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Geometric Properties of Special Spacelike Curves in Three-Dimension Minkowski Space-Time

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Abstract

In this paper, we introduce a special spacelike Smarandache curves φ reference to the Bishop frame of a regular spacelike curve ζ in Minkowski 3-space $\mathbb{R}_1^3$. From that point, we investigate the Frenet invariants of a special case in $\mathbb{R}_1^3$ and we obtain some properties of these curves when the base curve ζ is contained in a plane. Lastly, we shall give two examples to illustrate these curves.

Keywords: smarandache curve, bishop frame, Minkowski 3-space

AMS Subject Classification (2010):

Primary: 53A04; 53A05; 53C50; 51B20. **Secondary**: 53Z05; 53Z99.

1. Introduction

When considering the theory of curves in the Euclidean spaces $\mathbb{R}^3$ and Minkowski spaces $\mathbb{R}_1^3$, we discovered that the Smarandache curves are this regular curve whose position vector is composed of Frenet frame vectors on other regular curves(M. M. Wageeda, E. M. Solouma, & M. Bary., 2019)(C. Ashbacher., 1997).

When considering in the reference (C. Ashbacher., 1997) (M. A. Soliman, W. M. Mahmoud, E. M. Solouma, & M. Bary., 2019) (H. Iseri, 2002)(L. Mao.,2006) we find the Smarandache geometries are a generalization of classical geometries, and the Smarandache geometries can be either partially Euclidean and partially Non-Euclidean. Then recently, special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors (O. Bektas, & S. Yuce., 2013) (M. Cetin, Y. Tuncer, & M. K. Karacan., 2014)(E. M. Solouma, 2017a) (E. M. Solouma, 2017b) (K. Taskopru, & M. Tosun, 2014).

In this work, we mention spacelike special curves (Smarandache curves) according to Bishop frame of a spacelike curve ζ in the three-dimension Minkowski space $\mathbb{R}_1^3$. In Section 2, we give the basic conceptions of three-dimension Minkowski 3-space $\mathbb{R}_1^3$ and give of Bishop frame that will be used during this work. In Section 3, we investigate the Bishop special spacelike $T\,B_1, T\,B_2\,, B_1B_2\ and\ T\,B_1\,B_2-curves$ in terms of the curvature functions $\kappa_1(\sigma)\ and\ \kappa_2(\sigma)$ of the base curve in $\mathbb{R}_1^3$. On top of that, we obtain some properties on these special curves when the curve ζ is contained in a plane. Finally, in Section 4, we give two examples to clarify these curves.

2. Preliminaries

The Minkowski 3-space $\mathbb{R}_1^3$ is three-dimensional Euclidean space provided with the Lorentzian inner product,

$$\mathcal{D} = -d\,\varsigma_1^2 + d\,\varsigma_2^2 + d\,\varsigma_3^2$$

where $(\varsigma_1\,,\ \varsigma_2\,,\ \varsigma_3)$ is a rectangular coordinate system of $\mathbb{R}_1^3$. An arbitrary vector $u \in \mathbb{R}_1^3$ can have one of three characters; it can be spacelike if $\mathcal{D}(u\,,u) > 0\ or\ u = 0$, timelike if $\mathcal{D}(u\,,u) < 0$ and null if $\mathcal{D}(u\,,u)0\ \ and\ u \neq 0$. Similarly, an arbitrary curve $\zeta = \zeta(\sigma)$ can be locally spacelike, timelike or null if all of its velocity vectors $\zeta' = \zeta(\sigma)$ are spacelike, timelike or null, respectively (R. Lopez., 2014).(B. O'Neill., 1983).

Let {T, N, B} denote that Frenet frame, and suggest that {T, N, B} moving along the spacelike special curve ζ with arc-length parameter σ. The Frenet trihedron consists of the following: (1. the tangent vector {T}, 2. the principal normal vector {N}, 3. the binormal vector {B}). Then this frame (Frenet frame) has the following properties: (B. O'Neill., 1983).

$$\begin{pmatrix} T'(\sigma) \\ N'(\sigma) \\ B'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\sigma) & 0 \\ -\varepsilon\kappa(\sigma) & 0 & \tau(\sigma) \\ 0 & \tau(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix} \tag{1}$$

Where $\varepsilon = \pm 1$, $\mathcal{D}(T(\sigma), T(\sigma)) = 1$, $\mathcal{D}(N(\sigma), N(\sigma)) = \varepsilon$, $\mathcal{D}\,(B(\sigma), B(\sigma)) = -\varepsilon$ & $\mathcal{D}(T(\sigma), N(\sigma)) = \mathcal{D}(T(\sigma), B(\sigma)) = \mathcal{D}(N(\sigma), B(\sigma)) = 0$. If $\varepsilon = 1$, then $\zeta(\sigma)$ is a spacelike curve, and the $\zeta(\sigma)$ consists of the following: (spacelike principal normal $\{N\}$ and timelike binormal $\{B\}$. Also, if $\varepsilon = -1$, then $\zeta(\sigma)$ is a spacelike curve with timelike principal normal $\{N\}$ and spacelike binormal $\{B\}$.

Let $\zeta = \zeta(\sigma)$ be a regular curve in $\mathbb{R}_1^3$. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve (M. P. Do Carmo, 1976).

The Lorentzian sphere of radius $r > 0$ and with a center in the origin in the space $\mathbb{R}_1^3$ is defined by,

$$S_1^2 = \{\, p \in \mathbb{R}_1^3 : \mathcal{D}(p,p) = r^2\}.$$

The parallel transport (or Bishop) frame we can say is an alternative approach to defining a moving frame that is well defined even when the curve has vanished the second derivative(L. R. Bishop, 1975) (B. Bukcu, & M. K. Karacan, 1975).

