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On the Deformation Retractions of Frenet Curves in Minkowski 4 - Space

A.E. El-Ahmady[1] & A.T. M-Zidan[2]

[1]Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt

[2] Mathematics Department, Faculty of Science, Damietta University, Damietta, Egypt

Correspondence: A.T. M-Zidan, Mathematics Department, Faculty of Science, Damietta University, Egypt. E-mail: atm_zidan@yahoo.com

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Abstract

In this paper, the position vector equation of the Frenet curves with constant curvatures in Minkowski 4 -space has been presented. New types for retractions and deformation retracts of Frenet curves in E_1^4 are deduced. The relations between the Frenet apparatus of the Frenet curves before and after the deformation retracts are obtained.

Keywords: Minkowski 4-space E_1^4, Frenet curves, retraction, deformation retracts

AMS Subject Classification(2010):

Primary: 53A35, 53A04, 58C05, 53B30. **Secondary**: 53Z05; 53Z99.

1. Introduction and Definitions

Minkowski space time in E_1^4 is an Euclidean space provided with the standard flat metric given by $\langle X,Y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4) are rectangular coordinate system in E^4. Since $\langle\,,\rangle$ is an indefinite metric, recall that a vector $u \in E_1^4$ can have one of the three casual characters; it can be space like, if $< u,u >> 0$ or $u = 0$, time like, if $< u,u >< 0$, null or light like if $< u,u >= 0$ and $u \neq 0$. The norm of a vector v is given by $\parallel v \parallel = \sqrt{|\langle v,v\rangle|}$. Space like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s),\alpha'(s)) = \pm 1$. The velocity of α at $t \in I$ is $\alpha' = \frac{d\alpha(u)}{du}\Big|_t$. Next, $v,w\ in\ E_1^4$ are said to be orthogonal vectors if $g(v,w) = 0$ (M. Turgut & S. Yilmaz.2008) (R. Lopez. 2008) (A. E. El-Ahmady. 2007).

In this paper, we introduce some characterizations of retraction and deformation retract of Frenet curves in E_1^4 by the components of the position vector according to the Frenet equations. Also we obtain some relations among curvatures of Frenet curves and their deformation retracts.

2. Main results

Definition: Denoted by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal vector fields respectively. Let $\alpha(s)$ is a curve in the space-time in E_1^4 parameterized by arc length function *s* Lopez .Then for the unit speed curve $\alpha(s)$ with non-null frame vectors, such that the Frenet equations are,

$$\begin{pmatrix} T' \\ N' \\ B_1' \\ B_2' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ \mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\ 0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\ 0 & 0 & \mu_5 k_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

$case$ 1. If α is a time like curve in E_1^4. Then T is a time like vector, so the Frenet equations, $\mu_i (1 \leq i \leq 5)$ read, $\mu_3 = \mu_5 = -1$, $\mu_1 = \mu_2 = \mu_4 = 1$, where T, N, B_1, B_2 are mutually orthogonal vectors with $g(T,T) = -1$, $g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1$.

$case$ 2. If α is a space like curve in E_1^4.

Then T is a space like vector, so depending on N, then B_1 can have all three causal characters,

***Case*2.1.** If N is space-like, then B_1 have the next subcases

***Case*2.1.1** If B_1 be space like, then $\mu_i (1 \le i \le 5)$ read

$$\mu_1 = \mu_3 = -1, \mu_2 = \mu_4 = \mu_5 = 1,$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfies

$$g(T,T) = g(N,N) = g(B_1,B_1) = 1, g(B_2,B_2) = -1.$$

***Case*2.1.2** If B_1 is time like, then $\mu_i (1 \le i \le 5)$ read

$$\mu_1 = -1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1,$$

where T, N, B_1, B_2 satisfying equations,

$$g(T,T) = g(N,N) = g(B_2,B_2) = 1, g(B_1,B_1) = -1.$$

***Case*2.1.3** If B_1 be a null vector, then the Frenet frame equations read

$$\begin{pmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

where T, N, B_1, B_2, satisfying equations,

$$g(T,T) = g(N,N) = 1, g(B_1,B_1) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, g(B_1,B_2) = 1.$$

***Case*2.2** If N is time-like, then $\mu_i (1 \le i \le 5)$ read

$$\mu_5 = -1, \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1,$$

where T, N, B_1, B_2 are satisfying equations,

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1, g(N,N) = -1.$$

Remark. The curves which satisfy the previous cases called Frenet curves.

