Maximum and Minimum Works Performed by $\widetilde{T}_n$

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Abstract

Let X_n and X_n^* be the finite sets $\{1, 2, 3, \ldots, n\}$ and $\{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$ respectively. A map $\alpha: X_n \to X_n$ is called a transformation on X_n. We call α a signed transformation if $\alpha: X_n \to X_n^*$. Let T_n and $\tilde{T}_n$ be the sets of full and signed full transformations on X_n respectively. The work, $w(\alpha)$ performed by a transformation α is defined as the sum of all the distances $|i - i\alpha|$ for each $i \in dom(\alpha)$. In this paper, we present a range for the values of $w\ (\alpha)$ for all $\alpha \in T_n$. Further, we characterize elements of $\tilde{T}_n$ that attain minimum and maximum works and provide formulas for the values of these minimum and maximum.

Keywords: transformation, full transformation, signed full transformation, work

1. Introduction

Let $X_n = \{1, 2, 3, \ldots, n\}$ and $X_n^* = \{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$ be finite sets. A transformation α on X_n is a map $\alpha: X_n \to X_n$. α is a full transformation if its domain ($dom\ (\alpha)$) is the entire X_n. We denote by T_n the set of all full transformations on X_n. Now, if $\alpha: X_n \to X_n^*$, α will be called a signed transformation and in the same way, if $dom(\alpha) = X_n$, α will be called a signed full transformation. Here we denote the set of all signed full transformation on X_n by $\tilde{T}_n$ and the image of $i \in dom\ (\alpha)$ by $i\alpha$.

The work performed by a transformation semigroup was studies in East and McNamara (2011). It was stated in East and McNamara (2011) that if the elements of X_n are thought of as points, equally spaced, then the point $i \in X_n$ has been moved a distance of $|i - i\alpha|$ units. Summing these values as i varies over the $dom(\alpha)$ gives the (total) work performed by α and denoted by $w(\alpha)$. Although East and McNamara (2011) considered various subsemigroups of the partial transformation semigroup, it is noteworthy that the operation that qualifies the set of transformation to be a semigroup is not playing any role when calculating the work performed by any semigroup. Hence, their work can still be done in the sets of transformations. It is in this regard that Imam and Tal (2019) studied maximum work performed by elements of the sets of full and partial transformations. They characterized elements of the full and partial transformations that perform maximum work with respect to other elements in the sets. They found a formula for these maximums and further determined the number of maps that attain the maximum.

Researchers have over the years discussed concepts that is related to work although with slight variations. It was noted in Imam and Tal (2019) that Knuth (1973) considered the total displacement of a permutation π and defined it by $\sum_{i=1}^{n} |\ i - \pi(i)\ |$ where $i \in dom(\pi)$. Concepts closely related to work were discussed by Diaconis and Graham (1977), Aitken (1999), (Gallero, Montorsi, Benedetto & Cancellieri, 2001) and Ravichandran and Srinivasan (2003). The study on signed full transformation semigroup, $\tilde{T}_n$ was initiated in Richard (2008) which is the semigroup analogue of the signed symmetric group, $\tilde{S}_n$ that was studied in James and Kerber (1981).

In this paper, we extend the work of Imam and Tal (2019) to the signed full transformation since here also, the operation that makes $\tilde{T}_n$ a semigroup isn't playing any role in our study. The next sections will be for preliminary definitions and results while we present in the third section the findings of this paper.

2. Preliminaries

We present below existing definitions and results needed to understand the result of this paper.

Definition 2.1 (East and McNamara, 2011) The work performed by a (partial) transformation $\alpha \in P_n$ in moving a point $i \in dom(\alpha)$ is defined to be:

$$w_i(\alpha) = \begin{cases} |i - i\alpha| \; if \; i \in dom(\alpha) \\ 0 \quad\quad otherwise, \end{cases}$$

The (total) work performed by α is given by

$$w(\alpha) = \sum_{i \in X_n} w_i(\alpha)$$

Definition 2.2: Let $X_n = \{1, 2, 3, \ldots, n\}$ and $X_n^* = \{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$ be finite sets. A mapping $\alpha: X_n \to X_n$ is called a transformation on X_n. If $dom(\alpha) = X_n$, α is called a full transformation. The set of all full transformations on X_n is denoted by T_n. If the codomain of α is equal to $X_n^* = \{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$, and $dom\,(\alpha) = X_n$, α will be called a signed full transformation on X_n and the set of all signed full transformations on X_n is denoted by $\tilde{T}_n$.