Suppose that we consider the parallel transport (or Bishop) frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$ of the special spacelike curve $\zeta(\sigma)$ such that $T(\sigma)$ the spacelike unit tangent vector, $B_1(\sigma)$ is spacelike unit normal vector, and $B_1(\sigma)$ the timelike unit binormal vector. The Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$ is expressed as (B. Bukcu, & M. K. Karacan, 1975) (B. Bukcu, & M. K. Karacan, 2010).

$$\begin{pmatrix} T'(\sigma) \\ {B_1}'(\sigma) \\ {B_2}'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(\sigma) & -\kappa_2(\sigma) \\ -\varepsilon\kappa_1(\sigma) & 0 & 0 \\ -\varepsilon\kappa_2(\sigma) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ B_1(\sigma) \\ B_2(\sigma) \end{pmatrix} \tag{2}$$

Where $\mathcal{D}(T(\sigma), T(\sigma)) = 1$, $\big(B_1(\sigma), B_1(\sigma)\big) = \varepsilon$, $\mathcal{D}\,\big(B_2(\sigma), B_2(\sigma)\big) = -\varepsilon$ & $\mathcal{D}\big(T(\sigma), B_1(\sigma)\big) = \mathcal{D}\big(T(\sigma), B_2(\sigma)\big) = \mathcal{D}\big(B_1(\sigma), B_2(\sigma)\big) = 0$. Here, we shall call $\kappa_1(\sigma)$ and $\kappa_1(\sigma)$ as Bishop curvatures. The relation matrix may be expressed as,

$$\begin{pmatrix} T(\sigma) \\ B_1(\sigma) \\ B_2(\sigma) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta(\sigma) & \sinh\theta(\sigma) \\ 0 & \sinh\theta(\sigma) & \cosh\theta(\sigma) \end{pmatrix} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix} \tag{3}$$

Where

$$\begin{cases} \theta(\sigma) = \operatorname{arctanh}\left(\frac{\kappa_2}{\kappa_1}\right);\ \kappa_1 \neq 0, \\ \tau(\sigma) = -\varepsilon\,\frac{d\theta(\sigma)}{d\,\sigma}, \\ \kappa(\sigma) = \sqrt{|\kappa_1^2(\sigma) - \kappa_2^2\,(\sigma)|}\,. \end{cases} \tag{4}$$

And

$$\begin{cases} \kappa_1(\sigma) = \kappa(\sigma)\cosh\theta(\sigma), \\ \kappa_1(\sigma) = \kappa(\sigma)\sinh\theta(\sigma)\,. \end{cases}$$

Let $\zeta = \zeta(\sigma)$ be a regular non-null curve parametrized by arc-length in three-dimension Minkowski space $\mathbb{R}_1^3$ with its Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then $\{TB_1, TB_2, B_1B_2\ \&\ TB_1B_2\} - curves$ of ζ are defined, respectively as follows:

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}}\,(\mathrm{T}(\sigma) + B_1(\sigma)),$$

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}}\,(\mathrm{T}(\sigma) + B_2(\sigma)),$$

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}}\,(B_1(\sigma) + B_2(\sigma)),$$

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{3}}\,(\mathrm{T}(\sigma) + B_1(\sigma) + B_2(\sigma)).$$

3. Main Results

In this section, we introduce the special spacelike curves reference to the parallel transport frame in three-dimension Minkowski space $\mathbb{R}_1^3$. by the same token, we obtain the natural curvature functions of these curves and studying some properties on it when the base curve $\zeta = \zeta(\sigma)$ especially is contained in a plane.

3.1 Spacelike TB1-Special Curves

Definition 3.1. Let $\zeta = \zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}_1^3$ reference to moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then the special spacelike $T\,B_1 - curves$ (Smarandache curve) are defined by,

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}}\left(\mathrm{T}(\sigma) + B_1(\sigma)\right). \tag{5}$$

Theorem 3.1. Let $\zeta = \zeta(\sigma)$ be a spacelike curve in $\mathbb{R}_1^3$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. If the base curve ζ is contained in a plane, then the spacelike $T\,B_1 - curve$ (Smarandache curve) is a circular helix with $\varepsilon\kappa_2^2(\sigma) \neq (1+\varepsilon)\kappa_1^2(\sigma)$ and its natural curvature functions satisfying the following equations;

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\sqrt{(\kappa_1^2-\kappa_2^2)^2\left[(\kappa_2^2-(1+\varepsilon)\kappa_1^2)^2-\varepsilon\kappa_1^2\kappa_2^2+1\right]+2\kappa_1^4(\kappa_1^2-\kappa_2^2)(1-\kappa_2^2)+\varepsilon\kappa_1^6(\kappa_1^2+\kappa_2^2)}}{[(1+\varepsilon)\kappa_1^2-\varepsilon\,\kappa_2^2]^2},$$

$$\tau_\varphi(\sigma^*) = \frac{\sqrt{2}\left[\kappa_1^4(\kappa_1^2+\kappa_2(\varepsilon\,\kappa_1-1)+\varepsilon\,\kappa_1\kappa_2\,(2-\kappa_1)(\kappa_2^2-\kappa_1^2)\right]}{\kappa_1^2[\kappa_2^2-(\varepsilon+1)\kappa_1^2]^2-\varepsilon\kappa_1^2[(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2]^2+\varepsilon\,\kappa_2^2\,(2\,\kappa_1^2-\kappa_2^2)^2}. \tag{6}$$

Proof. Let $\varphi = \varphi(\sigma^*)$ be a spacelike $T\,B_1 - Smarandache\ curve$ with base curve $\zeta = \zeta(\sigma)$. From Eq. (5) and using Eq. (2), we get

$$\varphi'(\sigma^*) = \frac{d\,\varphi}{d\,\sigma^*}\frac{d\,\sigma^*}{d\,\sigma} = \frac{1}{\sqrt{2}}\left(-\,\varepsilon\,\kappa_1 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2\,B_2(\sigma)\right). \tag{7}$$

Hence

$$T_\varphi(\sigma^*) = \frac{1}{\sqrt{(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2}}\left(-\,\varepsilon\,\kappa_1 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2\,B_2(\sigma)\right). \tag{8}$$

Where

$$\frac{d\,\sigma^*}{d\,\sigma} = \frac{\sqrt{(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2}}{\sqrt{2}}. \tag{9}$$

Differentiating Eq. (8) with respect to σ, we have

$$T_\varphi{}'(\sigma^*) = \frac{\sqrt{2}}{[(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2]^2}\left(\xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3\,B_2(\sigma)\right).$$

Where

$$\begin{cases} \xi_1 = (\kappa_1^2-\kappa_2^2)[\kappa_2^2-(\varepsilon+1)\kappa_1^2-\kappa_1'] + \kappa_1\kappa_1' - \kappa_2\kappa_2', \\ \xi_2 = (\kappa_1^2-\kappa_2^2)(\varepsilon\kappa_1'-1) - \varepsilon\kappa_1(\kappa_1^3+\kappa_1\kappa_1'-\kappa_2\kappa_2'), \\ \xi_3 = \kappa_1\kappa_2(\kappa_1^2-\kappa_2^2) + \varepsilon\kappa_1^2(\kappa_1\kappa_2-\kappa_2') + (1+\varepsilon)\,\kappa_1\kappa_1'\kappa_2. \end{cases}$$

Then, the curvature and the principal normal vector field of φ are respectively,

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\sqrt{\xi_1^2+\varepsilon(\xi_2^2-\xi_3^2)}}{[(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2]^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3\,B_2(\sigma)}{\sqrt{\xi_1^2+\varepsilon(\xi_2^2-\xi_3^2)}}.$$

