A Special Case of a Mathematical Model for Ecological Community: A Predator-Prey System

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Abstract

This paper presents a mathematical model of an ecological community in which different populations of organisms with interactions involving predator-prey encounters are considered. Here, a concept of nucleolar solution is used to represent the optimal best position in the ecosystem. Some notions of game theory is applied, where the players are the populations and the benefits from the interactions are represented by means of the payoff functions.

Keywords: nucleolar solution, predator-prey system, ecological community

1. Introduction

The One of the most relevant problems in applied ecology is related of resource management in which, when one species disappears, earth is affected in a negative way. It is important to highlight that when the human population continues to increase, the conservation of nature has a time limit. Here, it is essential to understand the world of living things in order to manage nature in a sustainable way. Then, it is also important to design strategies (rational) for preserving somehabitats and exploiting others (Cárdenas et al, 2017). However, hunting, fishing and deforestation are necessary resources for men, but if these activities are done in a no rational way, the results obtained can be catastrophic (extinction of species).

Not only men affect the ecosystem, for example natural perturbations are frequent in the real world. So, the densities of natural populations are constantly influenced by a variety factors. Thus, natural systems may reduce species through emigration, destruction by weather, epidemics, predators. On the other hand, we can see that other species can increase (drastically), becoming a plague, affecting the other populations of the ecosystem.

The concept of stability of biological systems is well know in the literature. For example, systems are deemed stable when all species densities return to an equilibrium following a perturbation. But, to determine the stability of multispecies models in not very easy. Then, of the classical analysis, we can solve the differential equations, this means to know the population densities as functions of time. However, these solutions cannot be found except (in general) in very special cases. We know that other methods that produce linear approximations to nonlinear differential equations for population densities near the multi-species equilibrium can be used and the simple linear equations can be solved (explicitly). However, in this case the solutions are valid only for a small region near the equilibrium (Cárdenas et al, 2018).

Now, techniques through Lyapunov functions to determine whether a system is globally stable are complicated for multispecies models, consequently this approach has a limited utility. Then, the system should not be destroyed by perturbations. Natural systems can be controlled by men, which can positively influence populations.

2. Method

2.1 Model Description

Using game theory concepts, we wish to explain here a method to study a concept of equilibrium for populations that may avoid some of the problems mentioned before. A deep knowledge of the world, especially of the

populations of living things is needed to preserve nature. Then, the study of food webs gives us a helpful tool to know communities and their trophic interactions. Here, webs describe how a community works, in other words, a food web is a collection of trophic species, together with their feeding relations. Each arrow in a web goes from food to eater or from prey to predator (Stuart, 2002).

Now, even if animals are not rational, they can be considered as if they were, because their movements, environment and behavior can be described by men, who can control them. We can say that in terms of game theory, predators of a community can be considered as if they were players in a game, competing over some natural resource. This resource can be a place, food or any other source. It is important to say that also prey populations can be considered as players who try to avoid the predators. In this case, all situations can be considered explicitly. Moreover, if players include prey and predators, the method can be used even if there is only one consumer.

Therefore, species that are consumers, resources (or both of them), are considered as players having two task feed and avoid their predators. Thus, we consider the set of predators as $P_1 = \{1, \ldots, n\}$ and the set of prey $P_2 = \{1, \ldots, k\}$. As we consider that the conflict is over food, then the strategies should be connected with this source, because this is done considering the biomasses consumed by each predator as variables that from the following payoff matrix (represents the strategies of the populations):

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \tag{1}$$

where x_{ij} represents what i eats of j, in other words, for each i, the row i corresponds to predator i. Analogously, for each j, the column j corresponds to prey j.

Now, $\sum_{j=1}^{k} x_{ij} = x_i$, where x_i is the total consumption of population i. In the same way, $\sum_{i=1}^{n} x_{ij} = x_i \leq y_j$, where y_j is the weight of all organisms belonging to population j.

Therefore, $x_{ij} = 0$ iff i does not feed upon j in the food web R. For the other cases $x_{ij} \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, that is, usually j is a resource of i, but it is also considered the possibility that i does not eat j. The strategy set X_R is defined as the set of all strategies given in (1). This set (X_R) is compact.

