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Asymptotic and Geodesic Bishop Curves in Three Dimensions Minkowski Space E_1^3

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Abstract

In the following paper, we deal with the surfaces which are expressed as a linear combination of the components of Bishop frame for a given some special spacelike Smarandache curves in three dimensions Minkowski Space E_1^3 . We analyzed the problem of constructing a family of surfaces from these curves in E_1^3 , and derive the sufficient conditions for coefficients to satisfy the iso-asymptotic requirements. Additionally, we derive sufficient conditions for coefficients to satisfy both the geodesic and iso-parametric needs.

Keywords: Smarandache curve, bishop frame, Minkowski Spacetime, Iso-parametric curve, Iso-asymptotic, Iso-geodesic curve

1. Introduction

There are many important properties and consequences of curves in differential geometry (O'Neill 1983, 1966). Investigates take after works about the curves. In the light of the current examinations, creators dependably present new curves, the special Smarandache curves are one of them, it is have been researched by some differential geometers (O'Neill 1966), (Karakus, et. al. 2016). This curve is characterized as, a standard curve in Minkowski space-time, whose position vector is created by Frenet frame vectors on another regular curve, is Smarandache curve (Bükcü, et. al. 2010). (Ali 2010) has presented some special Smarandache curves in the Euclidean space. Common Smarandache curves as per Bishop Frame in Euclidean 3-space have been explored by (Çetin, et. al. 2014). Likewise, Darboux Smarandache bends as indicated in three dimensions Euclidean space has presented in (Bektas, et. al. 2013). They discovered a few properties of these special curves and discovered normal curvature, geodesic curvature and geodesic torsion of these curves. In addition, they explore special Smarandache curves in three dimensions Minkowski Space, (Karakus, et. al. 2016), (López2014), (Yilmaz, et. al. 2016), (Okuyucu, et. al. 2019). Besides, they discover a few properties of these curves and them curvature and torsion of those curves. Special Smarandache curves, for example, Smarandache curves as indicated by Sabban frame in Euclidean unit sphere has presented in (Tasköprü, et. al. 2012). Likewise, they give some portrayal of Smarandache curves and outline cases of their results. On the Quaternionic Smarandache curves in three dimensions Euclidean Space have been researched in (Çetin, et. al. 2013).

A standout amongst the foremost vast curve on a surface is an asymptotic curve. Asymptotic on a surface has been an extended haul investigate purpose in Differential Geometry (O'Neill 1983), (Tai, et. al. 2004). A curve on a surface is called an asymptotic curve gave its speed faithfully focuses an asymptotic way, that is the directing in which the normal curvature is zero.

Asymptotic curves are likewise experienced in space science, astronomy and computer-aided design in engineering. The idea of the family of surfaces having a given characteristic curve was first introduced by (Karakus et. al. 2016), (Li, et. al. 2011) in Euclidean 3-space. (Akyildiz, et.al.2008), summed up the work of Li by presenting new kinds of marching-scale functions, coefficients of the Frenet frame showing up in the parametric portrayal of surfaces. With the motivation of work of (Li et.al. 2011), changed the characteristic curve from geodesic to a line of curvature and characterized the surface pencil with a curvature. As of late, (Bayram, et. al. 2012) characterized the surface pencil with a common asymptotic curve.

In this paper, we investigate a family of surfaces that created by a spacelike curve as a common iso-geodesic and spacelike Smarandache curve in three dimension Minkowski space, and we derive the necessary and sufficient conditions for a given spacelike curve in E_1^3 to satisfy both iso-asymptotic and Smarandache curve with moving Bishop frame. In section 2, we give some basic concepts about Smarandache curves in Minkowski 3-space and define an iso-asymptotic and isogeodesic curves. In Section 3, we express a family of surfaces S as a linear combination of the Bishop frame of the given curve and derive necessary and sufficient conditions on matching scale functions to satisfy both iso-asymptotic and Smarandache requirements, and we illustrate the method by giving an example. In Section 4, we introduce the family of surfaces as a linear combination of the Bishop frame of the given spacelike curve and derive necessary and sufficient conditions on marching-scale functions to satisfy both iso-geodesic and spacelike Smarandache curve, finally, we illustrate the method by giving an example.

2. Basic Concepts

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\mathcal{H} = -(dz_1)^2 + (dz_2)^2 + (dz_3)^2$$

where (z_1, z_2, z_3) is a rectangular Cartesian coordinate system of E_1^3 Since $\mathcal{H}$ is an indefinite metric, recall that a nonzero vector $v \in E_1^3$ can have one of three characters; it can be spacelike if $\mathcal{H}(v,v) > 0$ or $v = 0$, timelike if $\mathcal{H}(v,v) < 0$, and null if $\mathcal{H}(v,v) = 0$ and $v \neq 0$. In particular, the norm of a vector $v \in E_1^3$ is given by $\|v\| = \sqrt{|\mathcal{H}(v,v)|}$. Similarly, an arbitrary curve $\beta = \beta(\varrho)$ in E_1^3 can be locally spacelike, timelike or null if all of its velocity vectors $\beta'(\varrho)$ are spacelike, timelike or null, respectively (O'Neill 1966).

Let $\omega = \omega(\varrho)$ be a regular curve parametrized by arc-length in E_1^3 and $\{T, N, B, \kappa, \tau\}$be its Frenet invariants, where $\{T, N, B\}, \kappa$ and τ are moving Frenet frame, curvature and torsion of $\omega(\varrho)$, respectively. If $\omega(\varrho)$ is spacelike curve in E_1^3, the Frenet formulae are define as L ́opez, (2014):

$$\begin{pmatrix} T'(\varrho) \\ N'(\varrho) \\ B'(\varrho) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\varrho) & 0 \\ -\varepsilon\,\kappa(\varrho) & 0 & \tau(\varrho) \\ 0 & \tau(\varrho) & 0 \end{pmatrix} \begin{pmatrix} T(\varrho) \\ N(\varrho) \\ B(\varrho) \end{pmatrix} \tag{1}$$

Where $\varepsilon = \pm 1$. If $\varepsilon = 1$, then $\omega(\varrho)$ is a spacelike curve with spacelike principal normal N and timelike binormal B. Also, $\mathcal{H}(T,T) = \mathcal{H}(N,N) = 1$,$\mathcal{H}(B,B) = -1$,