Also, the binormal vector of φ is

$$B_\varphi(\sigma^*) = \frac{\{-[\xi_3\,\kappa_1+\xi_2\,\kappa_2]T(\sigma) + [\varepsilon\,\xi_3\,\kappa_1-\xi_1\,\kappa_2]B_1(\sigma) - \kappa_1[\xi_1+\varepsilon\,\xi_2]B_2(\sigma)\}}{\sqrt{(1+\varepsilon)\kappa_1^2-\varepsilon\kappa_2^2}\sqrt{\xi_1^2+\varepsilon(\xi_2^2-\xi_3^2)}}.$$

Now, from Eq. (7) we have,

$$\varphi''(\sigma^*) = \frac{1}{\sqrt{2}}\{\varepsilon[\kappa_1^2 - \kappa_1^2 - \kappa_1']T(\sigma) + [\kappa_1' - \varepsilon\,\kappa_1^2]\,B_1(\sigma) + [\kappa_1\,\kappa_2 - \kappa_2']B_2(\sigma)\},$$

and

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{2}}\{\alpha_1 T(\sigma) + \alpha_2\,B_1(\sigma) + \alpha_3 B_2(\sigma)\},$$

where

$$\begin{cases}\alpha_1 = \kappa_1(\kappa_1^2 - \kappa_2\kappa_2') + \varepsilon\,(\kappa_2^2 + 2\,\kappa_2\kappa_2' - 3\,\kappa_1\kappa_1' - \kappa_1''),\\ \alpha_2 = \kappa_1'' + \varepsilon\,\kappa_1\,(\kappa_2^2 - \kappa_1^2 - 3\,\kappa_1'),\\ \alpha_3 = -\varepsilon\,\kappa_1\,(\kappa_2^2 - \kappa_1^2 - 3\,\kappa_1') - \kappa_2''.\end{cases}$$

Then the torsion of φ is given by formulae

$$\tau_\varphi = \sqrt{2}\left\{\frac{\kappa_1^2\,(\alpha_3\,\kappa_1 + \alpha_2\,\kappa_2) - \varepsilon\,\kappa_1\,(\alpha_3\,\kappa_1' + \alpha_2\,\kappa_2') + \alpha_1\,\kappa_1\,(\kappa_1' - \kappa_1 - \varepsilon\,\kappa_1^2) + \varepsilon\,[\alpha_3\,\kappa_1^2 + \alpha_2\,\kappa_2\,(\kappa_2^2 - \kappa_1^2 - \kappa_1')\,]}{[\varepsilon\,\kappa_1\,(\kappa_2^2 - \kappa_1^2) - \kappa_1^3]^2 + \varepsilon\,[\kappa_2\,(2\,\kappa_1^2 - \kappa_2^2 + \kappa_1') - \kappa_1\,\kappa_2']^2 - \varepsilon\,[\kappa_1^3 - \varepsilon\,\kappa_1\,(\kappa_2^2 - \kappa_1^2 - \kappa_1' + \kappa_1\,\kappa_1']^2}\right\}.$$

So, if $\zeta(\sigma)$ is contained in a plane, then κ_φ and τ_φ are constants which implies that the spacelike $T\,B_1 - Smarandache\ curve$ is a circular helix and Eq. (6) holds which complete the proof.

3.2 Spacelike $T\,B_2 - special\ curves$

Definition 3.2. Let $\zeta = \zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}_1^3$ reference to moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then the special spacelike $T\,B_2$ −Smarandache curves are defined by,

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}}\,(\mathrm{T}(\sigma) + B_2(\sigma)). \tag{10}$$

Theorem 3.2. Let $\zeta = \zeta(\sigma)$ be a spacelike curve in $\mathbb{R}_1^3$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. If the base curve ζ is contained in a plane, then the spacelike $T\,B_2 - Smarandache\ curve$ is a circular helix with $\varepsilon\kappa_1^2(\sigma) + (1-\varepsilon)\kappa_2^2(\sigma) \neq 0$ and its curvature functions are satisfying the following equations;

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2(\kappa_1^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2+1)}}{\varepsilon\kappa_1^2(\sigma)+(1-\varepsilon)\kappa_2^2(\sigma)},$$

$$\tau_\varphi(\sigma^*) = \frac{\sqrt{2}\,\kappa_1\kappa_2^2\,(\kappa_2+\varepsilon\,\kappa_1)\,(\kappa_2^2-\kappa_1^2)}{[\varepsilon\kappa_1^2(\sigma)+(1-\varepsilon)\kappa_2^2(\sigma)]^2}. \tag{11}$$

<u>Proof</u>. Let $\varphi = \varphi(\sigma^*)$ be a spacelike $T\,B_2 - Smarandache\ curve$ curves according to the base curve $\zeta = \zeta(\sigma)$. From Eq. (10), we get

$$\varphi'(\sigma^*) = \frac{d\,\varphi}{d\,\sigma^*}\frac{d\,\sigma^*}{d\,\sigma} = \frac{1}{\sqrt{2}}\,(-\varepsilon\,\kappa_2 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2\,B_2(\sigma)). \tag{12}$$

Hence

$$T_\varphi(\sigma^*) = \frac{1}{\sqrt{\varepsilon\kappa_1^2(\sigma)+(1-\varepsilon)\kappa_2^2(\sigma)}}\,(-\varepsilon\,\kappa_2 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2\,B_2(\sigma)). \tag{13}$$

Where

$$\frac{d\,\sigma^*}{d\,\sigma} = \frac{\sqrt{\varepsilon\kappa_1^2(\sigma)+(1-\varepsilon)\kappa_2^2(\sigma)}}{\sqrt{2}}. \tag{14}$$

Then

$$T_\varphi{}'(\sigma^*) = \frac{\sqrt{2}}{[\varepsilon\kappa_1^2(\sigma) + (1-\varepsilon)\kappa_2^2(\sigma)]^2}\,(\lambda_1 T(\sigma) + \lambda_2 B_1(\sigma) + \lambda_3\,B_2(\sigma)).$$

Where

$$\begin{cases}\lambda_1 = (\kappa_2^2 - \kappa_1^2 - \kappa_2')[\kappa_1^2 + \varepsilon(1-\varepsilon)\kappa_2^2] + \kappa_2[\kappa_1\kappa_1' + \varepsilon(1-\varepsilon)\kappa_2\kappa_2'],\\ \lambda_2 = (\kappa_1' - \varepsilon\,\kappa_1\,\kappa_2)(\varepsilon\,\kappa_1^2 + (1-\varepsilon)\,\kappa_2^2) - \kappa_1[\varepsilon\,\kappa_1\kappa_1' + (1-\varepsilon)\kappa_2\kappa_2'],\\ \lambda_3 = (\varepsilon\,\kappa_1^2 - \kappa_2^2)(\varepsilon\,\kappa_1^2 + (1-\varepsilon)\,\kappa_2^2) + \kappa_2[\varepsilon\,\kappa_1\kappa_1' + (1-\varepsilon)\kappa_2\kappa_2'].\end{cases}$$