***Case*2.3** If N is light-like (null), then the Frenet equations read

$$\begin{pmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

where $k_1 = 0$, when α is a straight line or $k_1 = 1$, in all other cases. With T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations,

$$g(T,T) = g(B_1,B_1) = 1, g(N,N) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N, \quad B_1) = g(B_1,B_2) = 0, g(N,B_2) = 1.$$

***case* 3. If α is light-like (null) curve in $\mathbf{E}_1^4$.**

Then T is a null vector, so the Frenet equations has the form,

$$\begin{pmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & -k_1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

where $k_1 = 0$, when α is a straight line or $k_1 = 1$, in all other cases. With T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations,

$$g(T,T) = g(N,N) = g(B_1,B_1) = 0, g(B_2,B_2) = 1,$$

$$g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0, g(T,B_1) = 1.$$

Where the functions $k_1 = k_1(s), k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called respectively the first, second and third curvature of the curve $\alpha(s)$ (J. Walrave. 1995).

Definition 2.1. A subset A of a topological space X is called retract of X if there exists a continuous map $r: X \to A$ called a retraction such that $r(a) = a$ for any $a \in A$ (A. E. El-Ahmady & A.T.M. Zidan. 2019).

Definition 2.2. A subset A of a topological space X is a deformation retracts of X if there exists a retraction $r: X \to A$ and a homotopy $\varphi: X \times I \to X$ such that:

$\begin{cases} \varphi(x,0) = x \\ \varphi(x,1) = r(x) \end{cases}$ $x \in X, \quad \varphi(a,t) = a, \quad a \in A, t \in [0,1]$ (A. E. El-Ahmady & A.T.M. Zidan. 2018) (A. E. El-Ahmady. 2014).

Definition 2.3. Time like curves and space like curves with space like or time like normal vector (curves with non-null frame vectors) are called Frenet curves, where $g(T,T) \neq 0, g(N,N) \neq 0, \quad g(B_1,B_1) \neq 0$ and $g(B_2,B_2) \neq 0$.

3. Position vector of the Frenet curves in E_1^4.

Frenet equations of the Frenet curves are,

$$\begin{pmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ \mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\ 0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\ 0 & 0 & \mu_5 k_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix} \qquad (1).$$

Let $\eta(s)$ be a Frenet curve in E_1^4, whose position vector satisfies the parametric equation,

$$\eta(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_3(s)B_1(s) + \nu_4(s)B_2(s) \qquad (2).$$

For some differentiable functions $\nu_j(s), 1 \le j \le 4,$ and for $\mu_i (1 \le i \le 5), \mu_i \in \{1,-1\}$.

By differentiating equation(2) with respect to arc-length parameter s and using the Frenet equations (1), for Frenet curves in E_1^4, we get

$$\begin{aligned} \eta'(s) &= (\nu_1' + \mu_1 k_1 \nu_2)T(s) \\ &+ (\nu_2' + k_1 \nu_1 + \mu_3 k_2 \nu_3)N(s) \\ &+ (\nu_3' + \mu_2 k_2 \nu_2 + \mu_5 k_3 \nu_4)B_1(s) \\ &+ (\nu_4' + \mu_4 k_3 \nu_3)B_2(s) \end{aligned} \qquad (3),$$

then we get

$$\begin{aligned} \nu_1' + \mu_1 k_1 \nu_2 &= 1 \\ \nu_2' + k_1 \nu_1 + \mu_3 k_2 \nu_3 &= 0 \\ \nu_3' + \mu_2 k_2 \nu_2 + \mu_5 k_3 \nu_4 &= 0 \\ \nu_4' + \mu_4 k_3 \nu_3 &= 0 \end{aligned} \qquad (4).$$

4. Deformation retracts of Frenet curves in E_1^4.

We introduce types of retraction on Frenet curves with non-zero curvature in E_1^4.