And $\mathcal{H}(T,N) = \mathcal{H}(T,B) = \mathcal{H}(N,B)$ =0. If $\varepsilon = -1$, then $\omega(\varrho)$ is a spacelike curve with timelike principal normal N and spacelike binormal B. Besides, $\mathcal{H}(T,T) = \mathcal{H}(B,B) = 1$,$\mathcal{H}(N,N) = -1$ and $\mathcal{H}(T,N) = \mathcal{H}(T,B) = \mathcal{H}(N,B) = 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express Bishop of an orthonormal frame along a curve simply by parallel transporting each component of the frame (Ali 2010), (Bukcu, et. al. 2010). For a unite spacelike curve $\omega(\varrho)$ in the space E_1^3 , the Bishop frame is expressed as

$$\begin{pmatrix} T'(\varrho) \\ M_1'(\varrho) \\ M_2'(\varrho) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(\varrho) & -\kappa_2(\varrho) \\ -\varepsilon\,\kappa_1(\varrho) & 0 & 0 \\ -\varepsilon\,\kappa_2(\varrho) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(\varrho) \\ M_1(\varrho) \\ M_2(\varrho) \end{pmatrix} \tag{2}$$

The relation matrix may be expressed as

$$\begin{pmatrix} \mathrm{T}(\varrho) \\ \mathrm{M}_1(\varrho) \\ \mathrm{M}_2(\varrho) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\vartheta(\varrho) & \sinh\vartheta(\varrho) \\ 0 & \sinh\vartheta(\varrho) & \cosh\vartheta(\varrho) \end{pmatrix} \begin{pmatrix} \mathrm{T}(\varrho) \\ \mathrm{N}(\varrho) \\ \mathrm{B}(\varrho) \end{pmatrix} \tag{3}$$

where

$$\begin{cases} \vartheta(\varrho) = \arg\tanh\left(\frac{\kappa_2}{\kappa_1}\right), \kappa_1 \neq 0 \\ \tau(\varrho) = -\varepsilon\frac{d\vartheta(\varrho)}{d\varrho}, \\ \kappa(\varrho) = \sqrt{\kappa_1^2(\varrho) + \kappa_2^2(\varrho)}. \end{cases} \tag{4}$$

and

$$\begin{cases} \kappa_1(\varrho) = \kappa(\varrho)\cosh\vartheta(\varrho). \\ \kappa_2(\varrho) = \kappa(\varrho)\sinh\vartheta(\varrho). \end{cases} \tag{5}$$

Let $\omega(\varrho)$ and $\xi(\varrho)$ are a regular curve, such that $\omega \subset \Psi$ and $\xi \subset S$. If ϱ is a geodesic then ω is a iso-parametric curve on a surface $\Psi = \Psi(\varrho, v)$, there exist $\omega(\varrho) = \Psi(\varrho, v_0)$ or $\omega(v) = \Psi(\varrho_0, v)$, then we call $\omega(\varrho)$ an iso-geodesic of a surface Ψ. At the same time if ϱ is an asymptotic direction, then ξ is a iso-parametric curve on a surface $S = S(\varrho, v)$ is that has a constant ϱ or v-parameter value, then we call $\xi(\varrho)$ an iso-asymptotic of a surface S if it is both a asymptotic and an iso-parametric curve on S .

Suppose that $\xi = \xi(\varrho)$ be a regular non-null curve parametrized by arc-length in Minkowski 3-space E_1^3 with its Bishop frame $\{T, M_1, M_2\}$. Then $T M_1$, TM_2 , $M_1 M_2$ and TM_1M_2-Smarandache curve of ϱ are defined, respectively as follows Turgut,et.al. (2008):

$$\zeta = \zeta(\varrho) = \frac{1}{\sqrt{2}}\big(\mathrm{T}(\varrho) + \mathrm{M}_1(\varrho)\big),$$

$$\zeta = \zeta(\varrho) = \frac{1}{\sqrt{2}}\big(\mathrm{T}(\varrho) + \mathrm{M}_2(\varrho)\big),$$

$$\zeta = \zeta(\varrho) = \frac{1}{\sqrt{2}}\big(\mathrm{M}_1(\varrho) + \mathrm{M}_2(\varrho)\big),$$

$$\zeta = \zeta(\varrho) = \frac{1}{\sqrt{3}}\big(\mathrm{T}(\varrho) + \mathrm{M}_1(\varrho) + \mathrm{M}_2(\varrho)\big).$$

3. Surfaces with Common Spacelike Asymptotic Curves

Let $S = S(\varrho, v)$ be a family of spacelike parametric surfaces defined by a given curve ξ as follow

$$S(\varrho, v) = \xi(\varrho) + [\alpha(\varrho, v)T(\varrho) + \gamma(\varrho, v)\, M_1(\varrho) + \delta(\varrho, v)\, \mathrm{M}_2(\varrho)],\ v_1 \le \varrho \le v_2,\ \ t_1 \le v \le t_2. \quad (6)$$

where $\alpha(\varrho, v)$, $\gamma(\varrho, v)$ and $\delta(\varrho, v)$ are C^1 marching-scale functions. We want to derive the necessary and sufficient conditions for which the some special Smarandache curves of the unit speed spacelike curve $\xi(\varrho)$ is a parametric curve and an asymptotic curve on the surface $S(\varrho, v)$. Since Smarandache curve of $\xi(\varrho)$ is a parametric curve on the surface $S(\varrho, v)$, there exists a parameter $v_0 \in [t_1, t_2]$ such that

$$\alpha(\varrho, v_0) = \gamma(\varrho, v_0) = \delta(\varrho, v_0) = 0,\ \ v_1 \le \varrho \le v_2,\ t_1 \le v \le t_2 \quad (7)$$

Also, the curve $\xi(\varrho)$ is an asymptotic curve on the surface $S(\varrho; v)$ if and only if the normal curvature$\kappa_n = \kappa \cos\vartheta = 0$, where ϑ is the angle between the surface normal $n(\varrho, v_0)$ and the principal normal $N(\varrho)$ of the curve $\xi(\varrho)$. Since $n(\varrho, v_0) \cdot T(\varrho) = 0, v_1 \le \varrho \le v_2$, then we have

$$\frac{\partial n}{\partial \varrho}(\varrho, v_0) \cdot T(\varrho) = 0, \quad (8)$$

for the curve $\xi(\varrho)$ to be an asymptotic curve on the surface $S(\varrho, v)$,where " $\cdot$ " denotes the standard inner product.