So,

$$\kappa_{\varphi}(\sigma^{*}) = \frac{\sqrt{2}\,\sqrt{\lambda_1^2+\varepsilon(\lambda_2^2-\lambda_3^2)}}{[\varepsilon\kappa_1^2\,(\sigma)+(1-\,\varepsilon)\kappa_2^2\,(\sigma)]^2},$$

and

$$N_{\varphi}(\sigma^{*}) = \frac{\lambda_1 T(\sigma)+\lambda_2 B_1(\sigma)+\lambda_3\,B_2(\sigma)}{\sqrt{\lambda_1^2+\varepsilon(\lambda_2^2-\lambda_3^2)}}.$$

Also,

$$B_{\varphi}(\sigma^{*}) = \frac{-[\lambda_3\,\kappa_1+\,\lambda_2\,\kappa_2]T(\sigma)+(\varepsilon-1)\,\lambda_1\,\kappa_2 B_1(\sigma)-[\lambda_1\,\kappa_1+\,\varepsilon\,\lambda_2\,\kappa_2]B_2(\sigma)}{\sqrt{\varepsilon\kappa_1^2\,(\sigma)+(1-\,\varepsilon)\kappa_2^2\,(\sigma)}\,\sqrt{\lambda_1^2+\varepsilon(\lambda_2^2-\lambda_3^2)}}.$$

Now, from Eq. (12) we have

$$\varphi''\,(\sigma^{*}) = \frac{1}{\sqrt{2}}\,\{\varepsilon[\kappa_2^2-\,\kappa_1^2-\,\kappa_2']T(\sigma)+[\kappa_1'-\,\varepsilon\,\kappa_1\,\kappa_2]\,B_1(\sigma)+[\varepsilon\,\kappa_2^2-\,\kappa_2']B_2(\sigma)\},$$

and

$$\varphi'''\,(\sigma^{*}) = \frac{1}{\sqrt{2}}\,\{\beta_1 T(\sigma)+\beta_2\,B_1(\sigma)+\beta_3 B_2(\sigma)\},$$

where

$$\begin{cases}\beta_1=\varepsilon\,[3\kappa_2\kappa_2'+\varepsilon\kappa_2\,(\kappa_1^2-\kappa_2^2)-2\,\kappa_1\kappa_1'-\,\kappa_2''],\\ \beta_2=\,\kappa_1''-\,\varepsilon\,\kappa_1\,(\kappa_1^2+\,\kappa_2^2-\kappa_2'+(\kappa_1\,\kappa_2)'\,),\\ \beta_3=\,\varepsilon\,\kappa_2\,(\kappa_1^2+\,\kappa_2^2-\kappa_2'+(\kappa_1\,\kappa_2)'\,)-\,\kappa_2''\,.\end{cases}$$

Then,

$$\tau_{\varphi} = \sqrt{2}\left\{\frac{\begin{array}{c}(\varepsilon\,\kappa_2^2-\kappa_2')\,(\varepsilon\,\alpha_2\,\kappa_2+\,\alpha_1\,\kappa_1)+\,\varepsilon\,\kappa_2(\alpha_1-\alpha_2)\,\,(\kappa_1'-\varepsilon\,\kappa_1\,\kappa_2)\\ +\varepsilon(\alpha_3\,\kappa_1+\alpha_2\,\kappa_2)\,\,(\kappa_1^2-\kappa_2^2-\kappa_2'\,)\end{array}}{\begin{array}{c}[\,\kappa_1\,(\varepsilon\,\kappa_2^2-\,\kappa_2')+\kappa_2(\kappa_1'-\,\varepsilon\,\kappa_1\kappa_2)]^2+\,\varepsilon\,\kappa_2^2\,[\,\kappa_1^2-(1+\varepsilon)\kappa_2^2]^2\\ -\,\varepsilon\,[\kappa_2(\kappa_1'-\varepsilon\,\kappa_1\,\kappa_2)+\,\kappa_1\,(\kappa_1^2-\,\kappa_2^2-\,\kappa_2')]^2\end{array}}\right\}.$$

Now, if the base curve $\zeta(\sigma)$ is contained in a plane, then the spacelike $T\,B_2-Smarandache\;curve$ is a circular helix and Eqs. (11) holds which complete the proof.

3.3 Spacelike $B_1\;B_2-special\;curves$

Definition 3.3. Let $\zeta=\zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}_1^3$ reference to moving Bishop frame $\{T(\sigma),B_1(\sigma),B_2(\sigma)\}$. Then the special spacelike $B_1\,B_2$ −Smarandache curves are defined by,

$$\varphi(\sigma)=\varphi(\sigma^{*})=\tfrac{1}{\sqrt{2}}\,(B_1(\sigma)+B_2(\sigma)). \tag{15}$$

Theorem 3.3. Let $\zeta=\zeta(\sigma)$ be a spacelike curve in $\mathbb{R}_1^3$ with the moving Bishop frame $\{T(\sigma),B_1(\sigma),B_2(\sigma)\}$. If the base curve ζ is contained in a plane, then the spacelike $B_1\,B_2-Smarandache\;curve$ is also contained in a plane with $\kappa_1(\sigma)+\kappa_2(\sigma)\neq 0$ and its curvature satisfying the following equation;

$$\kappa_{\varphi}(\sigma^{*}) = \tfrac{\sqrt{2\,\varepsilon\,(\,\kappa_1-\kappa_2)}}{\kappa_1+\kappa_2}. \tag{16}$$

***Proof*.** Let $\varphi=\varphi(\sigma^{*})$ be a spacelike $B_1\,B_2-Smarandache\;curve$ according to the base curve $\zeta=\zeta(\sigma)$. From Eq. (15), we get

$$\varphi'^{(\sigma^{*})}=\tfrac{d\,\varphi}{d\,\sigma^{*}}\tfrac{d\,\sigma^{*}}{d\,\sigma}=-\tfrac{\varepsilon}{\sqrt{2}}\,(\kappa_1+\kappa_2)T(\sigma). \tag{17}$$

Hence

$$T_{\varphi}(\sigma^{*}) = -\,\varepsilon\,T(\sigma). \tag{18}$$

Where

$$\tfrac{d\,\sigma^{*}}{d\,\sigma}=\tfrac{\kappa_1+\kappa_2}{\sqrt{2}}. \tag{19}$$

