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On Null Curves in Minkowski 3-Space and Its Fractal Folding

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Abstract

In this paper, a form for Frenet equations of all null curves in Minkowski 3-space has been presented. New types of foldings of curves are obtained. The connection between folding, deformation and Frenet equations of curves are also deduced.

Keywords: Minkowski 3-space, null curves, conditional fractal folding, deformation, Frenet equations

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1. Introduction

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system in E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ is said space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null (light-like) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be space-like, time- like or null(light-like), if all of its velocity vectors $\alpha'(s)$ are respectively, space-like, time-like or null (light-like) respectively. Space-like or time-like curve $\alpha(s)$ is said to be parameterized by arc length function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. A curve α is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$, $\alpha \in L^n$ is space-like if its velocity vectors $\alpha'$are space-like for all $t \in I$, similarly for time-like and null. If α is a null curve, we can re-parameterize it such that, $\langle \alpha'(t), \alpha'(t) \rangle = 0$ and $\alpha'(t) \neq 0$, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$.

Given a unit speed curve $\alpha(s)$ in Minkowski space E_1^3 $we\ can\ possible\ define\ a\ Frenet\ frame$ $\{T(s), N(s), B(s)\}$ associated for each point s. Where $T(s), N(s)$ and $B(s)$ are the tangent, normal and binormal vector field (A. E. El-Ahmady & A.T.M. Zidan. 2019) (A. E. El-Ahmady & E. Al-Hesiny. 2013) (R. Lopez. 2008) (R. Aslaner, A. Ihsan Boran. 2009).

2. Preliminary Notes

Let $\alpha(s)$ be a curve in E_1^3. Then for the unit speed curve $\alpha(s)$ with non-null frame vectors, we distinguish three cases depending on the causal character of $T(s)$ and its Frenet equations are as follows,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \mu_1 k & 0 & \mu_2 \tau \\ 0 & \mu_3 \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We write the following subcases,

Case 1. If $\alpha(s)$ is time-like curve in E_1^3, then T is time-like vector and T' is space-like vector. Then $\mu_i (1 < i < 3)$, read $\mu_1 = \mu_2 = 1$, $\mu_3 = -1, T, B$ and N are mutually orthogonal vectors satisfying the equations, $g(N,N) = g(B,B) = 1, g(T,T) = -1$.

Case 2. If $\alpha(s)$ is space like curve in E_1^3, then T is space like vector, since $T'(s)$ is orthogonal to the space like vector $T(s), T'(s)$ may be space like, time-like or light like. Thus we distinguish three cases according to $T'(s)$.

Case 2.1. If the vector $T'(s)$ is space-like, N is space like vector and B is time-like vector. Then $\mu_i (1 < i < 3)$ $read\ \mu_1 = -1, \mu_2 = \mu_3 = 1, T, N\ and\ Bare\ mutually\ orthogonal\ vectors\ satisfying\ g(T,T) = g(N,N) = 1, g(B,B) = -1.$

Case 2.2. If the vector $T'(s)$ is time-like, N is time-like vector and B is space-like vector. Then $\mu_i (1 < i < 3)$ read $\mu_1 = \mu_2 = \mu_3 = 1$, where the orthogonal vectors T, N and B are satisfying $g(T,T) = g(B,B) = 1, g(N,N) = -1$.

Case 2.3. If the vector $T'(s)$ is light like for all s, $N(s) = T'(s)$ is light like vector and $B(s)$is unique light like vector such that $g(N,B) = -1$ and it is orthogonal to T. The Frenet equations are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ 1 & 0 & -\tau \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Case 3. If $\alpha(s)$ is light like curve in E_1^3, $g(N,N) > 0$, when the parameterization is pseudo- arc so $g(N,N) = 1$ with $g(T,T) = 0$, $g(B,B) = 0, g(T,N) = 0$, and $B(s)$ is unique light like vector such that $g(T,B) = -1$ and it is orthogonal to N the pseudo torsion of $\alpha(s)$ be $\tau = -\langle N', B \rangle$, then the Frenet equations of $\alpha(s)$ are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & k \\ 0 & \tau & 0 \end{pmatrix} *$$

Where the curvature k can take only two values, 0 when α is a straight null line, or 1 in all other cases (J. Walrave. 1995).

A regular curve $\alpha : I \to E_1^3$ is called a null curve if α' is light like, that is $\langle \alpha', \alpha' \rangle = 0$ (M. P. Docarmo. 1992).

Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f o\gamma : I \to N$ is piecewise geodesic and of the same length as γ, if f does not preserve the length it is called topological folding (A. E. El-Ahmady. 2007) (A. E. El-Ahmady & E. Al-Hesiny. 2013).

A map $d\colon M \to M^*$ *such that* $M^* = d(M)$ where M and M^* are two smooth Riemannian manifolds is called deformation map if d is differentiable and has differentiable inverse. A deformation map $d\colon M \to M^*$ where M and M^* are two smooth Riemannian manifolds is called regular deformation if $\forall\, x, y \in M$, $K(x) = K(y)) \Leftrightarrow K(d(x)) = K(d(y)), K(x)$ is the curvature at the point $x \in M$, when $(x) = K(d(x))$ $\forall\, x \in M$, it is the identity deformation which is regular deformation (M. P. Docarmo. 1992).