3.1 Surfaces with Common TM1-Spacelike Curves

Theorem 3.1. Let $\xi = \xi(\varrho)$ be a unit speed spacelike curve reference to Bishop frame in Minkowski 3-space E_1^3. Then TM_1-spacelike Smarandache curve of $\xi(\varrho)$ is Iso-asymptotic on the surface $S(\varrho, v)$ if and only if the following conditions are satisfied:

$$\begin{cases} \alpha(\varrho, v_0) = \gamma(\varrho, v_0) = \delta(\varrho, v_0) = 0, \\ \dfrac{\partial \delta}{\partial v}(\varrho, v_0) = -\tanh\vartheta(\varrho)\ \dfrac{\partial \gamma}{\partial v}(\varrho, v_0). \end{cases}$$

Proof. Let $\zeta = \zeta(\varrho)$ be a spacelike TM_1-spacelike Smarandache curve on surface $S(\varrho, v)$. From Eqn. (6), $S(\varrho, v)$ parametric surface is defined by a given TM_1-spacelike Smarandache curve of the curve $\xi(\varrho)$ as follows:

$$S(\varrho, v) = \frac{1}{\sqrt{2}}\,[T(\varrho) + M_1(\varrho)] + [\alpha(\varrho, v)T(\varrho) + \gamma(\varrho, v)\, M_1(\varrho) + \delta(\varrho, v)\, \mathrm{M}_2(\varrho)]$$

If the TM_1-spacelike Smarandache curve is an parametric curve on this surface, $S(\varrho, v_0) = \frac{1}{\sqrt{2}}\,[T(\varrho) + M_1(\varrho)]$ for some $v = v_0$ that is,

$$\alpha(\varrho, v_0) = \gamma(\varrho, v_0) = \delta(\varrho, v_0) = 0. \quad (9)$$

Then

$$\begin{aligned} n(\varrho,v) = &\left[\frac{\partial\delta(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)\right.\\ &\left.-\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)\right]T(\varrho)\\ &+\left[\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)\right.\\ &\left.+\frac{\partial\delta(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\,\kappa_1(\varrho)}{\sqrt{2}}\right)\right]M_1(\varrho)\\ &-\left[\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\,\kappa_1(\varrho)}{\sqrt{2}}\right)\right.\\ &\left.+\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)\right]M_2(\varrho) \end{aligned} \tag{10}$$

Using Eqn. (9), if we let

$$\begin{cases} \Phi_1(\varrho,v_0)=\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\delta(\varrho,v_0)}{\partial v}+\frac{\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\gamma(\varrho,v_0)}{\partial v}\\ \Phi_2(\varrho,v_0)=\frac{\varepsilon\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\delta(\varrho,v_0)}{\partial v}-\frac{\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\alpha(\varrho,v_0)}{\partial v}\\ \Phi_3(\varrho,v_0)=\frac{\varepsilon\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\gamma(\varrho,v_0)}{\partial v}-\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\alpha(\varrho,v_0)}{\partial v} \end{cases} \tag{11}$$

Then from Eqns. (3) and (11), we get

$$\begin{aligned} n(\varrho,v)=&\,\Phi_1(\varrho,v_0)T(\varrho)+(\cosh\vartheta(\varrho)\,\Phi_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Phi_3(\varrho,v_0))N(\varrho)\\ &+(\sinh\vartheta(\varrho)\Phi_2(\varrho,v_0)+\cosh\vartheta(\varrho)\Phi_3(\varrho,v_0))B(\varrho) \end{aligned}$$

According to Eqn. (8), we should have

$$\frac{\partial n}{\partial\varrho}(\varrho,v_0)\cdot T(\varrho)=\frac{\partial\Phi_1}{\partial\varrho}(\varrho,v_0)-\varepsilon\kappa(\varrho)(\cosh\vartheta(\varrho)\,\Phi_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Phi_3(\varrho,v_0))=0 \tag{12}$$

Now Eqn. (9), implies that $\frac{\partial\Phi_1}{\partial\varrho}(\varrho,v_0)=0$. Using Eqns. (5) and (11) in Eqn. (12), we get

$$\frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\frac{\partial\gamma}{\partial v}(\varrho,v_0) \tag{13}$$

which completes the proof. □

3.2 Surfaces with Common TM_2-Spacelike Curves

Theorem 3.2. Let $\xi=\xi(\varrho)$ be a unit speed spacelike curve reference to Bishop frame in Minkowski 3-space E_1^3. Then TM_2-spacelike Smarandache curve of $\xi(\varrho)$ is Iso-asymptotic on a surface $S(\varrho,v)$ if and only if the following conditions are satisfied:

$$\begin{cases} \alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0,\\ \frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\,\frac{\partial\gamma}{\partial v}(\varrho,v_0). \end{cases}$$

Proof. Let $\zeta=\zeta(\varrho)$ be a spacelike TM_2-spacelike Smarandache curve on surface $S(\varrho,v)$. According to Eqn. (6), $S(\varrho,v)$ parametric surface is defined by a given TM_2-spacelike Smarandache as follows:

$$S(\varrho,v)=\frac{1}{\sqrt{2}}\,[T(\varrho)+M_2(\varrho)]+[\alpha(\varrho,v)T(\varrho)+\gamma(\varrho,v)\,M_1(\varrho)+\delta(\varrho,v)\,M_2(\varrho)]$$

If the TM_2-spacelike Smarandache curve is an parametric curve on this surface, then there exist a parameter $v=v_0$ such that $S(\varrho,v_0)=\frac{1}{\sqrt{2}}\,[T(\varrho)+M_2(\varrho)]$ that is,

$$\alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0. \tag{14}$$

Also the normal vector can be written as

$$n(\varrho,v)=\left[\frac{\partial\delta(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)-\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)\right]T(\varrho)$$
$$+\left[\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)+\frac{\partial\delta(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\,\kappa_2(\varrho)}{\sqrt{2}}\right)\right]M_1(\varrho)$$
$$-\left[\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\,\kappa_2(\varrho)}{\sqrt{2}}\right)+\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)\right]M_2(\varrho). \tag{15}$$

If we let

$$\begin{cases}\Lambda_1(\varrho,v_0)=\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\delta(\varrho,v_0)}{\partial v}+\frac{\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\gamma(\varrho,v_0)}{\partial v}.\\ \Lambda_2(\varrho,v_0)=\frac{\varepsilon\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\delta(\varrho,v_0)}{\partial v}-\frac{\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\alpha(\varrho,v_0)}{\partial v}.\\ \Lambda_3(\varrho,v_0)=\frac{\varepsilon\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial\gamma(\varrho,v_0)}{\partial v}+\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial\alpha(\varrho,v_0)}{\partial v}.\end{cases} \tag{16}$$

Then from Eqns. (9) and (16), we get

$$n(\varrho,v_0)=\Lambda_1(\varrho,v_0)T(\varrho)+(\cosh\vartheta(\varrho)\,\Lambda_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Lambda_3(\varrho,v_0))N(\varrho)+(\sinh\vartheta(\varrho)\Lambda_2(\varrho,v_0)+\cosh\vartheta(\varrho)\Lambda_3(\varrho,v_0))B(\varrho)$$

From Eqn. (8), we should have

$$\frac{\partial n}{\partial\varrho}(\varrho,v_0)\cdot T(\varrho)=\frac{\partial\Lambda_1}{\partial\varrho}(\varrho,v_0)-\varepsilon\kappa(\varrho)(\cosh\vartheta(\varrho)\,\Lambda_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Lambda_3(\varrho,v_0))=0 \tag{17}$$

