The Existences of Positive Solution

for Neutral Difference Equations with Muitiply Delay

Xiaozhu Zhong, Ning Li, Ping Yu, Wenxia Zhang & Shasha Zhang

Department of Mathematics

Yanshan University

Qinhuangdao 066004, China

E-mail: lining0229@163.com

Abstract

We prove the existences of positive solution for neutral difference equation with multiply delay.

Keywords: Difference equation, Positive solution, Much retard

1. Introduction

In recent years, Difference equations have beed applied in many areas,such as population dynamics, stability theory, circuit theory, bifurcation analysis, dynamical behavior of delayed network systems and so on. The oscillation or asymptotic behaviour of difference equations was the subject of invesigation by many authors.

In this paper, we are concerned with the following neutral difference equation with positive and negative coefficients.

$$\Delta\left[x(n)-\sum_{l=1}^{w}R_i(n)x(n-r_l)\right]+\sum_{i=1}^{m}P_i(n)x(n-\tau_i)-\sum_{j=1}^{k}Q_j(n)x(n-\sigma_j)=0 \quad (*)$$

We will investigate the existences of positive solution of this equation.

where $m\geq k$; $R_l, P_i, Q_j\in([n_0,\infty],R^+)$; r_l,τ_i,σ_j are non-negative and non-de-creasing about

l,i,j , $\tau_i\geq\sigma_i$.

Throughout the paper,we suppose the following assumptions

(1) $H_i(n)=P_i(n+\tau_i-\sigma_i)-Q_i(n)\geq 0$, $n\geq n_0, i=1,2,...m$

(2) $H(n)=\sum_{i=1}^{m}H_i(n)$.

(3) $Q_j(n)\equiv 0, \sigma_j\equiv 0, j=k+1,k+2,...m$

When $w=m=k=1$,the equations changed to

$$\Delta[x(n)-R(n)x(n-r)]+P(n)x(n-\tau)-Q(n)x(n-\sigma)=0 \quad n=0,1,... \qquad (4)$$

Where $\{P(n)\},\{Q(n)\},\{R(n)\}$ are sequence of nonnegative real numbers, τ,σ,r are integers

with $0\leq\sigma\leq\tau-1,r>0$ .The oscillatory and non-oscillatory solutions of Eq(4) have been investigateed by several authors. The aim of the present is to investigate the behavior of eventually positive solutions of Eq(1)On the bases of the references of [2].

2. Some lemmas

Lemma1 Assume that (1)(2) holds $n\geq n_0$ and $\sum_{l=1}^{w}R_i(n)+\sum_{i=1}^{m}\sum_{u=n}^{n+\tau_i-\sigma_i}P_i(n)\leq 1$ .if $x(n)$ is an eventually positive solutions of

the following inequality

$$\Delta\left[x(n)-\sum_{l=1}^{w}R_i(n)x(n-r_l)\right]+\sum_{i=1}^{m}P_i(n)x(n-\tau_i)-\sum_{j=1}^{k}Q_j(n)x(n-\sigma_j)\leq 0$$

Setting $$y(n)=x(n)-\sum_{l=1}^{w}R_i(n)x(n-r_l)+\sum_{i=1}^{m}\sum_{u=n}^{n+\tau_i-\sigma_i}P_i(u)x(u-\tau_i)$$

then we eventually have $\Delta y(n)\leq 0,\ y(n)>0$.

Lemma 2 Assume (1) $\{p_n\}$ are sequences of nonnegative real numbers; (2) k and l are integers; (3) $\{q_n\}$ are sequences of nonnegative real numbers and $p_n + lq_n > 0 \ (n \geq N), l > 0;$ or $l > 0, q_s \geq 0 \ s \in [n,n+1]$

Setting $b = \max\{k,l\}$ assume the inequality

$$y_n \geq p_n y_{n-k} + \sum_{s=n}^{\infty} q_s \max_{u \in [s-l,s]} y_u \quad n \geq N$$

has an eventually positive solution $\{y_n\}_{N-b}^{\infty}$, then the corresponding difference equation

$$x_n = p_n x_{n-k} + \sum_{s=n}^{\infty} q_s \max_{u \in [s-l,s]} x_u \quad n \geq N$$

has an eventually positive solution $\{x_n\}_{N-b}^{\infty}$

3. Main results and proof

Theorem 1 Assume $\sum_{l=1}^{w} R_i(n) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u) \equiv 1$, equation(*) has an eventually positive solution if and only if the inequality

$$\Delta\left[x(n) - \sum_{l=1}^{w} R_i(n)x(n-r_l)\right] + \sum_{i=1}^{m} P_i(n)x(n-\tau_i) - \sum_{j=1}^{k} Q_j(n)x(n-\sigma_j) \leq 0 \tag{5}$$

has an eventually positive solution.

