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A System Matrix Equations over an Arbitrary Skew Field

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Abstract

In this paper, we give a practical solving method and an expression of general solutions of a system matrix equations $A_1XB_1 = C_1andA_2XB_2 = C_2$ over an arbitrary skew field by using some matrix techniques and elementary operations on matrices.

Keywords: skew field, System matrix equations, Elementary operations on matrices

1. Introduction

It is well known that matrix equation is one of important contents of matrix study. In [1] an expression of general solutions of the matrix equation AXB=CYD over an arbitrary skew field was given. In this paper we consider the mentioned system matrix equations $A_1XB_1 = C_1andA_2XB_2 = C_2$ over an arbitrary skew field and give an expression of general solutions and a practical solving method of the matrix equation by using some matrix techniques and elementary operations on matrices with entries from an arbitrary skew field.

Throughout this paper we denote an arbitrary skew field by F, the set of all $m \times n$ matrixes over F by $F^{m \times n}$, the set of all matrices in $F^{m \times n}$ with rank r by $F_r^{m \times n}$, a $m \times m$ identity matrix by I_m , the rank of matrix A by rank A.

Now we introduce the following known lemmas.

Lemma 1.1^[2] Let $A \in F^{m \times n}$. Then rankA=r if and only if there exist $P \in F_m^{m \times m}$ and $Q \in F_n^{m \times n}$ such that $PAQ = \begin{bmatrix} I_r & 0\\ 0 & 0 \end{bmatrix}$

Lemma
$$1.2^{[3]}$$
 Suppose *P* and *Q* be invertible matrices over *F*. If the multiplication of matrices can be performed, then rank *A*=rank *PA*=rank *PA*=rank *PA*O=rank *PA*O

for any matrix A with entries from F.

Lemma 1.3^[4] Let $A \in F_r^{r \times r}$. Then

rank
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r + \operatorname{rank} (D - CA^{-1}B)$$

Lemma 1.4^[5] Let $A \in F^{n \times n}$. Then the following conditions are equivalent:

(i) A is invertible;

(ii) *A* is a product of elementary matrices;

(iii) rank A=n

Lemma $1.5^{[5]}$ Let $A \in F^{m \times n}$, $E_m(\text{resp. } E_n)$ be the elementary matrix obtained by performing an elementary row[resp. column] operation T on $I_m(\text{resp. } I_n)$. Then $E_m A(\text{resp. } AE_n)$ is the matrix obtained by performing the operation T on A.

2. Main Results

Now we consider the matrix equation

exist

$$\begin{cases} A_1 X B_1 = C_1 \\ A_2 X B_2 = C_2 \end{cases}$$
(1)

where $A_1 \in F_r^{m \times n}$, $A_2 \in F_t^{p \times n}$, $B_1 \in F_s^{l \times k}$, $B_2 \in F_w^{l \times q}$, $C_1 \in F^{m \times k}$, $C_2 \in F^{p \times q}$. (1) is equivalent to the following systems matrix equations

$$\begin{cases} PA_{1}QQ^{-1}XV_{0}^{-1}V_{0}B_{1}Q_{0} = PC_{1}Q_{0} \\ VA_{2}QQ^{-1}XV_{0}^{-1}V_{0}B_{2}U = VC_{2}U \end{cases}$$
(2)
Where $P \in F_{m}^{m\times m}, Q \in F_{n}^{n\times n}, Q_{0} \in F_{k}^{k\times k}, V \in F_{p}^{p\times p}, U \in F_{q}^{q\times q}, V_{0} \in F_{l}^{l\times l}.$
Theorem 2.1 For matrices $A_{1}A_{2}B_{2}B_{2}$ in (1), there

$$P \in F_m^{m \times m}, Q \in F_n^{n \times n}, Q_0 \in F_k^{k \times k}, V \in F_p^{p \times p}, U \in F_q^{q \times q}, V_0 \in F_l^{1 \times l}, \text{ such that}$$
(i) $PA_1Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m-r}^{r}, VA_2Q = \begin{bmatrix} 0 & 0 & I_{r_2} & 0 \\ I_{r_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{p-r_1-r_2}^{r_2}$
(ii) $V_0B_1Q_0 = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}_{l-s}^{s}, V_0B_2U = \begin{bmatrix} 0 & I_{s_2} & 0 \\ 0 & 0 & 0 \\ I_{s_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{l-s-s_1}^{s_2}$

Where $r_1 + r_2 = t, s_1 + s_2 = w$.