Then

$$T_{\varphi}{}'(\sigma^{*}) = \frac{\sqrt{2}}{\kappa_1+\kappa_2}\,(-\,\varepsilon\,\kappa_1\,B_1(\sigma)+\,\varepsilon\,\kappa_2\,B_2(\sigma)).$$

So,

$$\kappa_{\varphi}(\sigma^*) = \frac{\sqrt{2\,\varepsilon\,(\kappa_1-\kappa_2)}}{\kappa_1+\kappa_2}.$$

and

$$N_{\varphi}(\sigma^*) = \frac{-\,\varepsilon\,\kappa_1\,B_1(\sigma) + \,\varepsilon\,\kappa_2\,B_2(\sigma)}{\sqrt{\varepsilon(\kappa_1^2-\kappa_2^2)}}.$$

Also,

$$\mathrm{B}_{\varphi}(\sigma^*) = \frac{1}{\sqrt{\varepsilon(\kappa_1^2-\kappa_2^2)}}\,(\kappa_2\,\mathrm{B}_1(\sigma) + \,\kappa_1\,\mathrm{B}_2(\sigma)).$$

From Eq. (17) we have,

$$\varphi''(\sigma^*) = -\frac{\varepsilon}{\sqrt{2}}\,\{(\kappa_1'+\kappa_2')T(\sigma) + \kappa_1(\kappa_1+\kappa_2)B_1(\sigma) - \kappa_2(\kappa_1+\kappa_2)B_2(\sigma)\}.$$

And

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{2}}\,\{\mu_1 T(\sigma) + \mu_2\,B_1(\sigma) + \mu_3 B_2(\sigma)\}.$$

Where

$$\begin{cases}\mu_1 = (\kappa_1+\kappa_2)(\kappa_1^2-\,\kappa_2^2) - \varepsilon(\kappa_1''+\kappa_2''),\\ \mu_2 = -\,\varepsilon\,[\kappa_1'\,(\kappa_1+\kappa_2) + 2\,\kappa_1(\kappa_1'+\kappa_2')],\\ \mu_3 = \,\varepsilon\,[\kappa_2'\,(\kappa_1+\kappa_2) + 2\,\kappa_2(\kappa_1'+\kappa_2')].\end{cases}$$

Then,

$$\tau_{\varphi} = \left\{\frac{\sqrt{2}\,\,\varepsilon\,(\mu_2\kappa_2 + \mu_3\,\kappa_1)}{(\kappa_1-\kappa_2)(\kappa_1+\kappa_2)^3}\right\}.$$

So, if the base curve $\zeta(\sigma)$ is contained in a plane, then the spacelike $T\,B_2 - Smarandache\;curve$ is also contained in a plane and the Eq. (16) holds, this completes the proof.

3.4 Spacelike $T\;\;B_1\,B_2 - special\;curves$

Definition 3.4. Let $\zeta = \zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}_1^3$ reference to moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then the special spacelike $T\,B_1\,B_2$ −Smarandache curves are defined by,

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{3}}\,(\mathrm{T}(\sigma) + B_1(\sigma) + B_2(\sigma)). \quad (20)$$

Theorem 3.4. Let $\zeta = \zeta(\sigma)$ be a spacelike curve in $\mathbb{R}_1^3$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. If the base curve ζ is contained in a plane, then the spacelike $T\,B_1\,B_2 - Smarandache\;curve$ is also contained in a plane with $\kappa_1(\sigma), \kappa_2(\sigma) \neq 0$ and its natural curvature satisfying the following equation;

$$\kappa_{\varphi}(\sigma^*) = \frac{\sqrt{3}(\kappa_1+\kappa_2)\,\sqrt{(1+\varepsilon)\kappa_1^2+(1-\varepsilon)\kappa_2^2-2\,\kappa_1\,\kappa_2}}{(1+\varepsilon)\kappa_1^2+(1-\varepsilon)\kappa_2^2+2\,\kappa_1\,\kappa_2}. \quad (21)$$

<u>Proof</u>. Let $\varphi = \varphi(\sigma^*)$ be a spacelike $T\,B_1\,B_2 - Smarandache\;curve$ curves according to the base curve $\zeta = \zeta(\sigma)$. From Eq. (20), we get

$$\varphi'(\sigma^*) = \frac{d\,\varphi}{d\,\sigma^*}\frac{d\,\sigma^*}{d\,\sigma} = \frac{1}{\sqrt{3}}\,(-\,\varepsilon\,(\kappa_1+\kappa_2)\,T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2\,B_2(\sigma)). \quad (22)$$

Hence

$$T_{\varphi}(\sigma^*) = \frac{-\,\varepsilon\,(\kappa_1+\kappa_2)\,T(\sigma)+\kappa_1 B_1(\sigma)-\kappa_2\,B_2(\sigma)}{\sqrt{(1+\varepsilon)\kappa_1^2+(1-\varepsilon)\kappa_2^2+2\,\kappa_1\,\kappa_2}}. \quad (23)$$

Where

$$\frac{d\,\sigma^*}{d\,\sigma} = \frac{\sqrt{(1+\varepsilon)\kappa_1^2+(1-\varepsilon)\kappa_2^2+2\,\kappa_1\,\kappa_2}}{\sqrt{3}}. \quad (24)$$

Then, from Eq. (23) we get

$$T_{\varphi}{}'(\sigma^*) = \frac{\sqrt{3}\,(\gamma_1 T(\sigma) + \gamma_2 B_1(\sigma) + \gamma_3\,B_2(\sigma))}{[(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2]^2}.$$

Where

$$\begin{cases} \gamma_1 = \varepsilon\,[\kappa_1^2 - (\kappa_2^2 + \kappa_1' + \kappa_2')][(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2] \\ \quad +2\varepsilon\,(\kappa_1+\kappa_2)\,[(1+\varepsilon)\kappa_1\kappa_1' + (1-\varepsilon)\kappa_2\kappa_2' + (\kappa_1\kappa_2)']\,, \\ \gamma_2 = [(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2][\kappa_1' - \varepsilon\,\kappa_1(\kappa_1+\kappa_2)] \\ \quad -2\,\kappa_1\,[(1+\varepsilon)\kappa_1\kappa_1' + (1-\varepsilon)\kappa_2\kappa_2' + (\kappa_1\kappa_2)'], \\ \gamma_3 = [(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2][\kappa_2' - \varepsilon\,\kappa_2(\kappa_1+\kappa_2)] \\ \quad +\,2\,\kappa_1\,[(1+\varepsilon)\kappa_1\kappa_1' + (1-\varepsilon)\kappa_2\kappa_2' + (\kappa_1\kappa_2)']. \end{cases}$$