In the position vector equation of Frenet curve $\eta(s)$, in equation (2),

if we put $\nu_1(s) = 0$, then the Frenet retraction curve defined by $\eta_{r1}(s) = r_1(\eta(s))$ where,

$$\eta_{r1}(s) = \nu_2(s)N(s) + \nu_3(s)B_1(s) + \nu_4(s)B_2(s),$$

if we put $\nu_2(s) = 0$, then the Frenet retraction curve defined by $\eta_{r2}(s) = r_2(\eta(s))$ where,

$$\eta_{r2}(s) = \nu_1(s)T(s) + \nu_3(s)B_1(s) + \nu_4(s)B_2(s),$$

if we put $\nu_3(s) = 0$, then the Frenet retraction curve defined by $\eta_{r3}(s) = r_3(\eta(s))$ where

$$\eta_{r3}(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_4(s)B_2(s),$$

if we put $\nu_4(s) = \bar{c}, \quad \bar{c} \neq 0$ is constant, then the Frenet retraction curve defined by $\eta_{r4}(s) = r_4(\eta(s))$ where,

$$\eta_{r4}(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_4(s)B_2(s).$$

Theorem 4.1. Let $\eta_r(s) = \nu_2(s)N(s) + \nu_3(s)B_1(s) + \nu_4(s)B_2(s)$, be the position vector of the Frenet retracted curve of the Frenet curve $\eta(s)$ in E_1^4, by taking $\nu_1(s) = 0$, then $\eta_r(s)$ lies in the subspace NB_1B_2, and satisfies the differential equation

$$\frac{\mu_4 k_3}{\mu_3 k_2}\frac{d}{ds}\left(\frac{1}{\mu_1 k_1}\right)+\frac{d}{ds}\left\{\frac{1}{\mu_5\, k_3}\left(\frac{\mu_2 k_2}{k_1}-\frac{d}{ds}\left(\frac{1}{\mu_1\mu_3 k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right)\right)\right)\right\}=0.$$

Proof. The position vector of the Frenet retracted curve $\eta_r(\mathrm{s})$ of the Frenet curve $\eta(s)$ in E_1^4, by taking $\nu_1(s)=0$, in equation (2), cab be written as,

$$\eta_r(\mathrm{s})=\nu_2(s)N(s)+\nu_3(s)B_1(s)+\nu_4(s)B_2(s),$$

where $\eta_r(\mathrm{s})$ lies in the subspace NB_1B_2, and by taking $\nu_1(s)=0$, in equations (4),

$$\begin{aligned}\mu_1 k_1\nu_2&=1\\ \nu_2'+\mu_3 k_2\nu_3&=0\\ \nu_3'+\mu_2 k_2\nu_2+\mu_5 k_3\nu_4&=0\\ \nu_4'+\mu_4 k_3\nu_3&=0\end{aligned}\qquad (5).$$

By solving the system in equations (5), then the Frenet retracted curve $\eta_r(\mathrm{s})$ satisfies the differential equation in their curvatures and this completes the proof.

Theorem 4.2. The position vector equations of the Frenet retraction curves $\eta_{ri}(\mathrm{s})$ of the Frenet curve $\eta(s)$ with non-zero curvatures in E_1^4 can be written in the form,

$$\eta_{r1}(\mathrm{s})=\frac{1}{\mu_1 k_1}N(s)+\frac{k_1'}{\mu_1\mu_3 k_2 {k_1}^2}B_1(s)-\frac{1}{\mu_5 k_3}\left(\frac{\mu_2 k_2}{k_1}-\frac{d}{ds}\left(\frac{1}{\mu_1\mu_3 k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right)\right)\right)B_2(s),$$

$$\eta_{r2}(\mathrm{s})=(s+c)T(s)-\left(\frac{k_1(s+c)}{\mu_3 k_2}\right)B_1(s)+\frac{1}{\mu_3 k_3}\frac{d}{ds}\left(\frac{k_1(s+c)}{\mu_3 k_2}\right)B_2(s),$$

$$\eta_{r3}(\mathrm{s})=\frac{c\mu_3}{\mu_2\, k_1}\frac{d}{ds}\left(\frac{k_3}{k_2}\right)T(s)+\frac{c\mu_1 k_1}{\mu_3\, k_2}N(s)+\ cB_2(s),$$

$$\eta_{r4}(\mathrm{s})=\frac{\mu_5\bar{c}}{\mu_2 k_1}\frac{d}{ds}\left(\frac{k_3}{k_2}\right)T(s)-\frac{\mu_5\bar{c}k_3}{\mu_2 k_2}N(s)+\bar{c}B_1(s),$$

where $\bar{c}$ be non-zero constant.