Definition 2.1. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3,)$ be vectors in E_1^3, the vector product in Minkowski space-time E_1^3 is defined by the determinant

$$u \wedge v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Where e_1, e_2 and e_3 are mutually orthogonal vectors (coordinate direction vectors).

3. Form of Frenet Equations of Null Curves in Minkowski 3-Space

Theorem 3.1. Let $\xi(s)$ be a null curve in E_1^3 with the standard flat metric given by $g = dx_1^2 + dx_2^2 - dx_3^2$. Then the bi-normal vector of $\xi(s)$ can be calculated by the form,

$$B(s) = \left(\frac{-1}{\Delta_{1,2}} (\Delta_{2,3}\, b_3 + x_2''), \frac{1}{\Delta_{1,2}} (\Delta_{1,3}\, b_3 + x_1''), \frac{-(1 + x_3''^2)}{2x_3'} \right), \Delta_{1,2} \neq 0, x_3' \neq 0.$$

Where $\Delta_{2,3} = (x_2'\, x_3'' - x_3'\, x_2''), \Delta_{1,3} = (x_1'\, x_3'' - x_3'\, x_1'')$ *and* $\Delta_{1,2} = (x_1'\, x_2'' - x_2'\, x_1'')$.

Proof. Let $\xi(s) = (x_1(s), x_2(s), x_3(s))$, be the parametric equation of any null curve in E_1^3 where the tangent vector $T(s) = (x_1'(s), x_2'(s), x_3'(s))$ and the normal vector $N(s) = T'(s) = (x_1''(s), x_2''(s), x_3''(s))$. To calculate the bi-normal vector of the curve $\xi(s)$, let $B(s) = (b_1, b_2, b_3)$,

since $B(s)$ is unique light like vector, hence

$\langle B, B \rangle = 0$ *and so,*

$b_1{}^2 + b_2{}^2 - b_3{}^2 = 0.$ (1)

Also, $g(T,B) = -1$ and so,

$x_1' \, b_1 + x_2' \, b_2 - x_3' \, b_3 = -1. (2)$

Since B is orthogonal to N where $\langle N, B \rangle = 0$ so we get,

$x_1'' \, b_1 + x_2'' \, b_2 - x_3'' \, b_3 = 0. (3)$

Multiply equation (2) by x_1'' and equation (3) by x_1' and subtracting the product equations so we get,

$b_2 = \frac{1}{\Delta_{1,2}} \left(\Delta_{1,3} \, b_3 + x_1'' \right), \Delta_{1,2} \neq 0. (4)$

Multiply equation 2 by $x2''$ and equation 3 by $x2'$ and subtracting the product equations. Then,

$b_1 = \frac{-1}{\Delta_{1,2}} \left(\Delta_{2,3} \, b_3 + x_2'' \right), \Delta_{1,2} \neq 0. (5)$

By substituting equations 4 and 5 in equation 1. Then,

$\left({\Delta_{2,3}}^2 + {\Delta_{1,3}}^2 - {\Delta_{1,2}}^2 \right) {b_3}^2 + {x_1''}^2 + {x_2''}^2 + 2\left(\Delta_{1,3} \, x_1'' + \Delta_{2,3} \, x_2'' \right) b_3 = 0.$

But $\left({\Delta_{2,3}}^2 + {\Delta_{1,3}}^2 - {\Delta_{1,2}}^2 \right) = 0$ and so we get,

$b_3 = \frac{-\left({x_1''}^2 + {x_2''}^2 \right)}{2(\Delta_{1,3} \, x_1'' + \Delta_{2,3} \, x_2'')}. (6)$

Also, b_3 can be written in the form,

$b_3 = \frac{-[\, g(N,N) + {x_3''}^2 \,]}{2[\, g(T,N) x_3'' - g(N,N) x_3']} \, . (7)$

In equation (7), when the parameterization is pseudo-arc so $g(N, N) = 1, g(T, N) = 0$ and we get,

$b_3 = \frac{-(1 + {x_3''}^2)}{2x_3'}, x_3' \neq 0. (8)$

Where $\Delta_{2,3} = (x_2' \, x_3'' - x_3' \, x_2''), \Delta_{1,3} = (x_1' \, x_3'' - x_3' \, x_1'')$, and $\Delta_{1,2} = (x_1' \, x_2'' - x_2' \, x_1'')$. Then we get,

$$B(s) = \left(\frac{-1}{\Delta_{1,2}} \left(\Delta_{2,3} \, b_3 + x_2'' \right), \frac{1}{\Delta_{1,2}} \left(\Delta_{1,3} \, b_3 + x_1'' \right), \frac{-(1 + {x_3''}^2)}{2x_3'} \right). (9)$$

Where $\Delta_{1,2} \neq 0, x_3' \neq 0$ with curvature $k = 1$ and torsion $\tau = -\langle N', B \rangle = \frac{1}{2} \, g(\alpha''', \alpha''')$.