Equation (14) implies that $\frac{\partial\Lambda_1}{\partial\varrho}(\varrho,v_0)=0$. So, by using Eqns. (5) and (16) in Eqn. (17), since $\kappa(\varrho)\neq 0$, we get

$$\frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\,\frac{\partial\gamma}{\partial v}(\varrho,v_0). \tag{18}$$

which completes the proof. □

3.3 Surfaces with Common M_1M_2-Spacelike Curves

Theorem 3.3. Let $\xi=\xi(\varrho)$ be a unit speed spacelike curve reference to Bishop frame in Minkowski 3-space E_1^3. Then M_1M_2-spacelike Smarandache curve of $\xi(\varrho)$ is Iso-asymptotic on a surface $S(\varrho,v)$if and only if the following condition satisfied:

$$\begin{cases}\alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0,\\ \kappa_1(\varrho)+\kappa_2(\varrho)\neq 0,\\ \frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\,\frac{\partial\gamma}{\partial v}(\varrho,v_0).\end{cases}$$

Proof. Let $\zeta=\zeta(\varrho)$ be a spacelike M_1M_2-spacelike Smarandache curve on the surface $S(\varrho,v)$. According to Eqn. (6), the surface $S(\varrho,v)$ is defined by a given M_1M_2-spacelike Smarandache as follows:

$$S(\varrho,v)=\frac{1}{\sqrt{2}}\left[M_1(\varrho)+M_2(\varrho)\right]+\left[\alpha(\varrho,v)T(\varrho)+\gamma(\varrho,v)M_1(\varrho)+\delta(\varrho,v)M_2(\varrho)\right].$$

If spacelike M_1M_2-Smarandache curve is an parametric curve on $S(\varrho,v)$, then $S(\varrho,v_0)=\frac{1}{\sqrt{2}}\left[M_1(\varrho)+M_2(\varrho)\right]$ for some $v=v_0$ that is,

$$\alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0. \tag{19}$$

And

$n(\varrho,v)=\left[\frac{\partial\delta(\varrho,v)}{\partial v}\left(\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)-\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)\right]T(\varrho)+\left[\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)+\frac{\partial\delta(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{2}}\right)\right]M_1(\varrho)-$

$$\left[\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{2}}\right)+\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)\right] M_2(\varrho) \tag{20}$$

Now, we can write

$$\begin{cases}\Omega_1(\varrho,v_0)=0,\\ \Omega_2(\varrho,v_0)=\frac{-\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial\delta(\varrho,v_0)}{\partial v},\\ \Omega_3(\varrho,v_0)=\frac{\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial\gamma(\varrho,v_0)}{\partial v}.\end{cases} \tag{21}$$

According to Eqn. (21), we can write $n(\varrho,v_0)$ as follows:

$$n(\varrho,v_0)=\Omega_1(\varrho,v_0)T(\varrho)+\Omega_2(\varrho,v_0)M_1(\varrho)+\Omega_3(\varrho,v_0)M_2(\varrho)$$

From Eqn. (3), we get

$$n(\varrho,v_0)=\Omega_1(\varrho,v_0)T(\varrho)+(\cosh\vartheta(\varrho)\,\Omega_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Omega_3(\varrho,v_0))N(\varrho) + (\sinh\vartheta(\varrho)\,\Omega_2(\varrho,v_0)+\cosh\vartheta(\varrho)\Omega_3(\varrho,v_0))B(\varrho)$$

From the Eqn. (8), we should have

$$\frac{\partial n}{\partial\varrho}(\varrho,v_0)\cdot T(\varrho)=\frac{\partial\Omega_1}{\partial\varrho}(\varrho,v_0)-\varepsilon\kappa(\varrho)(\cosh\vartheta(\varrho)\,\Omega_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Omega_3(\varrho,v_0))=0 \tag{22}$$

Now, using Eqns. (5) and (21) in Eqn. (22), we get

$$\frac{\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial\delta}{\partial v}(\varrho,v_0)\cosh\vartheta(\varrho)=\frac{\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial\gamma}{\partial v}(\varrho,v_0)\sinh\vartheta(\varrho)$$

For $\kappa_1(\varrho)+\kappa_2(\varrho)\neq 0$, we obtain

$$\frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\,\frac{\partial\gamma}{\partial v}(\varrho,v_0) \tag{23}$$

which completes the proof. □

3.4 Surfaces with Common TM_1M_2-Spacelike Curves

Theorem 3.4. Let $\xi=\xi(\varrho)$ be a unit speed spacelike curve reference to Bishop frame in Minkowski 3-space E_1^3. Then TM_1M_2-spacelike Smarandache curve of $\xi(\varrho)$ is Iso-asymptotic on a surface $S(\varrho,v)$ if and only if the following conditions are satisfied:

$$\begin{cases}\alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0\\ \kappa_1(\varrho)+\kappa_2(\varrho)\neq 0\\ \dfrac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\,\dfrac{\partial\gamma}{\partial v}(\varrho,v_0)\end{cases}$$

Proof. Let $\zeta=\zeta(\varrho)$ be a spacelike TM_1M_2-spacelike Smarandache curve on surface $S(\varrho,v)$. According to Eqn. (6), then $S(\varrho,v)$ can can be expressed in terms of the TM_1M_2-spacelike Smarandache as follows:

$$S(\varrho,v)=\frac{1}{\sqrt{3}}\left[T(\varrho)+M_1(\varrho)+M_2(\varrho)\right]+\left[\alpha(\varrho,v)T(\varrho)+\gamma(\varrho,v)M_1(\varrho)+\delta(\varrho,v)M_2(\varrho)\right]$$

If spacelike TM_1M_2-spacelike Smarandache curve is an parametric curve on this surface, then there exist a parameter $v=v_0$ such that

$$S(\varrho,v_0)=\frac{1}{\sqrt{3}}\left[T(\varrho)+M_1(\varrho)+M_2(\varrho)\right]$$

that is,

$$\alpha(\varrho,v_0)=\gamma(\varrho,v_0)=\delta(\varrho,v_0)=0$$

Then

$$n(\varrho,v)=\left[\frac{\partial\delta(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{3}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)-\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{3}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)\right]T(\varrho)+\left[\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{-\kappa_2(\varrho)}{\sqrt{3}}+\kappa_2(\varrho)\alpha(\varrho,v)+\frac{\partial\delta(\varrho,v)}{\partial\varrho}\right)+\frac{\partial\delta(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{3}}\right)\right]M_1(\varrho)-\left[\frac{\partial\gamma(\varrho,v)}{\partial v}\left(\varepsilon\kappa_1(\varrho)\gamma(\varrho,v)+\varepsilon\kappa_2(\varrho)\delta(\varrho,v)-\frac{\partial\alpha(\varrho,v)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{3}}\right)+\frac{\partial\alpha(\varrho,v)}{\partial v}\left(\frac{\kappa_1(\varrho)}{\sqrt{3}}+\kappa_1(\varrho)\alpha(\varrho,v)+\frac{\partial\gamma(\varrho,v)}{\partial\varrho}\right)\right]M_2(\varrho) \tag{25}$$