Proof: We can see the sufficient condition of the theorem is obvious.we only proof the Necessary condition.
Assume $\{x(n)\}$ is an eventually positive solution of the inequality (5).

define $\qquad y(n) = x(n) - \sum_{l=1}^{w} R_i(n)x(n-r_l) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i) \tag{6}$

then in view of lemma 1, we have $\quad \Delta y(n) \leq 0$, $y(n) > 0$ .and $\lim_{n\to\infty} y(n) \geq 0$.

setting $y_{\infty} = \lim_{n\to\infty} y(n)$ in view of (6), we have

$$\Delta y(n) = \Delta(x(n) - \sum_{l=1}^{w} R_i(n)x(n-r_l) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i))$$

$$\leq \sum_{j=1}^{k} Q_j(n)x(n-\sigma_j) - \sum_{i=1}^{m} P_i(n)x(n-\tau_i) - \sum_{i=1}^{m} P_i(n+\tau_i-\sigma_i)x(n-\sigma_i) + \sum_{i=1}^{m} P_i(n)x(n-\tau_i)$$

$$\leq \sum_{j=1}^{k} Q_j(n)x(n-\sigma_j) - \sum_{i=1}^{m} P_i(n+\tau_i-\sigma_i)x(n-\sigma_i) = -\sum_{i=1}^{m} H_i(n)x(n-\sigma_i) \leq 0 \tag{7}$$

Summing both side of the inequality (7) form n to ∞ and takeing limt on both side of the resulting inequality, we have

$$y_{\infty} - y(n) \leq -\sum_{s=n}^{\infty}\sum_{i=1}^{m} H_i(s)x(s-\sigma_i)$$

Then we can see that $\qquad y(n) \geq \sum_{s=n}^{\infty}\sum_{i=1}^{m} H_i(s)x(s-\sigma_i) + y_{\infty} \geq \sum_{s=n}^{\infty}\sum_{i=1}^{m} H_i(s)x(s-\sigma_i)$

i.e $\qquad x(n) - \sum_{l=1}^{w} R_i(n)x(n-r_l) \geq \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i) + \sum_{s=n}^{\infty}\sum_{i=1}^{m} H_i(s)x(s-\sigma_i)$

Then the corresponding difference equation

$$z(n) = \sum_{l=1}^{w} R_i(n)z(n-r_l) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)z(u-\tau_i) + \sum_{s=n}^{\infty}\sum_{i=1}^{m} H_i(s)z(s-\sigma_i)$$

has an eventually positive solution $z(n)$ .obvious $\{z(n)\}$ are positive solutions of (*).

Theorem 2 Assume $\sum_{l=1}^{w} R_i(n) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u) \equiv 1$ .if $\Delta^2 y(n) + \frac{1}{k}\sum_{i=1}^{m} H_i(n)y(n) = 0$ (8)has an eventually positive solution,

the the equation(*) aslo has an eventually positive solution.

Proof: Assume $y(n)$ is an eventually positive solution of (8). then we have

$$\Delta^2 y(n) = -\frac{1}{k}\sum_{i=1}^{m} H_i(n)y(n) \leq 0 \tag{9}$$

Necessarily we have $\Delta y(n) > 0$ .if it is not true, there must be exist n_0 ,if $n \geq n_0$, we have $\Delta y(n) < 0$, then $y(n)$ exist an nonnegative limt.

Setting $\lim_{n\to\infty} y(n) = M > 0$

We have $\Delta^2 y(n) \le -\dfrac{M}{k}\sum_{i=1}^{m} H_i(n)$ Summing both side of the inequality   form n_2 to n twice

we can see that $\qquad y(n+2) - y(n_2+1) \le -\dfrac{M}{k}\sum_{s_2=n_2}^{n}\sum_{s_1=s_2}^{n}\sum_{i=1}^{m} H_i(n)$

then $\lim\limits_{n\to\infty} y(n) = -\infty$ which contradicts with $y(n)$ is an eventually positive solution. so $\Delta y(n) > 0$.

we eventually have $y(n) > 0, \Delta y(n) > 0, \Delta^2 y(n) \le 0$ .so $\{y(n)\}$ are sequences of nondecreasing, $\{\Delta y(n)\}$ are sequence of nonincreasing

setting $a(n) = \Delta y(n)$

we have $a(n) > 0, \Delta a(n) \le 0 \ n \ge n_1$

define a sequence $\{x(n)\}$ as follows

$$x(n) = \begin{cases} \dfrac{y(n_1)}{k}(n_1 \le n \le n_1+L) \\ a(n) + \sum\limits_{l=1}^{w} R_l(n)x(n-r_l) + \sum\limits_{i=1}^{m}\sum\limits_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i) \\ (n_1+L+tk \le n \le n_1+L+(t+1)k)(t=0,1,2,...) \end{cases}$$

then we have $x(n) > 0(n \ge n_1)$

$$a(n) = x(n) - \sum_{l=1}^{w} R_l(n)x(n-r_l) - \sum_{i=1}^{m}\sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i) \ n \ge n_1+L \qquad (10)$$

$$x(n) = \frac{y(n_1)}{k} \le \frac{y(n)}{k} = \frac{1}{k}\sum_{s=n_1}^{n-1} a_s + \frac{1}{k}y(n_1) \quad n_1 \le n \le n_1+L$$