Example 3.1. Let $\alpha(s) = \frac{1}{r^2}(cosh(rs), rs, sinh(rs))$ if we calculate 1st and 2nd order derivatives (with respect to s) of $\alpha(s)$ and so $T(s) = \frac{1}{r}(sinh(rs), 1, cosh(rs))$. Since $\langle T, T \rangle = 0$ so $\alpha(s)$ is a null curve and $N(s) = T'(s) = (cosh(rs), 0, sinh(rs)) \; so \; \langle N, N \rangle = 1$, since $B(s)$ is unique light like vector such that $g(T, B) = 1$and it is orthogonal to T, by substituting in the equation (9). We get $B(s) = \frac{r}{2}(sinh(rs), -1, cosh(rs))$ and so $\langle B, B \rangle = 0, N' = r(\sinh(rs), 0, \cosh(rs))$. The pseudo torsion is $\tau = -\langle N', B \rangle = \frac{-r^2}{2}$ where N is space like vector. Then $\alpha(s)$ is a null curve with curvature $k = 1$ and the Frenet equations of $\alpha(s)$ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & k \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-r^2}{2} & 0 & 1 \\ 0 & \frac{-r^2}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{r}(\sinh(rs), 1, \cosh(rs)) \\ (\cosh(rs), 0, \sinh(rs)) \\ \frac{r}{2}(\sinh(rs), -1, \cosh(rs)) \end{pmatrix}.$$

Corollary 3.1 Let $\xi(s)$ be a null curve in E_1^3 with non-zero curvature and pseudo torsion τ, then the bi-normal vector of $\xi(s)$ can be calculate by the form,

$$B(s) = \left(\frac{1}{k}\right) N'(s) - \left(\frac{\tau}{k}\right) T(s) = \left(\frac{1}{k}\right) \xi'''(s) - \left(\frac{\tau}{k}\right) \xi'(s).$$

Such that $\tau = -g(N', B) \; or \; \tau = \frac{1}{2} \, g(\xi''', \xi''')$.

Theorem 3.2. Let $\xi(s)$ be a null curve in E_1^3 with non-zero curvature and pseudo torsion $\tau(s)$. Then $\xi(s) \; satisfies \; a \; vector \; differential \; fourth \; order \; as \; follow,$

$$\frac{d4\xi}{ds^4} - 2\tau \left(\frac{d2\xi}{ds^2}\right) - \tau' \frac{d\xi}{ds} = 0.$$

Proof. Since $\xi(s)$ be a null curve in E_1^3 from the Frenet equation $(*)$. We get,

$$T'(s) = kN(s), N'(s) = \tau T(s) + kB(s)\ and\ B'(s) = \tau N(s),$$

with $k = 1$ and so we have,

$$T''(s) = N'(s) = \tau\, T + B(s)\ and\ T'''(s) = \tau'\, T + \tau\, T' + B'(s). Then,$$

$$T'''(s) = \tau'\, T + 2\tau\, T'\ and\ so\ T'''(s) - 2\tau\, T' - \tau'\, T = 0\ denoting\ T = \frac{d\xi}{ds},$$

$$\frac{d4\xi}{ds^4} - 2\tau\left(\frac{d2\xi}{ds^2}\right) - \tau'\frac{d\xi}{ds} = 0.$$

4. Folding of Null Curves

Theorem 4.1. Let $\xi(s)$ be a null curve in E_1^3 with non-zero curvature and $\Psi(s) = f(\xi(s))$ be a topological folding of $\xi(s)$ for all s where $s \in$ Domain $(\Psi(s)) = I \subset Domain\, \xi(s)$ defined by frame vectors. Then $\Psi(s) = f(\xi(s))$ is a null curve and the Frenet apparatus of the folded curve $\Psi(s)$ can be formed by the Frenet apparatus of $\xi(s)$.

Proof. Let $\xi = \xi(s)$ be a null curve in E_1^3 with non-zero curvature and $\Psi(s) = f(\xi(s))$, $s \in I \subset$ domain $\xi(s)$ is a topological folding of $\xi(s)$ with curvatures k_f and τ_f and so,

$\Psi(s) = f(\xi(s)), \Psi'(s) = f'(\xi)\xi'(s) = f'(\xi)\, T(s)$. And we get,

$\langle \Psi', \Psi' \rangle = \langle f'\xi'(s), f'\xi'(s) \rangle = f'^2 \langle T(s), T(s) \rangle = 0$. Since $\xi(s)$ is a null curve with $\langle T(s), T(s) \rangle = 0, f'^2 > 0$ for all s. Then $\Psi(s)$ is a null curve with curvatures $k_f = k = 1$ and $T_f = f'(s)\, T(s)$ where,

$\Psi'''(s) = T^3 f'''(\xi) + 3T\, N f''(\xi) + f'(\xi)\xi'''(s)$.