Proof. The position vector equations of the Frenet retraction curves $\eta_{ri}(\mathrm{s})$ of the Frenet curve $\eta(s)$ with non-zero curvatures in E_1^4 can be written in the form,

$$\eta_{ri}(\mathrm{s})=\textstyle\sum_{j=1}^4\nu_j\ W_j,\quad i,j\in\{1,2,3,4\},\quad \nu_j=0,\ \text{ when } i=j,$$

where $W_1=T$, $W_2=N$, $W_3=B_1$, and $W_4=B_2$, so we get,

$$\eta_{r1}(\mathrm{s})=\nu_2(s)N(s)+\nu_3(s)B_1(s)+\nu_4(s)B_2(s),$$

From equations (5), where $\nu_1(s)=0$, then we get,

$$\nu_2(s)=\frac{1}{\mu_1 k_1},\qquad \nu_3(s)=\frac{k_1'}{\mu_1\mu_3 k_2 {k_1}^2}\ \text{ and }\ \nu_4(s)=-\frac{1}{\mu_5 k_3}\left(\frac{\mu_2 k_2}{k_1}-\frac{d}{ds}\left(\frac{1}{\mu_1\mu_3 k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right)\right)\right),$$

and the position vector equations of the Frenet retraction curve $\eta_{r1}(\mathrm{s})$ of the Frenet curve $\eta(s)$ with non-zero curvatures can be written as follow,

$$\eta_{r1}(\mathrm{s})=\frac{1}{\mu_1 k_1}N(s)+\frac{k_1'}{\mu_1\mu_3 k_2 {k_1}^2}B_1(s)-\frac{1}{\mu_5 k_3}\left(\frac{\mu_2 k_2}{k_1}-\frac{d}{ds}\left(\frac{1}{\mu_1\mu_3 k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right)\right)\right)B_2(s).$$

Similarly, we can find the Frenet retraction curves $\eta_{r2}(\mathrm{s})$, $\eta_{r3}(\mathrm{s})$, $\eta_{r4}(\mathrm{s})$ and this completes the proof.

Corollary 4. 1. The Frenet equations of the Frenet curves with non-zero constant curvatures in the Euclidean space E^4, are coincide with the Frenet equations of the Frenet curves of constant curvatures in Minkowski 4-space E_1^4, if $\mu_1=\mu_3=\mu_5=-1$, and $\mu_2=\mu_4=1$.

Proof. The proof is clear by substituting $\mu_1=\mu_3=\mu_5=-1$ and $\mu_2=\mu_4=1$, in equations (4). with the same constant curvatures. Then we have

$$\begin{pmatrix} T' \\ N' \\ {B_1}' \\ {B_2}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

which they have the same position vector, and this completes the proof.

5. Frenet curves with constant curvatures in E_1^4 and their Deformation retracts.

The deformation retract ($D.R$) of $\eta(s) \subset \mathrm{E}_1^4$ into $\eta_{r1}(\mathrm{s}) = r_1(\eta(s))$ is given by

$$\mathrm{D(x, h)} = \mathrm{e^h}(1-h)\,\{\eta(\mathrm{s})\} \; + \; \frac{h}{2}(\mathrm{h}+1)\,\{\eta_6(s)\}\;, \quad m \in \mathbb{R} - \{0\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_1(s)\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\eta_{r2}(\mathrm{s}) = r_2(\eta(s))$ is given by

$$\mathrm{D(x,h)} = \frac{(1-h)}{2} 2^{(1-\mathrm{h})}\{\eta(s)\} + \left(\frac{2\mathrm{h}}{1+\mathrm{h}}\right)\{\eta_2(\mathrm{s})\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_2(s)\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\eta_{r3}(\mathrm{s}) = r_3(\eta(s))$ is given by

$$\mathrm{D(x,h)} = \left(\frac{1-\mathrm{h}}{1+\mathrm{h}}\right)\{\eta(s)\} + \left(\mathrm{h}\;\mathrm{e^{h-1}}\right)\{\eta_3(\mathrm{s})\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$ and $\mathrm{D(x,1)} = \{\eta_3(s)\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\eta_{r4}(\mathrm{s}) = r_4(\eta(s))$ is given by

$$\mathrm{D(x,h)} = \left(\frac{2h}{h+1}\left(\mathrm{e^{h-1}}\right)\right)\{\eta(s)\} + \{(|h-1|)\eta_3(\mathrm{s})\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_4(s)\}$.

Let the Frenet curves equation with constant curvatures be represented as follows:

$$\eta(\mathrm{s}) = v_1(s)T(s) + v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),$$

where k_1, k_2 and k_3 are non-zero constant curvatures.