In this cases, benefit for each species is represented by means of its payoff function, which depends (both) on its own strategy and on the strategies of the other players. The payoff functions can be represented in different ways, i.e., a payoff function for a population i, that depends on the strategies x will be named $S_i(x)$ (Schalk, 2018).

Let $x \in X_R, v(x) = (S_{i1}(x), \cdots, S_{ip}(x))$ is formed in a nondecreasing way. Then, the first components of this vector represent small benefits for the corresponding populations. Now, let $y \in X_R$, the vector $v(y) = (S_{j1}(y), \cdots, S_{jp}(y))$ is given in an analogous way. If we compare the first component of $v(x)$ with the first component $v(y)$, then $A_{i1}(x)$ is greater than $A_{j1}(y)$ we say that x dominates y and we can write it $x > y$ and so on. Here, the notation $x \sim y$ means that all the corresponding components of the two vectors are equal. We say that x is a nucleolar solution if $x \geq y$ for each $y \in X_R$ (Kurtis, 2016). The set of nucleolar solutions is represented as:

$$N(X_R) = \{x \in X_R : x \geq y, \forall y \in X_R\}$$

It is important to see that the payoff functions can move and this does not necessarily mean that the nucleolar solutions change. Now, the food webs help to represent what happens in reality, what one species wins or lost by other species that interact with it.

Therefore, we can say that a game is called a constant-sum game if there exists a number α such that $\sum_{l=1}^{p} A_{il}(x) = \alpha$ for all $x \in X_R$ and where $p = n + k$ is the number of species or players of the game. This case could be caused when the economical aspect is also considered, then α could be an economical parameter. If for example, $\alpha = 0$, the game is called a zero-sum game. This means that when the payoff of some species increase the benefits of others decrease. Thus, in the sum zero case for each decision x, the sum of the payoff values of all populations equal zero. This means that every value is zero or that there are positive and negative values, in such a way that they cancel each other out. On the other hand, if $\alpha > 0$, the payoff values can be all positive, at the same time.

Finally, if from the description of the community we obtain functions so that their sum is a constant number, the calculation of the nucleolar solutions is easier. In this case, the problem is reduced to the calculation of a system of equations, as we will see in theorem 1.

2.2 Variations in Time

Variations in time can also be introduced to our model, but we need the following lemmas (Amundrud, 2015).

Lemma 1. If the game is a constant-sum game, with $\sum_{l=1}^{p} A_{il}(x) = \alpha$ for each $x \in X_R$. Now, let x be an element such that

$$A_{i_1}(x) = A_{i_2}(x) = \cdots = A_{i_p}(x), \tag{2}$$

then $A_{il}(x) = \frac{\alpha}{p}$ for all $l = 1,2,\cdots,p$.

Lemma 2. If the game is a constant-sum game, where $\sum_{l=1}^{p} A_{il}(x) = \alpha$ for all $x \in X_R$. Let x be an element such that $A_{i_1}(x) = A_{i_2}(x) = \cdots = A_{i_p}(x)$, then

$$A_{i_1}(x) = A_{i_2}(x) = \cdots = A_{i_p}(x) \geq min\left(A_{i_1}(y), \cdots, A_{i_p}(y) \right)$$

for all $y \in XR$.

Therefore, with the lemmas given, we have the following theorem:

Theorem 1. If

$$\sum_{l_1 \in I_1} A_{i_{l_1}}(x) = \alpha_1, \cdots, \sum_{l_t \in I_t} A_{i_{l_t}}(x) = \alpha_t$$

with $l_1 \cup \cdots \cup l_t = \{1,\ldots,n+k\}$ **and** $l_i l_j = \emptyset$ **for all** $i \neq j, |l_s| = p_s$, **with** $s \in S = \{1,\ldots,t\}$. **If**

$D_t = \{x \in X_R : A_{i_{l_t}}(x) = \frac{\alpha_t}{p_t}, \forall l_t \in I_t\}$.

If the intersection $D_1 \cap \cdots \cap D_t \neq \emptyset$, then

$$N(X_R) = D_1 \cap \cdots \cap D_t.$$

Proof.