By substituting the value of $\xi'''(s)$ from the Frenet apparatus of the curve $\xi(s)$ in corollary 3.1. Then,

$T_f = T(s)\, f'(\xi)$,

$N_f = \Psi''(s) = N(s)\, f'(\xi) + T^2(s)\, f''(\xi)$,

$$B_f = \Psi'''(s) = f'(\xi)B(s) + T^3 f'''(\xi) + 3T\, N f''(\xi), \tau_f = \tau = 0,$$

$B_f = \Psi''' - \tau_f \Psi' = (\tau - \tau_f) f'(\xi) T + f'(\xi) B + T^3 f'''(\xi) + 3T\, N f''(\xi)$, for all $\tau \neq 0$ and $\tau_f \neq 0$.

Corollary 4.1. Let $\xi(s)$ be a null curve in E_1^3 and $\Psi(s) = f(\xi(s))$ be a topological folding of $\xi(s)$. Then the limit of folding's of $\xi(s)$ is a null point.

Proof. Let $\Psi(s) = f(\xi(s))$ be a topological folding of the null curve $\xi(s)$ in E_1^3 so $\Psi(s)$ be null curve and we have,

$$\Psi_1(s): f(\xi(s)) \to f(\xi(s)), \Psi_2(s): \Psi_1(f(\xi(s))) \to \Psi_1\left(f(\xi(s))\right),$$

$$\Psi_3(s): \Psi_2(\Psi_1(f(\xi(s)))) \to \Psi_2\left(\Psi_1\left(f(\xi(s))\right)\right), \dots,$$

$$\Psi_n: \Psi_{(n-1)}(\Psi_{(n-2)}(\dots \Psi_1 f(\xi)))) \dots) \to \Psi_{(n-1)}(\Psi_{(n-2)}\left(\dots \Psi_1(f(\xi))\right) \dots)$$

Then $\lim_{n\to\infty} \Psi_n = p = (0, 0, 0)$, which is a null point.

Definition 4.1. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$ be a null curve in E_1^3. Then $\Psi(s)$ be an isometric folding defined as follows,

$\Psi(s)$: $\xi(s) = \{x_1(s), x_2(s), x_3(s))\} \to \xi_f = \left\{\left(\frac{|x_1(s)|}{m}, \frac{|x_2(s)|}{m}, \frac{|x_3(s)|}{m}\right)\right\}$ for all s, $|m| > 1, m \neq 0$.

Theorem 4.2. Let $\xi(s) = (x_1(s), x_2(s), x_3(s))$ be a null curve in E_1^3 and $\Psi(\xi) = \left(\frac{|x_1(s)|}{m}, \frac{|x_2(s)|}{m}, \frac{|x_3(s)|}{m}\right)$ for all s be an isometric folding of $\xi(s), |m| > 1$. Then the folding $\Psi(s)$ be a null curve and, $\begin{pmatrix} T_f \\ N_f \\ B_f \end{pmatrix} = \begin{pmatrix} \frac{\delta}{m} & 0 & 0 \\ 0 & \frac{\delta}{m} & 0 \\ \delta m & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \delta = 1$ if $x_i(s) > 0$ and $\delta = -1$ if $x_i(s) < 0, i \in \{1, 2, 3\}$.

Proof. Let $\Psi(\xi): \xi(s) = (x_1(s), x_2(s), x_3(s)) \to (\frac{|x_1(s)|}{m}, \frac{|x_2(s)|}{m}, \frac{|x_3(s)|}{m}), |m| > 1,$ be an isometric folding of the null curve $\xi(s) = (x_1(s), x_2(s), x_3(s))$ in E_1^3. If $x_i(s) > 0,$ $i \in \{1,2,3\}$, then $\Psi' = \frac{d\Psi}{ds} = \frac{1}{m}(x_1'(s), x_2'(s), x_3'(s)),$ since $\xi(s)$ be a null curve where $\langle T(s), T(s)\rangle = 0$ and $\langle T'(s), T'(s)\rangle = 0,$ for the folded curve $\xi_f(s) = \left(\frac{x_1(s)}{m}, \frac{x_2(s)}{m}, \frac{x_3(s)}{m}\right)$ since $\langle T_f(s), T_f(s)\rangle = \frac{1}{m^2}\langle T(s), T(s)\rangle = 0$ and $\langle T_f'(s), T_f'(s)\rangle = \frac{1}{m^2}\langle T'(s), T'(s)\rangle = 0,$ then the folded curve $\xi_f(s)$ is a null curve. Since $B(s)$ is unique light like vector, also $g(T, B) = -1$ and B is orthogonal to N. Then,

$T_f(s) = \Psi'(s) = \frac{1}{m}T(s), N_f(s) = T_f' = \frac{1}{m}T'(s) = \frac{1}{m}N(s)$ and from theorem(1), we get,

$B_f(s) = mB(s).$

If $x_i(s) < 0,$ $i \in \{1,2,3\}$ and $\xi_f(s) = \left(\frac{-x_1(s)}{m}, \frac{-x_2(s)}{m}, \frac{-x_3(s)}{m}\right),$ so $T_f(s) = \frac{-1}{m}T(s),$ $N_f(s) = \frac{-1}{m}N(s)$ and $B_f(s) = -mB(s).$ Then the Frenet apparatus of the folding $\Psi(\xi)$ can be formed by the Frenet apparatus of $\xi(s)$.