Theorem 5.1. Let $\eta(\mathrm{s})$ be a Frenet curve in E_1^4 in equation (2) with non-zero constant curvatures, then the position vector of $\eta(\mathrm{s})$ has been presented by the curvature functions

$$v_1(s) = -\mu_1 k_1\left(\frac{-c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s}}{\lambda_1} + \frac{-c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s}}{\lambda_2}\right) + c_0\,,$$

$$v_2(s) = c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s} + c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s} + \frac{1}{\mu_1 k_1}, \qquad (6).$$

$$v_3(s) = \frac{1}{k_2}\left(\left(\frac{{\lambda_1}^2 + {k_1}^2}{\lambda_1}\right)\left(-c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s}\right) + \left(\frac{{\lambda_2}^2 + {k_1}^2}{\lambda_2}\right)\left(-c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s}\right)\right) + \frac{k_1}{k_2} c_5,$$

$$v_4(s) = -\mu_4\, k_3 \int v_3(s) ds$$

$$= -\frac{\mu_4\, k_3}{k_2}\left(\left(\frac{{\lambda_1}^2 + {k_1}^2}{{\lambda_1}^2}\right)\left(c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s}\right) + \left(\frac{{\lambda_2}^2 + {k_1}^2}{{\lambda_2}^2}\right)\left(c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s}\right)\right) + \frac{k_1}{k_2} c_5 s + c_6.$$

Where $c_l, (0 \le l \le 6)$ are integral constants and

$$A = -\left(\mu_1 {k_1}^2 + \mu_2 \mu_5 {k_2}^2 + \mu_4 \mu_5 {k_3}^2\right),$$

$$B = \mu_1 \mu_4 \mu_5 {k_1}^2 {k_3}^2, \qquad (7).$$

$$\lambda_1 = \frac{\sqrt{-2A - 2\sqrt{A^2 - 4B}}}{2}$$

$$\lambda_2 = \frac{\sqrt{-2A + 2\sqrt{A^2 - 4B}}}{2}$$

Proof. Let $\eta(s)$ be a constant curvatures Frenet curve in E_1^4, by differentiating the second and third equations in equations (4), for $\mu_i (1 \le i \le 5), \mu_i \in \{1, -1\}$, so we can get the system,

$$\begin{aligned} \nu_1' &= 1 - \mu_1 k_1 \nu_2 \\ \nu_2'' &= -\mu_5 k_2 \nu_3' - k_1(1 - \mu_1 k_1 \nu_2) \\ \nu_3'' &= \mu_4 \mu_5 {k_3}^2 \nu_3 - \mu_2 k_2 \nu_2' \\ \nu_4' &+ \mu_4 k_3 \nu_3 = 0 \end{aligned} \quad (8).$$

By solving the system in equations (8), which has non-trivial solution (6), and this completes the proof.

Corollary 5.1. Let $\eta(\mathrm{s})$ be a constant curvature time like curve in (2). Then the position vector of $\eta(\mathrm{s})$ has been presented by the curvature functions in (6), when $\mu_i (1 \le i \le 5)$ read, $\mu_3 = \mu_5 = -1$, $\mu_1 = \mu_2 = \mu_4 = 1$.

Corollary 5.2. The position vector of the Frenet retraction curves $\eta_{ri}(\mathrm{s})$ of the Frenet curve $\eta(s)$ with non-zero constant curvatures in E_1^4 can be written in the form,

$$\begin{aligned} \eta_{r1}(\mathrm{s}) &= \frac{1}{\mu_1 k_1} N(s) \\ \eta_{r2}(\mathrm{s}) &= (s + c)T(s) - \left(\frac{k_1(s+c)}{\mu_3 k_2}\right) B_1(s) \\ \eta_{r3}(\mathrm{s}) &= \frac{c\mu_1 k_1}{\mu_3 k_2} N(s) + cB_2(s) \\ \eta_{r4}(\mathrm{s}) &= \frac{\mu_5 \bar{c} k_3}{\mu_2 k_2} N(s) + \bar{c} B_2(s), \end{aligned} \quad (9),$$

where $\bar{c}$ be non-zero constant.

Now we introduce the retraction for the position vector of Frenet curves $\eta(\mathrm{s})$ as follow:

$$\eta(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_3(s)B_1(s) + \nu_4(s)B_2(s),$$

for some differentiable functions $\nu_j(s), 1 \le j \le 4$.