Using Eqn. (24), if we let

$$\begin{cases} \Gamma_1(\varrho,v_0)=\frac{\kappa_1(\varrho)}{\sqrt{3}}\ \frac{\partial\delta(\varrho,v_0)}{\partial v}+\frac{\kappa_2(\varrho)}{\sqrt{3}}\ \frac{\partial\gamma(\varrho,v_0)}{\partial v} \\ \Gamma_2(\varrho,v_0)=\frac{\varepsilon}{\sqrt{3}}\big(\kappa_1(\varrho)+\kappa_2(\varrho)\big)\ \frac{\partial\delta(\varrho,v_0)}{\partial v}-\frac{\kappa_2(\varrho)}{\sqrt{3}}\ \frac{\partial\alpha(\varrho,v_0)}{\partial v} \\ \Gamma_3(\varrho,v_0)=\frac{\varepsilon}{\sqrt{3}}\big(\kappa_1(\varrho)+\kappa_2(\varrho)\big)\ \frac{\partial\gamma(\varrho,v_0)}{\partial v}+\frac{\kappa_1(\varrho)}{\sqrt{3}}\frac{\partial\alpha(\varrho,v_0)}{\partial v} \end{cases} \quad (26)$$

We obtain

$$n(\varrho,v_0)=\Gamma_1(\varrho,v_0)T(\varrho)+\Gamma_2(\varrho,v_0)M_1(\varrho)+\Gamma_3(\varrho,v_0)M_2(\varrho)$$

From Eqn. (3), we get

$$n(\varrho,v_0)=\Gamma_1(\varrho,v_0)T(\varrho)+(\cosh\vartheta(\varrho)\,\Gamma_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Gamma_3(\varrho,v_0))N(\varrho) + (\sinh\vartheta(\varrho)\,\Gamma_2(\varrho,v_0)+\cosh\vartheta(\varrho)\Gamma_3(\varrho,v_0))B(\varrho)$$

Then applying Eqn. (8), we should have

$$\frac{\partial n}{\partial\varrho}(\varrho,v_0)\cdot T(\varrho)=\frac{\partial\Gamma_1}{\partial\varrho}(\varrho,v_0)-\varepsilon\kappa(\varrho)(\cosh\vartheta(\varrho)\,\Gamma_2(\varrho,v_0)+\sinh\vartheta(\varrho)\Gamma_3(\varrho,v_0))=0 \quad (27)$$

From Eqn. (24), we have $\frac{\partial\Gamma_1}{\partial\varrho}(\varrho,v_0)=0$. We using Eqn. (5) and Eqn. (26) in Eqn. (27), since $\kappa(\varrho)\neq 0$, we get

$$-\cosh\vartheta(\varrho)\left(\frac{\varepsilon}{\sqrt{3}}\big(\kappa_1(\varrho)+\kappa_2(\varrho)\big)\right)\frac{\partial\delta}{\partial v}(\varrho,v_0)=\sinh\vartheta(\varrho)\left(\frac{\varepsilon}{\sqrt{2}}\big(\kappa_1(\varrho)+\kappa_2(\varrho)\big)\right)\frac{\partial\gamma}{\partial v}(\varrho,v_0).$$

For $\kappa_1(\varrho)+\kappa_2(\varrho)\neq 0$, we obtain

$$\frac{\partial\delta}{\partial v}(\varrho,v_0)=-\tanh\vartheta(\varrho)\ \frac{\partial\gamma}{\partial v}(\varrho,v_0). \quad (28)$$

which completes the proof. □

Example 3.1. Let $\xi(s)=\big(\sinh s,\cosh s,\sqrt{2}\,s\big)$ be a spacelike curve parametrized by arc-length with timelike binormal $(\varepsilon=1)$. Then it is easy to show that $T(s)=\big(\cosh s,\sinh s,\sqrt{2}\,\big),\kappa=1\neq 0,\tau=-\sqrt{2}\neq 0$ and $\vartheta(s)=\sqrt{2}\,s+c, c=constant$. Here, we can take $c=0$. From Eqn. (5), we get $\kappa_1(s)=\cosh\big(\sqrt{2}\,s\big)$, $\kappa_2(s)=\sinh\big(\sqrt{2}\,s\big)$. From Eqn. (2), we get $M_1(s)=\int\kappa_1(s)\ T(s)\,ds$, $M_2(s)=\int\kappa_2(s)\ T(s)\,ds$, then we have

$$M_1(s)=\left(\frac{1}{2(\sqrt{2}+1)}\sinh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\sinh\Big(\big(\sqrt{2}-1\big)s\Big),\frac{1}{2(\sqrt{2}+1)}\cosh\Big(\big(\sqrt{2}+1\big)s\Big) -\frac{1}{2(\sqrt{2}-1)}\cosh\Big(\big(\sqrt{2}-1\big)s\Big),\sinh\big(\sqrt{2}\,s\big)\right)$$

$$M_2(s)=\left(\frac{1}{2(\sqrt{2}+1)}\cosh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\cosh\Big(\big(\sqrt{2}-1\big)s\Big),\frac{1}{2(\sqrt{2}+1)}\sinh\Big(\big(\sqrt{2}+1\big)s\Big) -\frac{1}{2(\sqrt{2}-1)}\sinh\Big(\big(\sqrt{2}-1\big)s\Big),\cosh\big(\sqrt{2}\,s\big)\right)$$

If we take $\alpha(s,v)=0$, $\gamma(s,v)=e^v-1$ and $\delta(s,v)=\sqrt{2}\,(e^v-1)$, we obtain a member of the surface with common spacelike curve $\xi(s)$ as

$$S_1(s,v)=\Bigg(\sinh(s)+(e^v-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\sinh\Big(\big(\sqrt{2}-1\big)s\Big)\right] +\sqrt{2}\,(e^v-1)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\cosh\Big(\big(\sqrt{2}-1\big)s\Big)\right],\cosh(s) +(e^v-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\sinh\Big(\big(\sqrt{2}-1\big)s\Big)\right] +\sqrt{2}\,(e^v-1)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\Big(\big(\sqrt{2}+1\big)s\Big)+\frac{1}{2(\sqrt{2}-1)}\cosh\Big(\big(\sqrt{2}-1\big)s\Big)\right],\sqrt{2}\,s +\sinh\big(\sqrt{2}\,s\big)\,(e^v-1)+\sqrt{2}\,(e^v-1)\,\cosh\big(\sqrt{2}\,s\big)\Bigg)$$

where $0 \leq s \leq 2\pi, -1 \leq v \leq 1$ (Fig. 1).