The result that follows presents a description of maps in T_n that performs maximum work when $n \in \mathbb{N}$ is either even or odd. It tells us about the value of this maximum and the number of maps that attain these maximums.

Theorem 2.1 (Imam and Tal, 2019) Let $\alpha \in T_n$. Then,

a. If n is even, α performs maximum work in T_n if and only if for each $i \in dom(\alpha)$,

$$i\alpha = \begin{cases} \mathrm{n} \;\; \text{if } 1 \leq \mathrm{i} \leq \frac{\mathrm{n}}{2}, \\ 1 \;\; \text{if } (\frac{\mathrm{n}}{2}) + 1 \leq \mathrm{i} \leq \mathrm{n}. \end{cases}$$

b. If n is odd, α performs the maximum work in T_n if and only if for each $i \in dom(\alpha)$,

$$i\alpha = \begin{cases} \mathrm{n} \quad \text{if } 1 \leq \mathrm{i} \leq \frac{\mathrm{n}-1}{2}, \\ \mathrm{n} \text{ or } 1 \text{ if } \mathrm{i} = (\frac{\mathrm{n}-1}{2}) + 1, \\ 1 \;\; \text{if } (\frac{\mathrm{n}-1}{2}) + 2 \leq \mathrm{i} \leq \mathrm{n}. \end{cases}$$

Moreover,

c.
$$\begin{cases} \frac{\mathrm{n}}{4}(3\mathrm{n} - 2) \quad\quad \text{if n is even}, \\ \frac{1}{4}(\mathrm{n} - 1)(3\mathrm{n} + 1) \quad \text{if n is odd}. \end{cases}$$

Finally, if

$\Delta(\mathrm{S}) = |\{\alpha \in \mathrm{S}: \mathrm{w}(\alpha) \text{ is maximum}\}|$. Then

d.
$$\Delta(T_n) = \begin{cases} 1 \quad \text{if n is even}, \\ 2 \quad \text{if n is odd}. \end{cases}$$

The proof to this result can found in (Imam and Tal, 2019).

3. Results

We present in this section the findings of this work. We begin by a result which is a consequence of theorem 2.1(c) and (d).

Theorem 3.1: If n is even, then $0 \leq w\,(\alpha) \leq {}^{n}/_{4}\,(3n - 2), \forall \alpha \in T_n \; (n \in \mathbb{N})$.

Proof:

Let n be even and $\alpha \in T_n$. Suppose $\forall i \in dom(\alpha), i\alpha = i,$ it follows from definition 2.1 that $w\,(\alpha) = 0.$ Now, since $\alpha \in T_n$ is unique, it follows that we cannot find any $\beta \in T_n$ for which $w\,(\beta) \leq w(\alpha)$. Thus, if $\forall i \in dom\,(\alpha), i\alpha = i,$ such α attains minimum work in T_n with the value 0 .

Further, let $\alpha \in T_n$ be as in theorem 2.1 (a). We notice that by theorem 2.1 (c), its work,$w\,(\alpha) = {}^{n}/_{4}\,(3n - 2)$. Now, by the uniqueness of such $\alpha \in T_n$ (theorem 2.1 (d)), we cannot find any $\gamma \in T_n$ for which $w\,(\gamma) \geq w\,(\alpha)$. Moreover, if $\alpha \in T_n$ is not such that $i\alpha = i, \forall i \in dom(\alpha)$, and the map $\alpha \in T_n$ as in theorem 2.1 (a), then $0 < w\,(\alpha) < {}^{n}/_{4}\,(3n - 2)$. This follows that $\forall \alpha \in T_n$, $0 \leq w\,(\alpha) \leq {}^{n}/_{4}\,(3n - 2)$.