Let $r_i: \{\eta(\mathrm{s}) - \delta\} \to \{\eta(\mathrm{s}) - \delta\}^*$. Where $\{\eta(\mathrm{s}) - \delta\}$ be open Frenet curve in E_1^4 and $\{\eta(\mathrm{s}) - \delta\}^*$ be the retraction of the position vector $\eta(\mathrm{s})$.

The retraction $r_5(\eta(s)) = \eta_5(s)$, by substituting $c_1 = 0$ in equations (6),

$$r_5(\eta(s)) = \eta_5(s) = \nu_{r5_1}(s)T(s) + \nu_{r5_2}(s)N(s) + \nu_{r5_3}(s)B_1(s) + \nu_{r5_4}(s)B_2(s).$$

The retraction $r_6(\eta(s)) = \eta_6(s)$, by substituting $c_2 = 0$ in equations (6),

$$r_6(\eta(s)) = \eta_6(s) = \nu_{r6_1}(s)T(s) + \nu_{r6_2}(s)N(s) + \nu_{r6_3}(s)B_1(s) + \nu_{r6_4}(s)B_2(s).$$

The retraction $r_7(\eta(s)) = \eta_7(s)$, by substituting $c_3 = 0$ in equations (6),

$$r_7(\eta(s)) = \eta_7(s) = \nu_{r7_1}(s)T(s) + \nu_{r7_2}(s)N(s) + \nu_{r7_3}(s)B_1(s) + \nu_{r7_4}(s)B_2(s).$$

The retraction $r_8(\eta(s)) = \eta_8(s)$, by substituting $c_4 = 0$ in equations (6),

$$r_8(\eta(s)) = \eta_8(s) = \nu_{r8_1}(s)T(s) + \nu_{r8_2}(s)N(s) + \nu_{r8_3}(s)B_1(s) + \nu_{r8_4}(s)B_2(s).$$

The retraction $r_9(\eta(s)) = \eta_9(s)$, by substituting $B_1 = 0$ in equation (2),

$$r_9(\eta(s)) = \eta_9(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_4(s)B_2(s).$$

The retraction $r_{10}(\eta(s)) = \eta_{10}(s)$, by substituting $B_2 = 0$ in equation (2),

$$r_{10}(\eta(s)) = \eta_{10}(s) = \nu_1(s)T(s) + \nu_2(s)N(s) + \nu_3(s)B_1(s).$$

The retraction $r_{11}(\eta(s)) = \eta_{11}(s)$, by substituting $B_1 = 0$ and $B_2 = 0$, in equation (2),

$$r_{11}(\eta(s)) = \eta_{11}(s) = \nu_1(s)T(s) + \nu_2(s)N(s).$$

The deformation retracts of Frenet curves with constant curvatures in Minkowski 4-space, where the deformation retract of the Frenet curve is defined as:

$$\varphi: \{\eta(\mathrm{s}) - \delta\} \times I \to \{\eta(\mathrm{s}) - \delta\},$$

where $\{\eta(\mathrm{s}) - \delta\}$ is open Frenet curve in E_1^4 and $\{\eta(\mathrm{s}) - \delta\}^*$ is the retraction of the position vector $\eta(\mathrm{s})$ and I is the closed interval $[0, 1]$, is presented by

$$\varphi(x, h): \{\eta(\mathrm{s}) - \delta\} \times I \to \{\eta(\mathrm{s}) - \delta\}.$$

The deformation retract $(D.R)$ of $\eta(s) \subset \mathrm{E}_1^4$ into the retraction $\mathrm{r}_1(\eta) = \eta_1(s)$ is

$$\mathrm{D(x,h)} = (1-\mathrm{h})^{\frac{m}{n}}\{\eta(s)\} + \mathrm{h}^{\frac{m}{n}}\{\eta_1(s)\},$$

where $\mathrm{D(x,0)} = \eta(s)$, and $\mathrm{D(x,1)} = \eta_1(\mathrm{s}), m, n \in \mathbb{N} - \{1\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_2(\eta) = \eta_2(s)$ be

$$\mathrm{D(x,h)} = \sin\left(\frac{\pi(1-\mathrm{h})}{2}\right)\{\eta(s)\} + \cos\left(\frac{\pi(1-\mathrm{h})}{2}\right)\{\eta_2(s)\}. n \in \mathbb{N},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_2(s)\}$.