Now we introduce a type of folding which make the null curves to be space like curves and time like curves and the converse as follows,

5. Conditional Fractal Folding of Null Curves

Definition 5.1 Let $\xi(s)$ be any curve in E_1^n the map which is defined as $\xi_f: (x_1(s), x_2(s), \dots, x_i(s), \dots, x_n) \to (x_1(s), x_2(s), \dots, \varepsilon x_i(s), \dots, x_n(s))$ for $\varepsilon \le 1, \varepsilon \ne 0$ is called conditional fractal folding of the coordinates x_i, ε depends on the type of the curve ξ_f (space like, time like and null curve) (M. EL-Ghoul & A. M. Soliman. 2002).

Theorem 5.1. Let $\xi(s)$ be a null curve in E_1^3. Under the conditional fractal folding $\Psi(s)$: $\xi(s) = \{x_1(s), x_2(s), x_3(s))\} \to \xi_f = (x_1(s), x_2(s), \varepsilon x_3(s)),$ $\varepsilon \ne 0$ for all s, then ξ_f is space like curve if $|\varepsilon| < 1, \xi_f$ is null curve if $\varepsilon = \pm 1$ and ξ_f is time like curve if $|\varepsilon| > 1$.

Proof. Let $\xi(s)= \{x_1(s), x_2(s), x_3(s))\}$ be a null curve in $E_1^3, \langle T, T\rangle = 0,$ so $x_1{}^2 + x_2{}^2 = x_3{}^2$ and $\Psi(s)$ be conditional folding defined as $\Psi(s)$: $\xi(s) \to \xi_f,$ if $\xi_f = (x_1(s),\ (x_2(s), \varepsilon x_3), \varepsilon \ne 0,$ so $\langle T_f, T_f\rangle = x_1{}^2 + x_2{}^2 - \varepsilon^2 x_3{}^2$ and then let $g(s) = \langle T_f, T_f\rangle,$ then we have $g'(s) = 2\langle T_f, T_f'\rangle = 2\langle T_f, k_f N_f\rangle = 0$ where $k_f \ne 0$ is constant, so $g'(s) = 0$ and $g(s) = c_1, c_1$ is constant.

If $c_1 > 0, \langle T_f, T_f\rangle > 0$ and $x_1{}^2 + x_2{}^2 - \varepsilon^2 x_3{}^2 > 0$ so $x_3{}^2 (1 - \varepsilon^2) > 0, \varepsilon^2 < 1,$ then ξ_f is space- like if $|\varepsilon| < 1.$

If $c_1 < 0$ we have $\langle T_f, T_f\rangle < 0$ and $\varepsilon^2 > 1,$ then ξ_f is time like curve if $|\varepsilon| > 1.$

If $c_1 = 0,$ $\langle T_f, T_f\rangle = 0$ and so $\varepsilon^2 = 1,$ then ξ_f is null curve if $\varepsilon = \pm 1.$

Corollary 5.1. Let $\xi(s)$ be a null curve in E_1^3. Under the conditional fractal folding which is defined as,

$\Psi(s)$: $\xi(s)= \{x_1(s), x_2(s), x_3(s))\} \to \xi_f = (x_1(s), x_2(s), \varepsilon x_3(s))$ for all $\varepsilon \le 1,$ $\varepsilon \ne 0.$

The Frenet equations of the folded curve ξ_f is depends on ε.

Corollary 5.2. Let $\xi(s)$ be a null curve in E_1^3 and $\Psi(t)$ be conditional fractal folding defined as $\Psi(s)$: $\xi(s) = \{x_1(s), x_2(s), x_3(s))\} \to \xi_f$ and $\xi_f = (\varepsilon x_1(s), \varepsilon x_2(s), x_3(s)), \varepsilon \ne 0, \varepsilon \le 1$ for all s. Then

ξ_f is space like curve if $|\varepsilon| > 1, \xi_f$ is time like curve if $|\varepsilon| < 1.$ and ξ_f is null curve if $\varepsilon = \pm 1.$

Corollary 5.3. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$ be any curve in E_1^3 under the conditional fractal folding $\Psi(s)$: $\xi(s) \to \xi_f, \xi_f = (\varepsilon x_1(s),\ x_2(s), x_3(s))$ or $\xi_f = (x_1(s), \varepsilon x_2(s),\ x_3(s)),,$ $\varepsilon \ne 0, |\varepsilon| < 1,$ for all s. Then the limit of a sequence of foldings of $\xi(s)$ is never being null curve.

Proof. Let the limit of a sequence of foldings of any curve $\xi(s)$ in E_1^3 be a null curve with $\xi_f = (0, x_2(s), x_3(s)),$ or $\xi_f = (x_1(s), 0,\ x_3(s))$ and $\xi_f = (x_1(s), x_2(s), 0),$ then from theorem 3.1, the bi-normal vector of the folded curve B_f undefined, also $N_f' = \tau T_f - kB_f$ undefined. The Frenet equations of ξ_f cannot appoints and so this contradict with ξ_f be null curve. Then ξ_f never being null curve.