Then

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{3}\,\sqrt{\gamma_1^2 + \varepsilon(\gamma_2^2 - \gamma_3^2)}}{[(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2]^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\gamma_1 T(\sigma) + \gamma_2 B_1(\sigma) + \gamma_3\,B_2(\sigma)}{\sqrt{\gamma_1^2 + \varepsilon(\gamma_2^2 - \gamma_3^2)}}.$$

Also, the binormal vector of φ is

$$B_\varphi(\sigma^*) = \frac{m_1 T(\sigma) + m_2 B_1(\sigma) + m_3\,B_2(\sigma)}{\sqrt{(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_2^2 + 2\,\kappa_1\,\kappa_2}\;\sqrt{\gamma_1^2 + \varepsilon(\gamma_2^2 - \gamma_3^2)}}.$$

Where

$$\begin{cases} m_1 = -\gamma_3\,\kappa_1 - \gamma_2\,\kappa_2\,, \\ m_2 = \varepsilon\,\gamma_3\,(\kappa_1+\kappa_1) - \gamma_2\,\kappa_2\,, \\ m_3 = -\gamma_1\,\kappa_1 - \varepsilon\,\gamma_2\,(\kappa_1+\kappa_1). \end{cases}$$

Now, from Eq. (21) we have

$$\varphi''(\sigma^*) = \frac{1}{\sqrt{3}}\,\{\varepsilon[\kappa_2^2 - (\kappa_1^2 + \kappa_1' + \kappa_2')]T(\sigma) + [\kappa_1' - \varepsilon\,\kappa_1\,(\kappa_1+\kappa_2)]\,B_1(\sigma) + [\varepsilon\,\kappa_2\,(\kappa_1+\kappa_2) + \kappa_2']\,B_2(\sigma)\}.$$

And

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{3}}\,\{\delta_1 T(\sigma) + \delta_2\,B_1(\sigma) + \delta_3 B_2(\sigma)\},$$

where

$$\begin{cases} \delta_1 = (\kappa_1+\kappa_2)(\kappa_1^2 - \kappa_2^2)^2 + \varepsilon\,[3(\kappa_1\kappa_1' + \kappa_2\kappa_2') - \kappa_1'' - \kappa_2''], \\ \delta_2 = \kappa_1'' - \varepsilon(\kappa_1+\kappa_2)\,[\kappa_1' + \kappa_1(\kappa_1-\kappa_2)]\,, \\ \delta_3 = \varepsilon(\kappa_1+\kappa_2)\,[\kappa_2' + \kappa_2(\kappa_1-\kappa_2)] - \kappa_2''\,. \end{cases}$$

Then,

$$\tau_\varphi = \sqrt{3}\left\{\frac{\begin{array}{c}[\kappa_1' - \varepsilon\,\kappa_1(\kappa_1+\kappa_2)][\delta_1\kappa_2 + \varepsilon\,\delta_3(\kappa_1+\kappa_2)] \\ +[\varepsilon\,\kappa_2(\kappa_1+\kappa_2) - \kappa_2{}'\,]\,[\delta_1\kappa_1 - \varepsilon\,\delta_2(\kappa_1+\kappa_2)] \\ -\,\varepsilon\,[\kappa_2^2 - (\kappa_1^2 + \kappa_1' + \kappa_2')][\delta_3\kappa_1 + \delta_2\kappa_2]\end{array}}{\begin{array}{c}[\kappa_1'\kappa_2 - \kappa_1\kappa_2']^2 \\ +\,\varepsilon\,[\,(\kappa_1+\kappa_2)[(1-\varepsilon)\kappa_2^2 + (1+\varepsilon)\kappa_1\kappa_2 - \kappa_2'] + (\kappa_1\kappa_2)']^2 \\ -\,\varepsilon\,[\,(\kappa_1+\kappa_2)[(1+\varepsilon)\kappa_1^2 + (1-\varepsilon)\kappa_1\kappa_2 + \kappa_2'] + \kappa_1(\kappa_1' + \kappa_2')]^2\end{array}}\right\}.$$

Now, if the base curve $\zeta(\sigma)$ is contained in a plane, then the spacelike $TB_1B_2 - Smarandache\ curve$ is also contained in a plane and the Eq. (21) holds. This completes the proof.

4. Examples

In this section, we construct two examples of the spacelike Smarandache curves in $\mathbb{R}_1^3$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$ of the base curve $\zeta(\sigma)$. The first example corresponds with the case $\varepsilon = 1$. In the second example, we assume $\varepsilon = -1$.

Example 4.1. Case $\varepsilon = 1$. Let $\zeta(\sigma) = (3\sinh\left(\frac{\sigma}{4}\right),\ 3\cosh\left(\frac{\sigma}{4}\right),\ \frac{5\,\sigma}{4})$ be a spacelike curve parametrized by arc-length with timelike binormal vector (see Figure 1). Then

$$T(\sigma) = \left(\frac{3}{4}\cosh\left(\frac{\sigma}{4}\right), \frac{3}{4}\sinh\left(\frac{\sigma}{4}\right), \frac{5}{4}\right).$$

This vector is spacelike and future-directed, we have $\kappa = \frac{3}{16} \neq 0$. Hence

$$N(\sigma) = (\sinh\left(\frac{\sigma}{4}\right), \cosh\left(\frac{\sigma}{4}\right), 0),$$

$$T(\sigma) = \left(\frac{5}{4}\cosh\left(\frac{\sigma}{4}\right), \frac{5}{4}\sinh\left(\frac{\sigma}{4}\right), \frac{3}{4}\right).$$

The torsion is $\tau = \frac{5}{16} \neq 0$ and $(\sigma) = -\int_0^\sigma \frac{5}{16}\, ds = \frac{-5\,\sigma}{16}$. From Eq. (4), we get $\kappa_1 = \frac{3}{16}\cosh\left(\frac{5\,\sigma}{16}\right)$, $\kappa_2 = \frac{3}{16}\sinh\left(\frac{5\,\sigma}{16}\right)$. Also from Eq. (2), we get $B_1(\sigma) = -\int \kappa_1(\sigma)\, T(\sigma)\, d\sigma$, $B_2(\sigma) = -\int \kappa_2(\sigma)\, T(\sigma)\, d\sigma$, the we have

$$B_1(\sigma) = \left(-\frac{1}{8}\sinh\left(\frac{9\,\sigma}{16}\right) - \frac{9}{8}\sinh\left(\frac{\sigma}{16}\right), -\frac{1}{8}\cosh\left(\frac{9\,\sigma}{16}\right) + \frac{9}{8}\cosh\left(\frac{\sigma}{16}\right), -\frac{3}{4}\sinh\left(\frac{5\,\sigma}{16}\right)\right),$$

$$B_2(\sigma) = \left(\frac{1}{8}\cosh\left(\frac{9\,\sigma}{16}\right) + \frac{9}{8}\cosh\left(\frac{\sigma}{16}\right), \frac{1}{8}\sinh\left(\frac{9\,\sigma}{16}\right) - \frac{9}{8}\sinh\left(\frac{\sigma}{16}\right), \frac{3}{4}\cosh\left(\frac{5\,\sigma}{16}\right)\right).$$