The $D.R$ of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_3(\eta) = \eta_3(s)$ is

$$\mathrm{D(x,h)} = |\mathrm{h}-1|\{\eta(s)\} + \frac{m\mathrm{h}}{m-1+\mathrm{h}}\{\eta_3(s)\}, \quad m \in \mathbb{R} - \{0\},$$

where $\mathrm{D(x,0)} = \{\eta(\mathrm{s})\}$, and $\mathrm{D(x,1)} = \{\eta_3(s)\}$.

The $D.R$ of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_4(\eta) = \eta_4(s)$ be

$$\mathrm{D(x,h)} = (1-\mathrm{h})\{\eta(s)\} + \mathrm{h}\{\eta_4(s)\},$$

where $\mathrm{D(x,0)} = \eta(s)$, and $\mathrm{D(x,1)} = \eta_4(\mathrm{s}),\ m, n \in \mathbb{N} - \{1\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_5(\eta) = \eta_5(s)$ is

$$\mathrm{D(x,h)} = \sqrt[m]{1-\mathrm{h}}\{\eta(s)\} + \sqrt[m]{\mathrm{h}}\{\eta_5(s)\},\ m \in \mathbb{N},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_5(s)\}$.

The $D.R$ of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_6(\eta) = \eta_6(s)$ is given by

$$\mathrm{D(x,h)} = |\mathrm{h}-1|\{\eta(s)\} + \frac{2\mathrm{he}^{(1-\mathrm{h})}}{1+\mathrm{h}}\{\eta_6(s)\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_6(s)\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_7(\eta) = \eta_7(s)$ be

$$\mathrm{D(x,h)} = \left(\frac{1-\mathrm{h}}{1+\mathrm{h}}\right)\{\eta\} + \left(\frac{2\mathrm{h}}{1+\mathrm{h}}\right)\{\eta_7(\mathrm{s})\},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_7(s)\}$.

The D.R of $\eta(s) \subset \mathrm{E}_1^4$ into $\mathrm{r}_8(\eta) = \{\eta_8(s)\}$ be

$$\mathrm{D(x,h)} = \cos(\left(\frac{\pi}{2} + 2n\pi\right)\mathrm{h})\{\eta(s)\} - \sin\left(\left(\frac{\pi}{2} + 2n\pi\right)\mathrm{h}\right)\{\eta_8(s)\}, n \in \mathbb{N},$$

where $\mathrm{D(x,0)} = \{\eta(s)\}$, and $\mathrm{D(x,1)} = \{\eta_8(s)\}$.

Theorem 5.2. The deformation retract of any Frenet curve in E_1^4 be a Frenet curve if and only if the Frenet apparatus $\{T_r, N_r, B_r, k_{1r}, k_{2r}, k_{3r}\}$ of the retracted curve $\Omega(s) = r(\eta(\mathrm{s}))$ can be formed by the Frenet apparatus $\{T, N, B, k_1, k_2, k_3\}$ of $\eta(\mathrm{s})$.

Proof. Let $D(s,h) = p(h)\eta(s) + q(h)r(\eta)$ be a deformation retract of the Frenet curve $\eta(s)$ where $D(s,0) = \eta(s)$ and $D(s,1) = r(\eta)$.

$$D'(s,h) = p(h)\eta'(s) + q(h)r'(\eta)\eta'(s) = p(h)T(s) + q(h)r'(\eta)T(s),$$

$$\langle D'(s,h), D'(s,h)\rangle = \langle T_D', T_D'\rangle = \langle p(h)T(s) + q(h)r'(\eta)T(s), p(h)T(s) + q(h)r'(\eta)T(s)\rangle \neq 0.$$

Then the deformation retract of any Frenet curve in E_1^4 be Frenet curve, since we can find that $\langle N_D', N_D'\rangle \neq 0$, and $\langle {B_1}_D', {B_1}_D'\rangle \neq 0$. Conversely this is clear by assume that the Frenet apparatus of the retracted curve $\phi(s) = r(\eta(\mathrm{s}))$ can be formed by the Frenet apparatus of $\eta(\mathrm{s})$ and by using the Frenet equations for the Frenet curves.

Conclusion. In this paper, the position vector equation of the Frenet curves with constant curvatures and non-zero curvatures in Minkowski 4 -space has been presented. The retractions and Frenet frame of Frenet curves in E_1^4 are deduced. The relations between the deformation retracts and Frenet Frame of Frenet curves are obtained.

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