Theorem 5.2. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$ be a null curve in E_1^3. Then the conditional folding $\xi_f = (\varepsilon x_1(s), \varepsilon x_2(s), \varepsilon\, x_3(s)), |\varepsilon| < 1,$ of $\xi(s)$ be null curve. And the Frenet equations of the folded curve ξ_f can be formed by the Frenet equations of $\xi(s)$.

Proof. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$, be null curve in E_1^3 and $\xi_f = (\varepsilon x_1(s), \varepsilon x_2(s), \varepsilon\, x_3(s)), |\varepsilon| < 1$ be a conditional fractal folding of $\xi(s)$ and so $\langle T_f, T_f\rangle = \varepsilon^2 \langle T(s), T(s)\rangle = 0$. Then the folded curve ξ_f is null curve, with curvature $k_f = k = 1$ and torsion $\tau_f = \tau$, by using the form of Frenet equations in theorem1. Then we have,

$$\left.\begin{array}{c} T_f(s) = \varepsilon T(s) \\ N_f(s) = \varepsilon N(s) \\ B_f(s) = \left(\frac{1}{\varepsilon}\right) B(s) \end{array}\right\}.$$

Corollary 5.4. Let $\xi(s) = \{(x_1(s), x_2(s), x_3(s))\}$ be a null curve in E_1^3. Then the conditional fractal folding $\xi_f = (\varepsilon_i x_1(s), \varepsilon_i x_2(s), \varepsilon_i x_3(s))$, $i \in \mathbb{N}, |\varepsilon_i| < 1$, $\varepsilon_i \neq 0$ be a null curve and the limit of a sequence of foldings of a null curve $\xi(s)$ be a null point.

Proof. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$ be a null curve in E_1^3. So $\langle T(s), T(s)\rangle$, since $\langle T_f(s), T_f(s)\rangle = {\varepsilon_i}^2 \langle T(s), T(s)\rangle = 0, \varepsilon_i \neq 0$, then ξ_f is a null curve.

Let $f\colon \xi \to \varepsilon_i \xi$ be a conditional fractal folding of the null curve ξ such that $\forall x, y \in \xi$, $d(x, y) \geq d(f(x), f(y))$ where $\xi(s)$ be a null curve. By successive steps of conditional fractal folding's we get,

$$f_1\colon \xi \to \varepsilon_1 \xi, |\varepsilon_1| < 1,$$
$$f_2\colon \varepsilon_1 \xi \to \varepsilon_2(\xi), \varepsilon_2 < \varepsilon_1,$$
$$f_3\colon \varepsilon_2 \xi \to \varepsilon_3(\xi), \varepsilon_3 < \varepsilon_2 \cdots,$$
$$f_n\colon \varepsilon_{(n-1)} \xi \to \varepsilon_n(\xi), \varepsilon_n < \varepsilon_{(n-1)} \ll 1,$$

$\lim_{n\to\infty} f_n(\xi) = p$ where $p = (0, 0, 0)$ is a null point.

Theorem 5.5. If $\xi(s)$ and $\bar{\xi}(s)$ are null curves with non-zero curvature in E_1^3 and $F_\tau\colon \xi \to \bar{\xi}$ is an isotorsion folding, then the torsion of $\bar{\xi}$ identically zero if and only if ξ is a part of the null cubic.

Proof. Let $\bar{\xi}$ be a null curve in E_1^3 has torsion identically zero. Since F_τ is an isotorsion folding from ξ into $\bar{\xi}$. Then the torsion of ξ is zero and the Maclaurin series can be written as,

$\xi\,(s) = \xi(0) + \xi'(0)s + \xi''(0)\frac{s^2}{2} + \xi'''(0)\frac{s^3}{6}$.

Since $B(s) = -\xi'''(s)$ when $\tau = 0$. So we get,

$\xi(s) = \xi(0) + T(0)s + N(0)\frac{s^2}{2} - B(0)\frac{s^3}{6}$. With Frenet frame $\{T, N, B\}$ of $\xi(s)$ in this case $g(T, T) = g(B, B) = 0, g(T, B) = g(N, N) = 1$. Without loss of generality,

assume that $T(0) = \frac{1}{\sqrt{2}}(1, 0, 1), N(0) = (0, 1, 0)$ and $B(0) = \frac{1}{\sqrt{2}}(1, 0, -1)$ so we get,

$\xi(s) = \frac{1}{6\sqrt{2}}\left(6s - s^3, 3\sqrt{2}\, s^2, 6s + s^3\right)$. Then $\xi(s)$ is a part of null cubic. Conversely let the curve $\xi(s)$ be a part of the null cubic, then the torsion of $\xi(s)$ identically zero. Since F_τ is an isotorsion folding and $\bar{\xi}$ has torsion identically zero.