Theorem 3.2: If n is odd, then $0 \leq w\,(\alpha) \leq {}^{1}/_{4}\,(n-1)(3n+1), \forall \alpha \in T_n\ (n \in \mathbb{N})$.

Proof:

Let n be odd and $\alpha \in T_n$. Suppose $\forall i \in dom\,(\alpha), i\alpha = i$, it follows from definition 2.1 that $w\,(\alpha) = 0$. Now, since $\alpha \in T_n$ is unique, it follows that we cannot find any $\beta \in T_n$ for which $w\,(\beta) \leq w\,(\alpha)$. Thus, if $\forall i \in dom\,(\alpha), i\alpha = i$, such α attains minimum work in T_n with the value 0.

Now, by the characterization in theorems 2.1 (b) and by theorem 2.1 (c), two maps $\alpha_1, \alpha_2 \in T_n$ will attain maximum work in T_n with $w\,(\alpha_1) = w\,(\alpha_2) = {}^{1}/_{4}\,(n-1)(3n+1)$. It follows by this that we cannot find any $\xi \in T_n$ for which $w\,(\alpha_1) \leq w\,(\xi) \geq w\,(\alpha_2)$. Notice that if $\alpha \in T_n$ is not such that $i\alpha = i, \forall i \in dom\,(\alpha)$, and the map $\alpha \in T_n$ as in theorem 2.1 (a), then $0 < w\,(\alpha) < {}^{1}/_{4}\,(n-1)(3n+1)$. This follows that $\forall \alpha \in T_n,\ 0 \leq w\,(\alpha) \leq {}^{n}/_{4}\,(3n-2)$.

The above results provide a range for values of work performed by every map in T_n in the even and odd cases respectively.

The next result characterizes elements of the signed full transformation that attain maximum work.

Theorem 3.3: Let $\alpha \in \tilde{T}_n$. α performs the maximum work in $\tilde{T}_n$ if and only if for each $i \in dom\,(\alpha)$, $i\alpha = -n$.

Proof:

Let $\alpha \in \tilde{T}_n$. Suppose α performs maximum work in $\tilde{T}_n$, then it follows by the definition of work performed by $\alpha \in \tilde{T}_n$, that for each $i \in dom\,(\alpha)$,

$$w\,(\alpha) = \sum_{i=1}^{n} |\, i - i\alpha \,|$$

Now, it is easy to notice that maximum $w\,(\alpha)$ can only be achieved when $|i - i\alpha|$ is made as large as possible for each $i \in dom\,(\alpha)$. Thus, we can clearly achieve this maximum if,

$$max\,|\, i - i\alpha| = |i - (-n)| = i + n, \quad \text{for each } i \in dom\,(\alpha).$$

We can easily deduce from above that $i\alpha = -n$.

Conversely, suppose $\alpha \in \tilde{T}_n$ is such that $i\alpha = -n$, then for each $i \in dom\,(\alpha)$, $|i - i\alpha|$ will clearly be at maximum for each $i \in dom\,(\alpha)$. This ultimately makes $w\,(\alpha)$ to be at maximum. And so $\alpha \in \tilde{T}_n$ performs maximum work in $\tilde{T}_n$.

It is not hard to see by the result above that $\alpha \in \tilde{T}_n$ will attain maximum work if and only if it is a constant map whose image is $-n\ (n \in \mathbb{N})$.

Next, we present the maximum value $\alpha \in \tilde{T}_n$ will attain for any n.

Theorem 3.4: Suppose $\alpha \in \tilde{T}_n$ is such that $i\alpha = -n\ \forall i \in dom\,(\alpha)$, then $w\,(\alpha) = \frac{3n^2+n}{2}, n \in \mathbb{N}$.