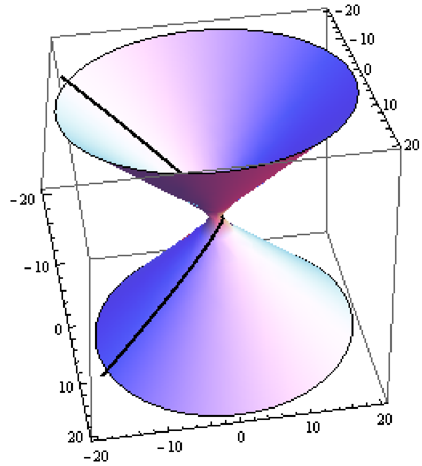

Figure 1. The spacelike $T\,B_1 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

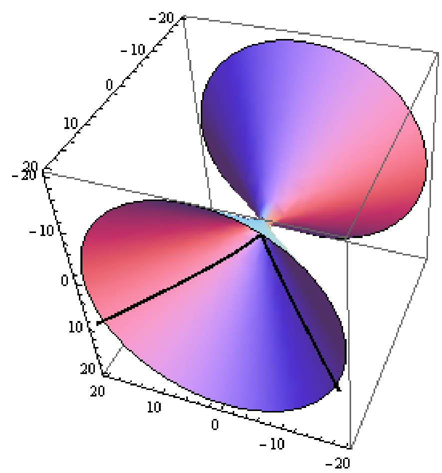

Figure 2. The spacelike $T\,B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

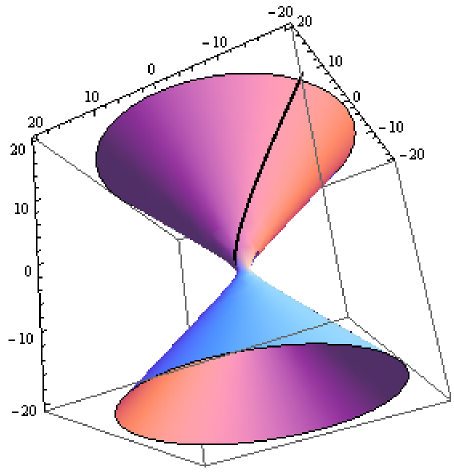

Figure 3. The spacelike $B_1\, B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

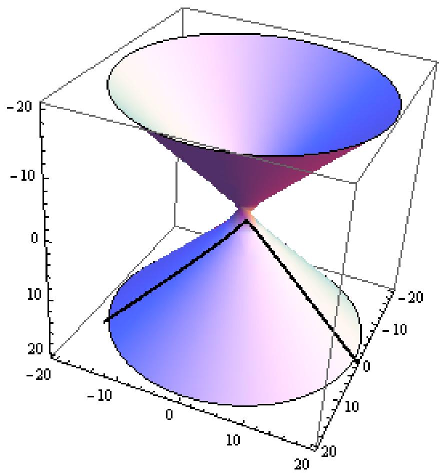

Figure 4. The spacelike $T\, B_1\, B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

Example 4.2. Case $\varepsilon = -1$. Let now Let $\omega(\sigma) = \frac{1}{\sqrt{2}}(\cosh(\sigma), \sinh(\sigma)\ , \sigma)$ be a spacelike curve parametrized by arc-length with timelike principal normal vector (see Figure 5). Then it is easy to show that $T(\sigma) = \frac{1}{\sqrt{2}}(\sinh(\sigma), \cosh(\sigma), 1)$, $\kappa = \frac{1}{\sqrt{2}} \neq 0$, $\tau = \frac{1}{\sqrt{2}} \neq 0\ and\ \theta(\sigma) = \int_0^\sigma \frac{1}{\sqrt{2}}\, dt = \frac{\sigma}{\sqrt{2}}$. From Eq. (4), we get $\kappa_1 = \frac{1}{\sqrt{2}}\cosh\left(\frac{\sigma}{\sqrt{2}}\right)$, $\kappa_2 = \frac{1}{\sqrt{2}}\sinh\left(\frac{\sigma}{\sqrt{2}}\right)$.

From Eq. (2), we get $B_1(\sigma) = \int \kappa_1(\sigma)\, T(\sigma)\, d\sigma$, $B_2(\sigma) = \int \kappa_2(\sigma)\, T(\sigma)\, d\sigma$. Then we have,

$$B_1(\sigma) = \left(\frac{\sqrt{2}}{4(\sqrt{2}+1)}\cosh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4(\sqrt{2}+1)}\cosh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right), \frac{\sqrt{2}}{4(\sqrt{2}+1)}\sinh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right) + \frac{\sqrt{2}}{4(\sqrt{2}+1)}\sinh\left(\frac{(\sqrt{2}+1)\,\sigma}{\sqrt{2}}\right), \frac{\sqrt{2}}{2}\sinh\left(\frac{\sigma}{\sqrt{2}}\right)\right),$$

$$B_2(\sigma) = \left(-\frac{\sqrt{2}}{4(\sqrt{2}+1)}\sinh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right) + \frac{\sqrt{2}}{4(\sqrt{2}+1)}\sinh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right), -\frac{\sqrt{2}}{4(\sqrt{2}+1)}\cosh\left(\frac{(\sqrt{2}+1)\sigma}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4(\sqrt{2}+1)}\cosh\left(\frac{(\sqrt{2}+1)\,\sigma}{\sqrt{2}}\right), -\frac{\sqrt{2}}{2}\cosh\left(\frac{\sigma}{\sqrt{2}}\right)\right),$$

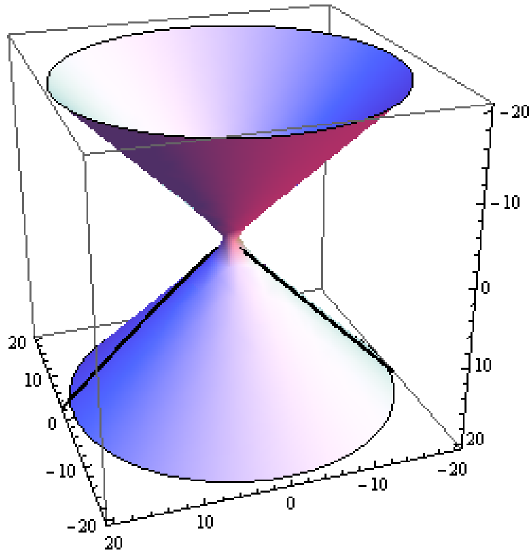

Figure 5. The spacelike curve $\omega = \omega(\sigma)$ on S_1^2.