Proof We proof (ii). It follows from Lemma 1.1 that there exist $V_1 \in F_l^{1 \times l}, Q_1 \in F_k^{k \times k}$ such that

$$V_{1}B_{1}Q_{1} = \begin{bmatrix} I_{s} & 0\\ 0 & 0 \end{bmatrix} \stackrel{s}{l-s}$$

$$\text{(3)}$$

$$\text{Let } V_{1}B_{2} = \begin{bmatrix} W_{1}\\ W_{2} \end{bmatrix}, \text{where } W_{1} \in F^{s \times q}, W_{2} \in F^{(l-s) \times q} \quad \text{. For } W_{2}, \text{ there exist } V_{2} \in F^{(l-s) \times (l-s)}_{l-s}, Q_{2} \in F^{q \times q}_{q}, \text{ such that}$$

$$V_{2}W_{2}Q_{2} = \begin{bmatrix} I_{s_{1}} & 0\\ 0 & 0 \end{bmatrix} \stackrel{s_{1}}{l-s-s_{1}}$$

$$\text{(4)}$$

Let $W_1Q_2 = \begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$, where $W_{11} \in F^{s \times s_1}, W_{12} \in F^{s \times (q-s_1)}$. It follows from Lemma 1.2 and 1.3 that $\begin{bmatrix} I & 0 \end{bmatrix}$

$$rankB_{2} = rank \begin{bmatrix} I_{s} & 0 \\ 0 & V_{2} \end{bmatrix} V_{1}B_{2}Q_{2} = rank \begin{bmatrix} I_{s} & 0 \\ 0 & V_{2} \end{bmatrix} \begin{bmatrix} W_{1}Q_{2} \\ W_{2}Q_{2} \end{bmatrix} = rank \begin{bmatrix} W_{1}Q_{2} \\ V_{2}W_{2}Q_{2} \end{bmatrix}$$
$$= rank \begin{bmatrix} W_{11} & W_{12} \\ I_{s_{1}} & 0 \\ 0 & 0 \end{bmatrix} = s_{1} + rank W_{12}$$

Hence rank $W_{12} = s - s_1 \equiv s_2$

For W_{12} there exist $V_3 \in F_s^{s \times s}$ and $Q_3 \in F_{q-s_1}^{(q-s_1) \times (q-s_1)}$ such that

$$V_{3}W_{12}Q_{3} = \begin{bmatrix} I_{s_{2}} & 0\\ 0 & 0 \end{bmatrix} s_{-}s_{2}$$
(5)

Let

$$V_{0} = \begin{bmatrix} V_{3} & (-V_{3}W_{11}, 0)V_{2} \\ 0 & V_{2} \end{bmatrix} V_{1}, \quad Q_{0} = Q_{1} \begin{bmatrix} V_{3}^{-1} & 0 \\ 0 & I_{k-s} \end{bmatrix}, \quad U = Q_{2} \begin{pmatrix} I_{s_{1}} & 0 \\ 0 & Q_{3} \end{pmatrix}$$

by (3)~(5)
$$V_{0}B_{1}Q_{0} = \begin{bmatrix} V_{3} & (-V_{3}W_{11}, 0)V_{2} \\ 0 & V_{2} \end{bmatrix} V_{1}B_{1}Q_{1} \begin{bmatrix} V_{3}^{-1} & 0 \\ 0 & I_{k-s} \end{bmatrix} = \begin{bmatrix} I_{s} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{split} V_0 B_2 U &= \begin{bmatrix} V_3 & (-V_3 W_{11}, 0) V_2 \\ 0 & V_2 \end{bmatrix} V_1 B_2 Q_2 \begin{bmatrix} I_{s_1} & 0 \\ 0 & Q_3 \end{bmatrix} = \begin{bmatrix} V_3 W_1 Q_2 + (-V_3 W_{11}, 0) V_2 W_2 Q_2 \\ V_2 W_2 Q_2 \end{bmatrix} \begin{bmatrix} I_{s_1} & 0 \\ 0 & Q_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & V_3 W_{12} \\ I_{s_1} & 0 \\ 0 & Q_3 \end{bmatrix} = \begin{bmatrix} 0 & V_3 W_{12} Q_3 \\ I_{s_1} & 0 \\ 0 & Q_3 \end{bmatrix} = \begin{bmatrix} 0 & I_{s_2} & 0 \\ 0 & 0 & 0 \\ I_{s_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{split}$$

 $-r_2$

Similarly,(i) may be shown.

Let

(iii)
$$\begin{cases}
Q^{-1}XV_{0}^{-1} = \begin{bmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
X_{21} & X_{22} & X_{23} & X_{24} \\
X_{31} & X_{32} & X_{33} & X_{34} \\
X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} n - r - r \\
s_{2} & s - s_{2} & s_{1} & l - s - s_{1} \\
PC_{1}Q_{0} = \begin{pmatrix}
G_{1} & G_{2} \\
G_{3} & G_{4}
\end{pmatrix} r \\
PC_{2}U = \begin{pmatrix}
G_{1} & G_{2} \\
G_{3} & G_{4}
\end{pmatrix} r \\
VC_{2}U = \begin{pmatrix}
E_{11} & E_{12} & E_{13} \\
E_{21} & E_{22} & E_{23} \\
E_{31} & E_{32} & E_{33}
\end{pmatrix} r_{3} \\
s_{1} & s_{2} & k - s
\end{cases}$$