6. Conditional Deformations of Null Curves in E_1^3

Theorem 6.1. Let $\xi(s)$ be a null curve in E_1^3 and $F(x) = M\,x + c, c \in \mathbb{R}, M \neq 0$ be a conditional deformation of $\xi(s)$ defined as $\mathrm{F}(s) = (Mx_1(\mathrm{s}) + \mathrm{c}\,, \mathrm{M}x_2(s) + c\,, \mathrm{M}x_3(s) + c)$.Then the deformation $\mathrm{F}(s)$ be a null curve and,

$$\begin{pmatrix} T_F \\ N_F \\ B_F \end{pmatrix} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & \frac{1}{M} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proof. Let$\xi(s)$ be a null curve in E_1^3 and $F(s)$ be a conditional deformation of $\xi(s)$ defined as $F(s)$= $M\xi(s) + c$, since $\xi(s)$ is a null curve so $\langle \xi', \xi'\rangle = 0$, $M \neq 0$ we get,

$F'(\mathrm{s}) = \mathrm{M}\xi'(s), \langle F', F'\rangle = M^2\langle \xi', \xi'\rangle = 0$. Then $F(s)$ is a null curve with $k = k_d = 1$ and we get,

$T_F(s) = F'(s) = M\xi'(s) = MT(s)$,

$N_F(s) = {T_F}'(s) = MT'(s) = MN(s)$,

$B_F(s) = \frac{1}{M}B$, where the torsion be $\tau_F = -\langle N_F{}', B_F\rangle = \tau$. Then the Frenet apparatus of $F(s)$ can be formed by the Frenet apparatus of $\xi(s)$.

Corollary 6.1. A null curve $\xi(s)$ in E_1^3 under the conditional deformation $F(s)= M\xi(s) + c$ of $\xi(s), M \neq 0$ has the first curvature identically zero if and only if $F(\xi)$ be a part of a straight line.

Proof. Assume that the conditional deformation $F(\xi)$ of the null curve $\xi(s)$ be $\mathrm{F}(s) = (Mx_1(\mathrm{s}) + \mathrm{c}, \mathrm{M}x_2(s) + c, \mathrm{M}x_3(s) + c)$ where $M \neq 0$ such that $\dim F(\xi) = \dim \xi(s)$ and from the Frenet equations with first curvature $\mathrm{k} = 0$, then $F''(\xi) = M\,N(s) = 0$ and this implies that $F(\xi)$ is a straight line, where $k_F = \|T_F'(s)\|, N_F(s) = T_F'(s)$. Conversely, let $F(\xi)$ be a straight line then $F''(\xi) = MN(s) = 0$ and $F(\xi)$ has the curvature k_F which is identically zero.

Remark 6.1. If $\alpha(\mathrm{s})$ be a light-like curve in E_1^3 with standard flat metric $g = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2, g(N,N) > 0$, when the parameterization is pseudo arc so $g(N,N) = 1$ with $g(T,T) = 0$, $g(B,B) = 0$ and $g(T,N) = 0$, and $B(s)$ is unique light like vector such that $g(T,B) = 1$ and it is orthogonal to N the pseudo torsion of $\alpha(\mathrm{s})$ be $\tau = -\langle N', B\rangle$, then the Frenet equations of $\alpha(s)$ are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & -k \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. 10.$$

Where the curvature k can take only two values 0 when α is a straight null line or 1 in all other cases.

Theorem 6.2. Let $\xi(s)$ be a null curve in E_1^3 with standard flat metric $g = dx^2 + dy^2 - dz^2$. Under the conditional deformation,

D: $\xi(s) = (x(s), y(s), z(s)) \rightarrow D(\xi) = (\bar{x}(s), \bar{y}(s), \bar{z}(s)) = (z(s), y(s), -x(s))$ which rotation the coordinates x and z in x-z plane with rotation angle $\theta = \frac{n\pi}{2}, n \in \mathbb{R}, n$ is odd integer. Then $D(\xi)$ be a null curve with standard flat metric $g = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$.

Proof. Let $\xi(s)$ be a null curve in E_1^3 with standard flat metric $g = dx^2 + dy^2 - dz^2$ since the equation of which rotation coordinates x and z in x-z plane can be written as,

$\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \bar{y} = y$. Under the conditional deformation $D(\xi)$ which rotation coordinates x and z with rotation angle $\theta = \frac{n\pi}{2}$, and $n \in \mathbb{R}, n$ is odd integer then $\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \bar{y} = y$, or $\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \bar{y} = y$,

$D(\xi) = (\bar{x}(s), \bar{y}(s), \bar{z}(s)) = \big(z(s), y(s), -x(s)\big)$ also,

$g(D', D') = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$,

$\langle D', D'\rangle = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 = dx^2 + dy^2 - dz^2 = \langle \xi', \xi'\rangle = 0$. Then the conditional deformation $D(\xi)$ be a null curve with standard flat metric $g = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$.

Theorem 6.3. Let $\xi(s)$ be a null curve in E_1^3 with the standard flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2$. Then the bi-normal vector of can be calculated by,

$B(s) = \left(\frac{1}{\Delta_{1,2}}\left(x_2'' - \Delta_{2,3}\, b_3\right), \frac{1}{\Delta_{1,2}}\left(\Delta_{1,3}\, b_3 - x_1''\right), \frac{-(1+ {x_3''}^2)}{2x_3'}\right), \Delta_{1,2} \neq 0, x_3' \neq 0.$

Where $\Delta_{2,3} = (x_2'\, x_3'' - x_3'\, x_2'')$, $\Delta_{1,3} = (x_1'\, x_3'' - x_3'\, x_1'')$ and $\Delta_{1,2} = (x_1'\, x_2'' - x_2'\, x_1'')$.