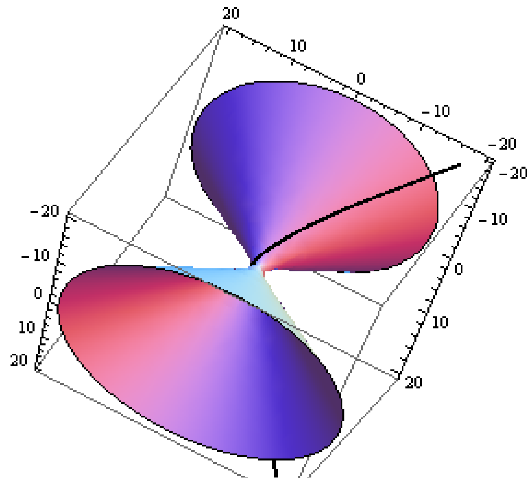

Figure 6. The spacelike $T\,B_1 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

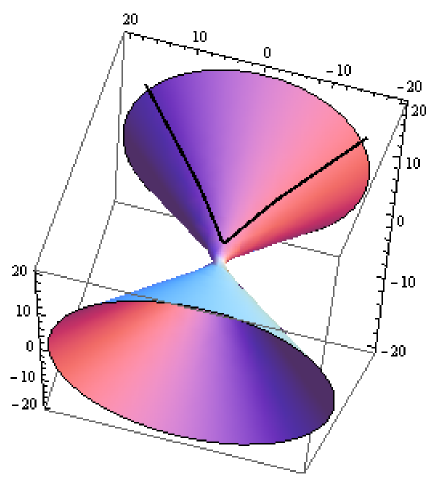

Figure 7. The spacelike $T\,B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

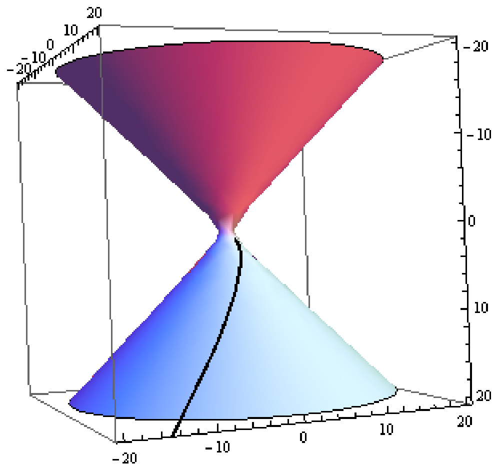

Figure 8. The spacelike $B_1\, B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

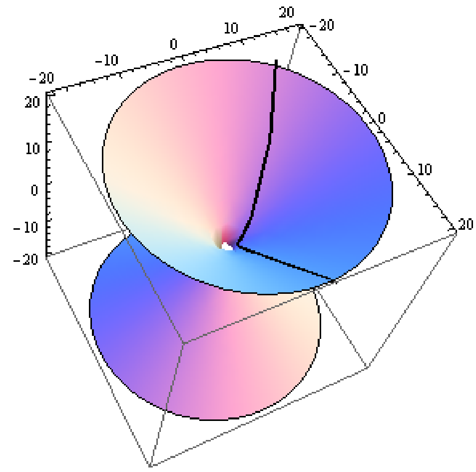

Figure 9. The spacelike $T\, B_1\, B_2 - Smarandache\ curve$ with base curve $\zeta(\sigma)$ on S_1^2.

References

B. Bukcu, & M. K. Karacan, (1975). Parallel transport frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 57*(1) x, 13-22.

B. Bukcu, & M. K. Karacan. (2010). parallel transport frame of the spacelike curve with a spacelike binormal in Minkowski 3-space. *Selcuk J. Appl. Math., 11*(1), 15-25.

B. O'Neill. (1983). Semi-Riemannian geometry with applications to relativity. Academic press, New York.

C. Ashbacher. (1997). Smarandache geometries. *Smarandache Notions Journal, 8*(1-3) , 212-215.

E. M. Solouma. (2017a). Special Smarandache curves recording by curves on a spacelike surface in Minkowski space-time. *PONTE Journal, 73*(2), 251-263.https://doi.org/10.21506/j.ponte.2017.2.20

E. M. Solouma. (2017b). Special equiform Smarandache curves in Minkowski space-time. *Journal of Egyptian Math. Society, 25*, 319-325. https://doi.org/10.1016/j.joems.2017.04.003

H. Iseri. (2002). Smarandache Manifolds, American Res. Press, Mansfield University, PA, 2002.

K. Taskopru, & M. Tosun. (2014). Smarandache curves according to Sabban frame on. *Boletim da Sociedade Paraneanse de Matematica, 32*(1), 51-59. https://doi.org/10.5269/bspm.v32i1.19242

L. Mao. (2006). Smarandache Geometries & Map Theory with Applications, Chinese Branch Xiquan House, Academy of Mathematics and Systems Chinese Academy of Sciences, Beijing, China.

L. R. Bishop. (1975). There is more than one way to frame a curve. Amer. *Math. Monthly, 82*(3), 246-251. https://doi.org/10.1080/00029890.1975.11993807

M. A. Soliman, W. M. Mahmoud, E. M. Solouma, & M. Bary. (2019). The new study of some characterization of canal surfaces with Weingarten and linear Weingarten types according to Bishop frame. *Journal of the Egyptian Mathematical Society.* 07 August. 27:26 https://doi.org/10.1186/s42787-019-0032-y

M. Cetin, Y. Tuncer, & M. K. Karacan. (2014). Smarandache curves according to Bishop frame in Euclidean 3-space. *General Mathematics Notes, 20*(2), 50-66.

M. M. Wageeda, E. M. Solouma, & M. Bary. (2019). Darboux Iso-Geodesic Special Curve in Euclidean Space. *Modern Applied Science,13*(9), 98-104. https://doi.org/10.5539/mas.v13n9p98

M. P. Do Carmo. (1976). Differential geometry of curves and surfaces. Prentice Hall, Englewood Cliffs, NJ.

O. Bektas, & S. Yuce. (2013). Special Smarandache curves according to Darboux frame in Euclidean 3- Space. *Romanian Journal of Mathematics and Computer Science, 3*, 48-59.

R. Lopez. (2014). Differential geometry of curves and surfaces in Lorentz-Minlowski space. *International Electronic Journal of Geometry, 7*(1), 44-107.