Theorem 2.2 Let $P, U, Q, Q_0, V, V_0, E_{ij}$ (*i*, *j* = 1,2,3), G_{ij} (*i*, *j* = 1,2), G_k (= 1,2,3,4) be matrices mentioned in theorem 2.1 if and (iii). Then (1) has solution and only if $X_{ij} = G_{ij}(i, j = 1, 2), X_{33} = E_{11}, X_{31} = E_{12}, X_{13} = E_{21}, X_{11} = E_{22} = G_{11}, G_i = 0 (i = 2, 3, 4) E_{j3} = E_{3j} = 0 (j = 1, 2, 3)$ Whence the general solution of (1) is

$$X = Q \begin{bmatrix} G_{11} & G_{12} & E_{21} & X_{14} \\ G_{21} & G_{22} & X_{23} & X_{24} \\ E_{12} & X_{32} & E_{11} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} V_0$$
(6)

where,

$$\begin{split} X_{14} &\in F^{r_1 \times (l-s-s_1)}, X_{23} \in F^{(r-r_1) \times s_1}, X_{24} \in F^{(r-r_1) \times (l-s-s_1)}, X_{32} \in F^{r_2 \times (s-s_2)}, \\ X_{34} &\in F^{r_2 \times (l-s-s_1)}, X_{41} \in F^{(n-r-r_2) \times s_2}, X_{42} \in F^{(n-r--r_2) \times (s-s_2)}, X_{43} \in F^{(n-r-r_2) \times s_1}, \\ X_{44} &\in F^{(n-r--r_2) \times (l-s-s_1)} \end{split}$$

are all any matrices over F with corresponding orders.

Proof For (1), i.e. for (2), by (iii), (2) is equivalent to the following systems matrix equations

$$\begin{pmatrix} I_{r} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{21} & X_{21} & X_{21} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} \begin{pmatrix} I_{s} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} & G_{2} \\ G_{3} & G_{4} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & I_{r_{2}} & 0 \\ I_{r_{1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{21} & X_{21} & X_{21} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} \begin{pmatrix} 0 & I_{s_{2}} & 0 \\ 0 & 0 & 0 \\ I_{s_{1}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}$$

$$(7)$$

(7) i.e.

(8)

$$\begin{cases} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} & G_{2} \\ G_{3} & G_{4} \end{pmatrix} \\ \begin{pmatrix} X_{33} & X_{31} & 0 \\ X_{13} & X_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & F_{32} & E_{33} \end{pmatrix}$$

This proof is completed.

To sum up the above results, we obtain the detailed steps of solving (1):

(i) Let $L = \begin{bmatrix} B_1 & B_2 & I_l \\ I_k & I_q & 0 \end{bmatrix}$, we apply a sequence of elementary row operations on the first *l* rows of *G* and apply a

sequence of elementary column operations on the first k columns of L and obtain the following form

$$L_1 = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ W_2 \end{pmatrix} \\ Q_0 & I_q & 0 \end{pmatrix}$$

Where, Λ is a non-degenerate upper (or lower) triangular matrix. Then, we apply a sequence of elementary row operations on the first s rows of L_1 and apply a sequence of elementary column operations on the next q columns of L_1 again till we obtain the following form

$$L_{2} = \begin{pmatrix} I_{s} & 0\\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{s_{2}} & 0\\ 0 & 0 & 0\\ I_{s_{1}} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \quad V_{0} \\ Q_{0} & U & 0 \end{pmatrix}$$

(ii) Let $M = \begin{pmatrix} A_1 & I_m \\ A_2 & I_p \\ I_n & 0 \end{pmatrix}$, we apply a sequence of elementary row operations on the first *m* rows of *M* and apply a sequence

of elementary column operations of the first n columns of M and obtain

$$M_1 = \begin{pmatrix} \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} & P \\ \begin{pmatrix} D_1 & D_2 \end{pmatrix} & I_p \\ Q_1 & 0 \end{pmatrix}$$

Where, Ω is a non-degenerate upper (or lower) triangular matrix. Then, we apply a sequence of elementary column operations on the first *r* columns of M_1 and apply a sequence of elementary row operations on the next *p* rows of M_1 again till we obtain the following form

 $M_{2} = \begin{pmatrix} \begin{pmatrix} I_{s} & 0 \\ 0 & 0 \end{pmatrix} & P \\ \begin{pmatrix} 0 & 0 & I_{s_{1}} & 0 \\ I_{s_{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & V \\ Q & 0 \\ \end{pmatrix}$

(iii) By theorem 2.2, we can discuss the all solution circumstances of (1) and obtain an expression of general solutions if it has solution.

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