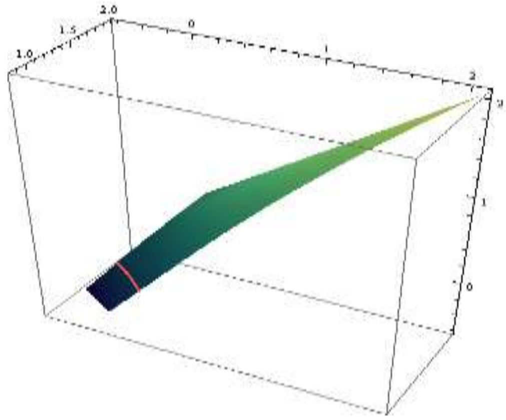

Figure 1. $S_1(s,v)$ as a member of surfaces and curve $\xi(s)$.

If we take $\alpha(s,v)=0$, $\gamma(s,v)=e^{-v}-1$ and $\delta(s,v)=\tanh(\sqrt{2}\,s)\,\sinh(v)$ and $v_0=0$, then the Eqns. (9) and (13) are satisfied. Thus, we obtain a member of the surface with common TM_1-spacelike Smarandache Iso-asymptotic curve as

$$\begin{aligned} S_2(s,v) = \Bigg(& \frac{1}{\sqrt{2}}\cosh(s) + \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) + \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right) \\ & + (e^{-v}-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) + \frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\ & + \tanh(\sqrt{2}\,s)\,\sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right) \right. \\ & \left. + \frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{\sqrt{2}}\sinh(s) \\ & + \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right) - \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right) + (e^{-v} \\ & -1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) + \frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\ & + \tanh(\sqrt{2}\,s)\,\sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right) \right. \\ & \left. + \frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{\sqrt{2}}\sinh(\sqrt{2}s) + \sinh(\sqrt{2}\,s)\,(e^{-v}-1) \\ & + \tanh(\sqrt{2}\,s)\,\sinh(v)\,\cosh(\sqrt{2}\,s)\Bigg) \end{aligned}$$

where $0 \leq s \leq 2\pi, -1 \leq v \leq 1$ (Fig. 2).

A member of the surface with common TM_2-spacelike Smarandache Iso-asymptotic curve as

$$
\begin{aligned}
S_3(s,v) = \Bigg(&\frac{1}{\sqrt{2}}\cosh(s) + \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right) + \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right) \\
&+ (e^{-v}-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) + \frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
&+ \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)\right. \\
&\left.+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{\sqrt{2}}\sinh(s) \\
&+\frac{1}{2\sqrt{2}(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) - \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right) + (e^{-v} \\
&- 1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right) + \frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
&+ \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)\right. \\
&\left.+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{\sqrt{2}}\cosh(\sqrt{2}s) + \sinh(\sqrt{2}\,s)\ (e^{-v}-1) \\
&+ \tanh(\sqrt{2}\,s)\ \sinh(v)\ \cosh(\sqrt{2}\,s)\Bigg)
\end{aligned}
$$

where $0 \le s \le 2\pi, -1 \le v \le 1$ (Fig. 3).

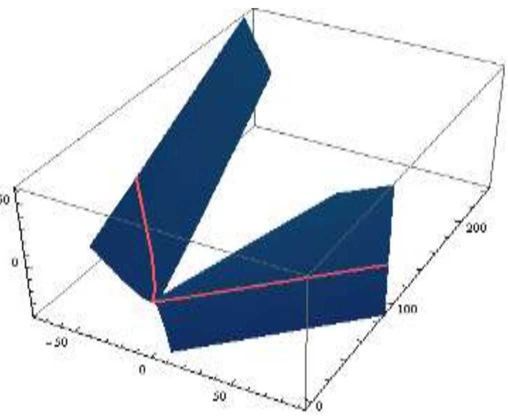

Figure 2. $S_2(s,v)$ as a member of surfaces and its TM1-spacelike Smarandache Iso-asymptotic curve $\xi(s)$

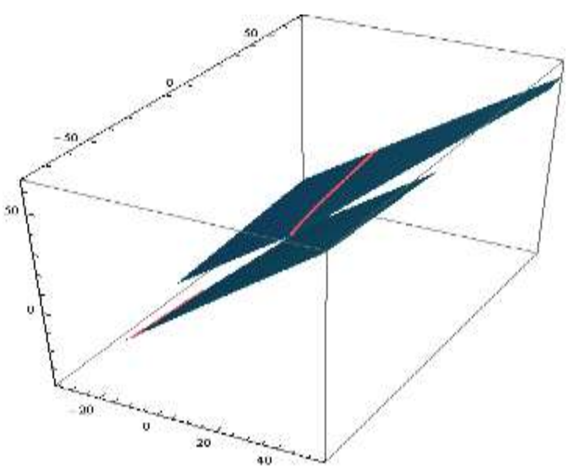

Figure 3. $S_3(s,v)$ as a member of surfaces and its TM2-spacelike Smarandache Iso-asymptotic curve $\xi(s)$.

A member of the surface with common M_1M_2-spacelike Smarandache Iso-asymptotic curve as

$$
\begin{aligned}
S_4(s,v) = \Bigg(& \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\left[\cosh\left((\sqrt{2}+1)s\right)+\sinh\left((\sqrt{2}+1)s\right)\right] \\
& + \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\left[\cosh\left((\sqrt{2}-1)s\right)+\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + (e^{-v}-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right)+\frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)\right. \\
& \left.+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\left[\cosh\left((\sqrt{2}+1)s\right)+\sinh\left((\sqrt{2}+1)s\right)\right] \\
& - \frac{1}{2\sqrt{2}(\sqrt{2}-1)}\left[\cosh\left((\sqrt{2}-1)s\right)+\sinh\left((\sqrt{2}-1)s\right)\right] + (e^{-v} \\
& - 1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right)+\frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)\right. \\
& \left.+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \sqrt{2}\ \cosh(\sqrt{2}s) + \ \sinh(\sqrt{2}\,s)\ (e^{-v}-1) \\
& + \tanh(\sqrt{2}\,s)\ \sinh(v)\ \cosh(\sqrt{2}\,s)\Bigg)
\end{aligned}
$$

where $0 \le s \le 2\pi, -1 \le v \le 1$ (Fig. 4).