Let x be an element of $N(X_R)$, we will see that $x \in D_1 \cap \cdots \cap D_t$. Then, let $v(x) = \left(A_{i1}(x), \cdots, A_{ip}(x) \right)$ a vector ordered (in a non decreasing way). Let us assume, relabelling the expressions if necessary, that

$$\frac{\alpha_1}{p_1} \leq \cdots \leq \frac{\alpha_t}{p_t}$$

Now, we form with $y \in D_1 \cap \cdots \cap D_t$ a vector $v(y)$ similarly to the other

$$v(y) = \left(\frac{\alpha_1}{p_1} \cdots \frac{\alpha_1}{p_1} \cdots \frac{\alpha_t}{p_t} \cdots \frac{\alpha_t}{p_t} \right)$$

where the components are the elements $\frac{\alpha_r}{p_r}$, written in a nondecreasing way.

Now, since that $x \in N(X_R)$ and $A_{i_1}(x) \geq \frac{\alpha_1}{p_1}$, then by construction of $v(x)$ we have

$$\sum_{l_1 \in I_1} A_{i_{l_1}}(x) > \alpha_i,$$

but by hypothesis this is not possible, then $A_{i1}(x) = \frac{\alpha_1}{p_1}$.

We can see that in a similar way

$$A_{i_1}(x) = \cdots = A_{i_p}(x) = \frac{\alpha_1}{p_1}$$

and

$$A_{i_{p_1+1}}(x) = A_{i_{p_1+2}}(x) = \frac{\alpha_2}{p_2}, \cdots, = A_{i_{n+k}}(x) = \frac{\alpha_r}{p_r}$$

Therefore, $x \in D_1 \cap \cdots \cap D_t$ as we wanted to show.

In order to prove the other inclusion, let x be an element of $x \in D_1 \cap \cdots \cap D_t$ and we will see that $x \in N(X_R)$. Let

$$v(x) = \left(A_{i_1}(x), \cdots, A_{i_{n+k}}(x) \right)$$

and

$$v(y) = \left(A_{j_1}(y), \cdots, A_{j_{n+k}}(y) \right)$$

be vectors ordered in a nondecreasing way as before. Now, to see that $x \in N(X_R)$ it is enough to show that for any $y \in X_R$ if the first $t - 1$ components of the vectors $v(x)$ and $v(y)$ are equal and the $n - t$

component is different, that is $A_{i_t}(x) \neq A_{i_t}$, with $t \geq 1$, then

$$A_{i_t}(x) > A_{i_t}(y) \tag{3}$$

Therefore, being x an element of $D_1 \cap \cdots \cap D_t$, with $\frac{\alpha_1}{p_1} \leq \cdots \leq \frac{\alpha_t}{p_t}$ it holds that $A_{i_1}(x) = \cdots = A_{i_{p_1}}(x) = \frac{\alpha_1}{p_1}$. Consequently, we have that if $0 = t - 1 = p_1$, then by the aforementioned it is fulfilled that $A_{i_1}(x) \neq A_{i_1}(y)$. Now, by Lemma 2 and the construction of $v(y)$ we have that

$$A_{i_1}(x) = \cdots = A_{i_p}(x) \geq min\left(A_{i_1}(y), \cdots, A_{i_p}(y)\right) \geq A_{j_1}(y)$$

By Eq.(3) $A_{i_1}(x) > A_{i_1}(y)$ then $x \in N(X_R)$. Now, if $0 < t - 1 < p_1$, by the assumptions we have just mentioned before we obtain

$$A_{j_1}(y) = \cdots = A_{j_{t-1}}(y) = \frac{\alpha_1}{p_1}$$

By the construction of $v(y)$, then $A_{j_1}(y) = \cdots = A_{j_{t-1}}(y) \leq A_{j_t}(y) \leq \cdots \leq A_{j_{p-1}}(y)$. If there is a first $t', t \leq t' \leq p_1$ such that $\frac{\alpha_1}{p_1} < A_{t'}(y)$ then

$$\sum_{l=1}^{p_1} A_{i_l}(y) > \alpha_1,$$

but this is not possible by hypothesis. Analogously, by construction of $v(y)$, it can not be that $\frac{\alpha_1}{p_1} > A_{t'}(y)$ then $A_{j_1}(y) = \cdots = A_{j_{p_1}}(y) = \frac{\alpha_1}{p_1}$. It is also true that:

$$\{j_1, \cdots, j_{p_1}\} = \{i_1, \cdots, i_{p_1}\}, \tag{4}$$

that can be proved as follows. Assume there is an element i_s is such that is $i_s \notin \{j_1, \cdots, j_{p_1}\}$ then $A_{i_s}(y) = \alpha$ with $\alpha > \frac{\alpha_1}{p_1}$. Then, we could include is $i_s \in \{j_1, \cdots, j_{p_1}\}$. But by hypothesis

$$\sum_{t=1}^{p_1} A_{i_t}(x) = \sum_{t=1}^{p_1} A_{j_t}(x) = \alpha 1$$

and

$$\sum_{t=1}^{p_1} A_{i_t}(y) = (p_1 - 1)\frac{\alpha_1}{p_1} + \alpha > \alpha 1$$

that gives a contradiction.

If we assume that there are two elements $i_{s'}$ and $i_{s''}$ such that $i_{s'}$ and $i_{s''} \notin \{j_1, \cdots, j_{p_1}\}$. Therefore, we have proved that (4) is true, that is useful to study the following components.

Now, in the case that $p_1 \leq t - 1 < p_2$ we study the following p_2 components in a similar way, if $p_2 \leq t - 1 < p_3$ we study the following p_3 components and so on.

Lemma 3. Let $A_{i_1}, \cdots, A_{i_{n+k}}$ be payoff functions such that

$$\sum_{l=1}^{n+k} A_{ij}(x) = \alpha$$

for each $x \in X_R$. Now, if $D = \{x \in X_R : A_{i_1}(x) = \cdots = A_{i_{n+k}}(x)\} \neq 0$, then $N(X_R) = D$.

For the proof, we can see this lemma as a particular case of Theorem 1 with $t = 1$.

3. Results

Here, we are wondering now whether it is natural to find payoff functions that give a constant sum. We have considered simple and complicated food webs and examined the corresponding payoff functions. We only give the following simple example of food webs:
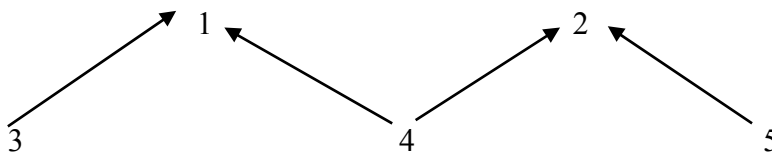

Figure 1. Food webs

Let for example, β_{ij} be the capture time, that is an average of number of attempts of components of i and time to capture an individual of j. Thus, we say that $\beta_{ij} \leq v_{ij}$, if j is a population faster than j'. The sum of the corresponding functions gives zero, that is

$$A_1(x) = \beta_{13}x_{13} + \beta_{14}x_{14} - \beta_{24}x_{24} - \beta_{25}x_{25}$$
$$A_2(x) = \beta_{24}x_{24} + \beta_{25}x_{25} - \beta_{14}x_{14} - \beta_{13}x_{13}$$
$$A_3(x) = \beta_{14}x_{14} - \beta_{13}x_{13}$$
$$A_4(x) = \beta_{13}x_{13} + \beta_{25}x_{25} - \beta_{14}x_{14} - \beta_{24}x_{24}$$
$$A_5(x) = \beta_{24}x_{24} - \beta_{25}x_{25}$$

In most of the food webs, we arrived to similar conditions, except in the cases of omnivore. It seems that in these cases it is not so simple to arrive to payoff functions as we are studying, since in general, they do not cancel each other out in every level. Nucleolar solutions can now be found using the given mathematical results.

4. Conclusion

In this article we present a mathematical model for a community, taking into account predator-prey relationships. We use the concepts of game theory to show that your interactions in the ecosystem can be represented through the use of payoff functions. The concept of nucleolar solution to represent the optimal solution of the ecosystem which can be associated with real situations present in nature. It is important to say that once the solutions are found, human control is helpful or even necessary to allow the individuals of the populations to reach the best position or the most adequate distribution of food. For example, one may move individuals of some populations to another place or introduce more individuals into the field.

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