Therefore $\quad x(n) = a(n) + \sum\limits_{l=1}^{w} R_l(n)x(n-r_l) + \sum\limits_{i=1}^{m}\sum\limits_{u=n}^{n+\tau_i-\sigma_i} P_i(u)x(u-\tau_i)$

$$\le a(n) + \left(\sum_{l=1}^{w} R_l(n) + \sum_{i=1}^{m}\sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u)\right)\left(\frac{1}{k}\sum_{s=n_1}^{n-k-1} a_s + \frac{1}{k}y(n_1)\right)$$

$$\le \frac{1}{k}\sum_{s=n-k}^{n-1} a(s) + \frac{1}{k}\sum_{s=n_1}^{n-k-1} a(s) + \frac{1}{k}y(n_1)$$

$$= \frac{1}{k}\sum_{s=n_1}^{n-1} a(s) + \frac{1}{k}y(n_1)$$

By induction,   we can show that

$$x(n) \le \frac{1}{k}\sum_{s=n_1}^{n-1} a(s) + \frac{1}{k}y(n_1) \quad (n_1+L+tk \le n \le n_1+L+(t+1)k)(t=0,1,2,...)$$

and   hence $\qquad x(n) \le \dfrac{1}{k}\sum\limits_{s=n_1}^{n-1} a(s) + \dfrac{1}{k}y(n_1) \qquad n \ge n_1$

$$x(n-\sigma_i) \le \frac{1}{k}\sum_{s=n_1}^{n-\tau_i-1} a(s) + \frac{1}{k}y(n_1) = \frac{1}{k}y(n-\tau_i) \le \frac{1}{k}y(n) \qquad n \ge n_1+L$$

Substituting this into (9) we obtain $\Delta a(n) + \sum\limits_{i=1}^{m} H_i(n)x(n-\sigma_i) \le 0 \ \ n \ge n_1+L$

So, it follows form (10) that

$$\Delta\left[x(n) - \sum_{l=1}^{w} R_l(n)x(n-r_l)\right] - \sum_{i=1}^{m} P_i(n+\tau_i-\sigma_i)x(n-\sigma_i) + \sum_{i=1}^{m} P_i(n)x(n-\tau_i)$$

$$+ \sum_{i=1}^{m} P_i(n+\tau_i-\sigma_i)x(n-\sigma_i) - \sum_{i=1}^{m} Q_i(n)x(n-\sigma_i) \le 0$$

i.e $\quad \Delta\left[x(n) - \sum\limits_{l=1}^{w} R_l(n)x(n-r_l)\right] + \sum\limits_{i=1}^{m} P_i(n)x(n-\tau_i) - \sum\limits_{j=1}^{k} Q_j(n)x(n-\sigma_j) \le 0$

By theorem 1,   Eq(*) has an eventually positive solution.

Lemma 3 Let $\{d(n)\}$ be a sequence of nonnegative real numbers,Assume that, for some integer n^* and sufficiently large n the inequality

$$(n - n^*)\sum_{s=n}^{\infty} d_s \le \frac{1}{4}$$

holds,Then,the following difference equation

$$\Delta^2 y(n) + d_n y(n) = 0$$

has an eventually positive solution.

Now we are ready to give out result :

Theorem 3 Assume $\sum_{l=1}^{w} R_l(n) + \sum_{i=1}^{m} \sum_{u=n}^{n+\tau_i-\sigma_i} P_i(u) \equiv 1$ holds,for some integer n^* and sufficiently large n, the

inequality $(n - n^*)\sum_{s=n}^{\infty} \sum_{i=1}^{m} H_i(s) \le \frac{k}{4}$

is satisfied, Then, Eq(*) has an eventually positive positive.

References

Dong, Weilei. (2006). The oscillation for neutral difference equations with multiply delay. *Journal of Hebei Normal University*(Natural Science Edition), 30(2): 136-139.

Tang, Xianhua, & Yu, Jianshe. (2000). Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients. *Computer Math Appl*, 39: 169-181.

Tian, J.C.(2003). Oscillation Criteria for neutral equations with positive and negative coefficients. Bol SocParana Math, 21: 19-30.

Wu, Dong. (2003). Oscillation and nonscillation of neutral difference equations with maxima term. *JOURNAL OF OCEAN UNIVERSITY OF QINGDAO*, 3(6): 975-982.

Zhang, B G, &Wang, H. (1996). The exience of oscillatory and non-oscillatory solutions of neutral difference equations. *Chinese J Math*, 24: 377-393.