A member of the surface with common TM_1M_2-spacelike Smarandache Iso-asymptotic curve as

$$
\begin{aligned}
S_5(s,v) = \Bigg(& \frac{1}{\sqrt{3}}\cosh(s) + \frac{1}{2\sqrt{3}(\sqrt{2}+1)}\left[\cosh\left((\sqrt{2}+1)s\right)+\sinh\left((\sqrt{2}+1)s\right)\right] \\
& + \frac{1}{2\sqrt{3}(\sqrt{2}-1)}\left[\cosh\left((\sqrt{2}-1)s\right)+\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + \ (e^{-v}-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right)+\frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)\right. \\
& \left.+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{1}{\sqrt{3}}\sinh(s) \\
& + \frac{1}{2\sqrt{3}(\sqrt{2}+1)}\left[\cosh\left((\sqrt{2}+1)s\right)+\sinh\left((\sqrt{2}+1)s\right)\right] \\
& - \frac{1}{2\sqrt{3}(\sqrt{2}+1)}\left[\cosh\left((\sqrt{2}+1)s\right)+\sinh\left((\sqrt{2}+1)s\right)\right] \\
& + (e^{-v}-1)\left[\frac{1}{2(\sqrt{2}+1)}\sinh\left((\sqrt{2}+1)s\right)+\frac{1}{2(\sqrt{2}-1)}\sinh\left((\sqrt{2}-1)s\right)\right] \\
& + \tanh(\sqrt{2}\,s)\ \sinh(v)\left[\frac{1}{2(\sqrt{2}+1)}\cosh\left((\sqrt{2}+1)s\right)+\frac{1}{2(\sqrt{2}-1)}\cosh\left((\sqrt{2}-1)s\right)\right], \frac{\sqrt{2}}{\sqrt{3}} \\
& + \frac{2}{\sqrt{3}}\cosh(\sqrt{2}s) + \sinh(\sqrt{2}\,s)\ (e^{-v}-1) + \tanh(\sqrt{2}\,s)\ \sinh(v)\ \cosh(\sqrt{2}\,s)\Bigg)
\end{aligned}
$$

where $0 \le s \le 2\pi, -1 \le v \le 1$ (Fig. 5).

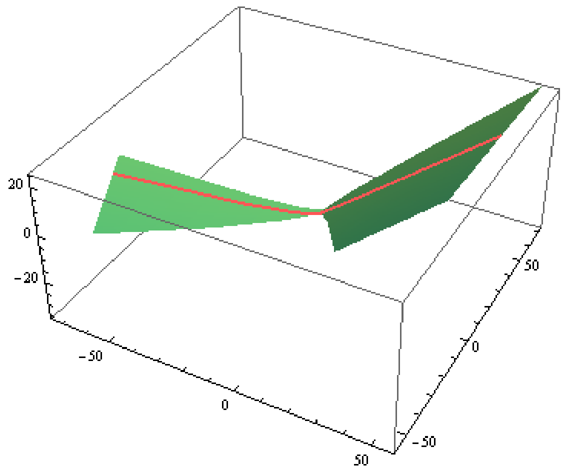

Figure 4. $S_4(s,v)$ as a member of surfaces and its M_1M_2 -spacelike Smarandache Iso-asymptotic curve $\xi(s)$.

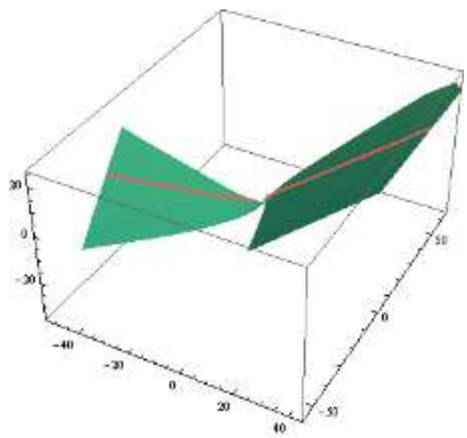

Figure 5. $S_5(s,v)$ as a member of surfaces and its TM_1M_2-Sspacelike marandache Iso-asymptotic curve $\xi(s)$

4. Surfaces with Common Spacelike Geodesic Curve

Let $\Psi=\Psi(\varrho,t)$ be a spacelike parametric surface defined by a given curve as

$$\Psi(\varrho,t)=\omega(\varrho)+[f(\varrho,t)T(\varrho)+g(\varrho,t)\,\eta(\varrho)+h(\varrho,t)\xi(\varrho)],\quad p_1\le\varrho\le p_1,q_1\le t\le q_1 \tag{29}$$

where $f(\varrho,t)$, $g(\varrho,t)$ and $h(\varrho,t)$ are C^1 marching-scale functions. Since the spacelike Smarandache curve of $\omega(\varrho)$ is a parametric curve on $\Psi(\varrho,t)$, then we get

$$f(\varrho,t_0)=\ g(\varrho,t_0)\ =\ h(\varrho,t_0)=0,\quad p_1\le\varrho\le p_1,\quad q_1\le t\le q_1. \tag{30}$$

for some $t_0\epsilon\,[q_1,\ q_2]$. Also, since the spacelike Smarandache curve of $\omega(\varrho)$ is an geodesic curve on $t_0\epsilon\,[q_1,\ q_2]$, then we get

$$n(s,v_0)=N(\varrho), \tag{31}$$

for some $t_0\epsilon\,[q_1,\ q_2]$ and $n(\varrho,t)$ is the normal vector of $\Psi(\varrho,t)$ and $N(\varrho)$ is the normal vector of $\omega(\varrho)$.

Theorem 4.1. Let $\omega=\omega(\varrho)$ be a unit speed spacelike curve with moving Bishop frame in Minkowski 3-space $\mathcal{R}_1^3$. Then $\eta\xi$-spacelike Smarandache curves of $\omega(\varrho)$ is iso-geodesic on the surface $\Psi(\varrho,t)$ if and only if the following conditions are satisfied:

i. $t_0)\ =\ g(\varrho,t_0)\ =\ h(\varrho,t_0)=0,$

ii. $\kappa_1(\varrho)+\kappa_2(\varrho)\neq 0,$

iii. $\frac{\partial h}{\partial t}(\varrho,t_0)\neq 0$

iv. $\frac{\partial g}{\partial t}(\varrho,t_0)=\tanh\theta(\varrho)\frac{\partial h}{\partial t}(\varrho,t_0)$

Proof. Let $\gamma(s(\varrho))$ be $\eta\xi$ -spacelike Smarandache curve on the surface $\Psi(\varrho,t)$. From Eqn. (29), $\Psi(s,t)$ parametric surfaces is defined in terms of γ of the curve $\omega(\varrho)$ as follows:

$$\Psi(\varrho,t)=\frac{1}{\sqrt{2}}[\eta(\varrho)+\xi(\varrho)]+[f(\varrho,t)T(\varrho)+g(\varrho,t)\,\eta(\varrho)+h(\varrho,t)\xi(\varrho)],$$

If the $\eta\xi$ -spacelike Smarandache curve is a parametric curve on $\Psi(\varrho,t)$, then $\Psi(s,t_0)=\frac{1}{\sqrt{2}}[\eta(\varrho)+\xi(\varrho)]$ for some $t=t_0$ that is,

$$f(\varrho,t_0)=g(\varrho,t_0)=h(\varrho,t_0)=0 \tag{32}$$

Since

$$n(\varrho,t)=\frac{\partial\Psi(\varrho,t)}{\partial\varrho}\times\frac{\partial\Psi(\varrho,t)}{\partial t}$$