Proof:

Suppose $\alpha \in \tilde{T}_n$ is such that $i\alpha = -n$, for each $i \in dom\,(\alpha)$, then by definition, the work $\alpha \in \tilde{T}_n$

Will perform is given by

$$\begin{aligned} w\,(\alpha) &= \sum_{i=1}^{n} |\, i - i\alpha \,| \\ &= \sum_{i=1}^{n} |\, i - (-n) \,| \\ &= \sum_{i=1}^{n} |\, n + i \,| \\ &= (n+1) + (n+2) + (n+3) + \ldots + 2n. \\ &= n^2 + \frac{n\,(n+1)}{2} \\ &= \frac{3n^2 + n}{2}. \end{aligned}$$

The result above provides us with the maximum value (formula) $\alpha \in \tilde{T}_n$ (as in theorem 3.3) will attain for any $n \in \mathbb{N}$.

Example 3.1:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -5 & -5 & -5 & -5 \end{pmatrix} \in \tilde{T}_5$

Now, $w(\alpha) = \sum_{i=1}^{n} |i - i\alpha| = |1-(-5)| + |2-(-5)| + |3-(-5)| + |4-(-5)| + |5-(-5)|$

$= 6+7+8+9+10 = 40.$

Since $n = 5$, we can verify Theorem 3.4,

$$w(\alpha) = \frac{3n^2+n}{2} = \frac{3\times 5^2+5}{2} = \frac{80}{2} = 40.$$

Example 3.2:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -4 & -4 & -4 & -4 \end{pmatrix} \in \tilde{T}_4$

Now, $w(\alpha) = \sum_{i=1}^{n} |i - i\alpha| = |1-(-4)| + |2-(-4)| + |3-(-4)| + |4-(-4)|$

$= 5+6+7+8 = 26.$

To verify Theorem 3.4,

$$w(\alpha) = \frac{3n^2+n}{2} = \frac{3\times 4^2+4}{2} = \frac{52}{2} = 26.$$

Example 3.3:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ -3 & -3 & -3 \end{pmatrix} \in \tilde{T}_3$

Now, $w(\alpha) = \sum_{i=1}^{n} |i - i\alpha| = |1-(-3)| + |2-(-3)| + |3-(-3)||$

$= 4+5+6 = 15.$

Theorem 3.4 can be verified as in examples 3.1 and 3.2.

Consider,

$\tilde{R}_n = \{\alpha \in \tilde{T}_n | i\alpha < 0, \forall i \in dom(\alpha), n \in \mathbb{N}\}$. $\tilde{R}_n$ is the set of all transformations in $\tilde{T}_n$ whose images are negative.

Remark 1: $\alpha \in \tilde{T}_n$ such as described by theorem 3.3 will attain maximum work in $\tilde{R}_n$ since such $\alpha \in \tilde{R}_n$. This is not hard to see.

We explore therefore in the next result and the one after next the nature of the map in $\tilde{R}_n$ that performs the minimum work, and the value (formula) for this minimum respectively.

Theorem 3.5: Let $\alpha \in \tilde{R}_n$. α performs minimum work in $\tilde{R}_n$ if and only if for each $i \in dom(\alpha)$, $i\alpha = -1$.

Proof:

Let $\alpha \in \tilde{R}_n$. Suppose α performs minimum work in $\tilde{R}_n$, then we know by the definition of work performed by $\alpha \in \tilde{R}_n$, that for each $i \in dom(\alpha)$,

$$w(\alpha) = \sum_{i=1}^{n} |i - i\alpha|$$

Now, we can achieve this minimum $w\,(\alpha)$ only when $|i - i\alpha|$ is made as small as possible for each $i \in dom\,(\alpha)$. This is thus possible if,

$$min\,|\,i - i\alpha| = |i - (-1)| = i + 1, \quad \text{for each } i \in dom\,(\alpha).$$

We can easily deduce from above that $i\alpha = -1$.

