $A=(1-a)I+aT$ is a nonexpansive mapping such that $F(T)=F(A)$. From x_t solves the fixed point equation $x_t=tf(x_t)+(1-t)Ax_t$, we have $\|x_t-x_n\|=\|(1-t)(Ax_t-x_n)+t[f(x_t)-x_n]\|$.

It follows from Lemma 1.2 that

$$\|x_t-x_n\|^2\le(1-2t+t^2)\|x_t-x_n\|^2+f_n(t)+2t\langle f(x_t)-x_t,x_t-x_n\rangle+2t\|x_t-x_n\|^2,$$ (2.14)

where $f_n(t)=(2\|x_t-x_n\|+\|x_n-Ax_n\|)\|x_n-Ax_n\|\to 0$, as $n\to\infty$. (2.15)

It follows that $\langle x_t-f(x_t),x_t-x_n\rangle\le\dfrac{t}{2}\|x_t-x_n\|^2+\dfrac{1}{2t}f_n(t)$. (2.16)

Let $n\to\infty$ in (2.16) and note (2.15) yields $\limsup\limits_{n\to\infty}\langle x_t-f(x_t),x_t-x_n\rangle\le\dfrac{t}{2}M$. (2.17)

where $M\ge 0$ is a constant such that $M\ge\|x_t-x_n\|^2$ for all $t\in(0,1)$ and $n\ge 1$. Taking $t\to 0$, from (2.17), we have $\limsup\limits_{t\to 0}\ \limsup\limits_{n\to\infty}\langle x_t-f(x_t),x_t-x_n\rangle\le 0$. Since H is a Hilbert space the order of $\limsup\limits_{t\to 0}$ and $\limsup\limits_{n\to\infty}$ is exchangeable, and hence (2.13) holds.

Finally, we show that $x_n\to q$ strongly and this concludes the proof. Indeed, we have

$$\|x_{n+1}-q\|^2=\|(1-\alpha_n)(y_n-q)+\alpha_n[f(x_n)-q]\|^2$$

$$\le(1-\alpha_n)^2\|y_n-q\|^2+2\alpha_n\langle f(x_n)-q,x_{n+1}-q\rangle$$

$$\le(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle f(x_n)-f(q),x_{n+1}-q\rangle+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle$$

$$\le(1-\alpha_n)^2\|x_n-q\|^2+\alpha\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle.$$

Therefore, we obtain

$$\|x_{n+1}-q\|^2\le\frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q,x_{n+1}-q\rangle$$

$$\le[1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}]\|x_n-q\|^2+\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}[\frac{M\alpha_n}{2(1-\alpha)}+\frac{1}{1-\alpha}\langle f(q)-q,x_{n+1}-q\rangle].$$

Now we apply Lemma 1.1 to see that $\|x_n-q\|\to 0$. This completes the proof.

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