Proof. Let $\xi(s) = (x_1(s), x_2(s), x_3(s))$ be a null curve in E_1^3 with tangent vector $T(s) = (x_1'(s), x_2'(s), x_3'(s))$ and the normal vector $N(s) = T'(s) = (x_1''(s), x_2''(s), x_3''(s))$, to calculate the bi-normal vector of the curve $\xi(s)$, let $B(s) = (b_1, b_2, b_3)$, since $B(s)$ is unique light-like vector. Then,

$\langle B, B\rangle = 0$ and we get,

$$-{b_1}^2 + {b_2}^2 + {b_3}^2 = 0. (11)$$

Also since $g(T, B) = 1$ we get,

$$-x_1'\, b_1 + x_2'\, b_2 + x_3'\, b_3 = 1. (12)$$

Since B be orthogonal to N, then $\langle N, B\rangle = 0$ and we get,

$$-x_1''\, b_1 + x_2''\, b_2 + x_3''\, b_3 = 0. (13)$$

By solving these equations as theorem1, we get the bi-normal vector be

$$B(s) = \left(\frac{1}{\Delta_{1,2}}\left(\Delta_{2,3}\, b_3 + x_2''\right), \frac{1}{\Delta_{1,2}}\left(\Delta_{1,3}\, b_3 + x_1''\right), \frac{(1 - x_3''^2)}{2x_3'}\right), \Delta_{1,2} \neq 0, x_3' \neq 0. (14)$$

Where $\Delta_{2,3} = (x_2'\, x_3'' - x_3'\, x_2'')$, $\Delta_{1,3} = (x_1'\, x_3'' - x_3'\, x_1'')$ and $\Delta_{1,2} = (x_1'\, x_2'' - x_2'\, x_1'')$ and so,

$$b_1 = \frac{1}{\Delta_{1,2}}\left(\Delta_{2,3}\, b_3 + x_2''\right), b_2 = \frac{1}{\Delta_{1,2}}\left(\Delta_{1,3}\, b_3 + x_1''\right) \text{ and } b_3 = \frac{\left(x_2''^2 - x_1''^2\right)}{2(\Delta_{1,3}\, x_1'' - \Delta_{2,3}\, x_2'')}.$$

Also b_3 can be written in the form,

$$b_3 = \frac{[\, g(N,N) - x_3''^2\,]}{2[\, g(N,N)x_3' - g(T,N)x_3''\,]}\ . (15)$$

In equation (15), when the parameterization is pseudo-arc so $g(N,N) = 1, g(T,N) = 0$. Then,

$$b_3 = \frac{(1 - x_3''^2)}{2x_3'}, x_3' \neq 0. (16)$$

Example 6.1. Let $\alpha(s) = \frac{1}{r^2}(\cosh(rs), rs, \sinh(rs))$ be a null curve in E_1^3 with standard flat metric $g = dx^2 + dy^2 - dz^2$ and $\alpha_D(s) = \frac{1}{r^2}(\sinh(rs), rs, \cosh(rs))$ be deformation of the null curve $\alpha(s)$ by rotation coordinates x and z with rotation angle $\theta = \frac{n\pi}{2}, n$ is odd integer with standard flat metric $g = -d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$. If we calculate 1st and 2nd order derivatives (with respect to s) of $\alpha_D(s)$ and so $T(s) = \frac{1}{r}(\cosh(rs), 1, \sinh(rs))$, since $\langle T,T\rangle = 0$ so $\alpha(s)$ is null a curve and $N(s) = T'(s) = (\sinh(rs), 0, \cosh(rs))$, so $\langle N,N\rangle = 1$, since $B(s)$ is unique light like vector such that $g(T,B) = 1$ and it is orthogonal to T by substituting in the equation (13). Then $B(s) = \frac{-r}{2}(\cosh(rs), -1, \sinh(rs))$, so $\langle B,B\rangle = 0, N' = r(\cosh(rs), 0, \sinh(rs))$, the pseudo torsion is $\tau = -\langle N',B\rangle = \frac{1}{2}\, g(\alpha''', \alpha''') = \frac{-r^2}{2}, N$ is space like vector. Then $\alpha(s)$ is a null curve with curvature $k = 1$ and the Frenet equations of $\alpha(s)$ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & -k \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-r^2}{2} & 0 & -1 \\ 0 & \frac{r^2}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{r}(\cosh(rs), 1, \sinh(rs)) \\ (\sinh(rs), 0, \cosh(rs)) \\ \frac{-r}{2}(\cosh(rs), -1, \sinh(rs)) \end{pmatrix}.$$

Corollary 6.2. Under the conditional deformation which is defined by,

D: $\xi(s) = (x(s), y(s), z(s)) \rightarrow D(\xi) = (z(s), y(s), x(s))$, the Frenet equations of $D(\xi)$ are invariant.

Proof. The proof is clear from theorem 6.3, the Frenet equations of $D(\xi)$ calculates from equation (10).

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