Conversely, suppose $\alpha \in \tilde{R}_n$, is such that $i\alpha = -1$, then for each $i \in dom\,(\alpha)$, $|i - i\alpha|$ will clearly be at minimum for each $i \in dom\,(\alpha)$, compared to any $\lambda \in \tilde{R}_n$ different from $\alpha \in \tilde{R}_n$. This clearly makes $w\,(\alpha)$ to be at minimum, and so $\alpha \in \tilde{R}_n$, performs minimum work in $\tilde{R}_n$.

By theorem 3.5, $\alpha \in \tilde{R}_n$ will attain minimum work if and only if it is a constant map whose image is -1.

Theorem 3.6: Suppose $\alpha \in \tilde{R}_n$ is such that $i\alpha = -1$, for all $i \in dom\,(\alpha)$, then $w\,(\alpha) = \frac{n^2+3n}{2}, n \in \mathbb{N}$.

Proof:

Suppose $\alpha \in \tilde{R}_n$ is such that $i\alpha = -n$, for each $i \in dom\,(\alpha)$, then by definition, the work $\alpha \in \tilde{R}_n$ will perform is given by

$$\begin{aligned} w(\alpha) &= \sum_{i=1}^{n} |\,i - i\alpha\,| \\ &= \sum_{i=1}^{n} |\,i - (-1)\,| \\ &= \sum_{i=1}^{n} |\,1 + i\,| \\ &= (1+1) + (1+2) + (1+3) + \ldots + (1+n). \\ &= n + \frac{n\,(n+1)}{2} \\ &= \frac{n^2 + 3n}{2}. \end{aligned}$$

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix} \in \tilde{R}_5 \in \tilde{T}_5$.

Now, $w(\alpha) = \sum_{i=1}^{n} |\,i - i\alpha\,| = |\,1 - (-1)\,| + |\,2 - (-1)\,| + |\,3 - (-1)\,| + |\,4 - (-1)\,| + |\,5 - (-1)\,|$

$$= 2 + 3 + 4 + 5 + 6 = 20.$$

Since $n = 5$, we can verify using $\frac{n^2 + 3n}{2}$ we obtain the value 20.

Example 3.5:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -1 & -1 \end{pmatrix} \in \tilde{R}_4$

Now, $w\,(\alpha) = \sum_{i=1}^{n} |\,i - i\alpha\,| = |\,1 - (-1)\,| + |\,2 - (-1)\,| + |\,3 - (-1)\,| + |\,4 - (-1)\,|$

$$= 2 + 3 + 4 + 5 = 14.$$

Since $n = 4$, we can verify using $\frac{n^2 + 3n}{2}$ we obtain the value 14.

Example 3.6:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \end{pmatrix} \in \tilde{R}_3$

Now, $w\ (\alpha) = \sum_{i=1}^{n} | i - i\alpha | \ = \ |1-(-1)| + |2-(-1)| + |3-(-1)|$

$= 2 + 3 + 4 = 9.$

We can verify $\frac{n^2 + 3n}{2}$ as in examples 3.4 and 3.5.

Remark 2: We remark here that the paper of Imam and Tal (2019) on maximum work performed by elements of T_n can be carried out in $\tilde{T}_n$ since $T_n \subset \tilde{T}_n$. The implication is that the identity map in $\tilde{T}_n$ will perform the minimum work in $\tilde{T}_n$. With this together with remark 1 in mind, we conclude thus:

Theorem 7:

(a). for all $\alpha \in \tilde{T}_n$, $0 \leq w\ (\alpha) \leq \frac{3n^2+n}{2}$.

(b). for all $\alpha \in \tilde{R}_n$, $\frac{n^2+3n}{2} \leq w\ (\alpha) \leq \frac{3n^2+n}{2}$.

Proof:

The proof to (a) and (b) will follow the same pattern as the proof of theorem 3.1.

4. Conclusion

This study provides a consequence to the result of East and McNamara (2011) by providing a range for the values of the work performed by all $\alpha \in T_n$. We further characterized elements of $\tilde{T}_n$ and $\tilde{R}_n$ that attain maximum and minimum and generalized the values of the maximum and minimum for any $n \in \mathbb{N}$.

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