So, it can be expressed as

$$n(\varrho,t)=\left[\frac{\partial h(\varrho,t)}{\partial t}\left(\kappa_1(\varrho)\,f(\varrho,t)+\frac{\partial g(\varrho,t)}{\partial\varrho}\right)-\frac{\partial g(\varrho,t)}{\partial t}\left(\frac{\partial h(\varrho,t)}{\partial\varrho}-\kappa_2(\varrho)f(\varrho,t)\right)\right]T(\varrho)+\left[\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{\partial h(\varrho,t)}{\partial\varrho}-\kappa_2(\varrho)\,f(\varrho,t)\right)+\frac{\partial h(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)\,g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{2}}\right)\right]\eta(\varrho)-\left[\frac{\partial g(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)}{\sqrt{2}}\right)+\frac{\partial f(\varrho,t)}{\partial t}\left(\kappa_1(\varrho)f(\varrho,t)+\frac{\partial g(\varrho,t)}{\partial\varrho}\right)\right]\xi(\varrho) \tag{33}$$

If

$$\begin{cases}\Gamma_1(\varrho,v_0)=0,\\ \Gamma_2(\varrho,v_0)=\frac{\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial h(\varrho,t_0)}{\partial t},\\ \Gamma_3(\varrho,v_0)=\frac{-\varepsilon}{\sqrt{2}}\left(\kappa_1(\varrho)+\kappa_2(\varrho)\right)\frac{\partial g(\varrho,t_0)}{\partial t}.\end{cases} \tag{34}$$

then we obtain

$$n(\varrho,t_0)=\Gamma_2(\varrho,t_0)\eta(\varrho)+\Gamma_3(\varrho,t_0)\,\xi(\varrho)$$

From Eqn. (34), we get

$$n(\varrho,t_0)=\left(\cosh\theta(\varrho)\;\Gamma_2(\varrho,t_0)+\sinh\theta(\varrho)\;\Gamma_3(\varrho,t_0)\right)N(\varrho)+\left(\sinh\theta(\varrho)\;\Gamma_2(\varrho,t_0)+\cosh\theta(\varrho)\;\Gamma_3(\varrho,t_0)\right)B(\varrho)$$

Also from Eqn. (30), we should have

$$\begin{cases}\left(\cosh\theta(\varrho)\;\Gamma_2(\varrho,t_0)+\sinh\theta(\varrho)\;\Gamma_3(\varrho,t_0)\right)\neq 0.\\ \left(\sinh\theta(\varrho)\;\Gamma_2(\varrho,t_0)+\cosh\theta(\varrho)\;\Gamma_3(\varrho,t_0)\right)=0.\end{cases} \tag{35}$$

Then

$$\Gamma_3(\varrho,t_0)=-\tanh\theta(\varrho)\,\Gamma_2(\varrho,t_0). \tag{36}$$

Using Eqn. (36) and $\cosh\theta(\varrho)\;\Gamma_2(\varrho,t_0)+\sinh\theta(\varrho)\;\Gamma_3(\varrho,t_0)\neq 0$, we get

$$\Gamma_2(\varrho,t_0)\neq 0 \tag{37}$$

which implies that

$$\frac{\partial g(\varrho,t_0)}{\partial t}=\tanh\theta(\varrho)\frac{\partial h(\varrho,t_0)}{\partial t}. \tag{38}$$

and

$$\kappa_1(\varrho)+\kappa_2(\varrho)\neq 0 \quad\text{and}\quad \frac{\partial h(\varrho,t_0)}{\partial t}\neq 0 \tag{39}$$

Combining the conditions (32) ,(38) and (39), we have complete the proof. □

Corollary 4.1. Let $\omega=\omega(\varrho)$ be a unit speed spacelike curve with moving Bishop frame in Minkowski 3-space $\mathcal{R}_1^3$. Then $T\eta$-spacelike Smarandache curves of the curve $\omega(\varrho)$ is not eodesic on the surface $\Psi(\varrho,t)$.

Proof. Let $\gamma(s(\varrho))$ be $T\eta$ -spacelike Smarandache curve on the surface $\Psi(\varrho,t)$. Thus from Eqn. (29), $\Psi(\varrho,t)$ can be expressed as

$$\Psi(\varrho,t)=\frac{1}{\sqrt{2}}[T(\varrho)+\eta(\varrho)]+[f(\varrho,t)T(\varrho)+\mathrm{g}(\varrho,t)\,\eta(\varrho)+h(\varrho,t)\xi(\varrho)],$$

If the $T\eta$ -spacelike Smarandache curve is a parametric curve on $\Psi(\varrho,t)$, $\Psi(\varrho,t)=\frac{1}{\sqrt{2}}[T(\varrho)+\xi(\varrho)]$ and

$$f(\varrho,t_0)=\mathrm{g}(\varrho,t_0)=h(\varrho,t_0)=0 \tag{40}$$

Then

$$n(\varrho,t)=\left[\frac{\partial h(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\,f(\varrho,t)+\frac{\partial \mathrm{g}(\varrho,t)}{\partial\varrho}\right)+\frac{\partial \mathrm{g}(\varrho,t)}{\partial t}\left(\frac{\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)f(\varrho,t)-\frac{\partial h(\varrho,t)}{\partial\varrho}\right)\right]T(\varrho)+\left[\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}-\kappa_2(\varrho)\,f(\varrho,t)+\frac{\partial h(\varrho,t)}{\partial\varrho}\right)+\frac{\partial h(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)\,\mathrm{g}(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\,\kappa_1(\varrho)}{\sqrt{2}}\right)\right]\eta(\varrho)-\left[\frac{\partial \mathrm{g}(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)\mathrm{g}(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\,\kappa_1(\varrho)}{\sqrt{2}}\right)+\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)f(\varrho,t)+\frac{\partial \mathrm{h}(\varrho,t)}{\partial\varrho}\right)\right]\xi(\varrho) \tag{41}$$

If we let

$$\begin{cases}\Phi_1(\varrho,t_0)=\frac{\kappa_1(\varrho)}{\sqrt{2}}\,\frac{\partial h(\varrho,t_0)}{\partial t}+\frac{\kappa_2(\varrho)}{\sqrt{2}}\,\frac{\partial \mathrm{g}(\varrho,t_0)}{\partial t}\\ \Phi_2(\varrho,t_0)=\frac{-\kappa_2(\varrho)}{\sqrt{2}}\,\frac{\partial f(\varrho,t_0)}{\partial t}+\frac{\varepsilon\kappa_1(\varrho)}{\sqrt{2}}\,\frac{\partial h(\varrho,t_0)}{\partial t}\\ \Phi_3(\varrho,t_0)=\frac{-\varepsilon\kappa_1(\varrho)}{\sqrt{2}}\,\frac{\partial \mathrm{g}(\varrho,t_0)}{\partial t}-\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial f(\varrho,t_0)}{\partial t}\end{cases} \tag{42}$$

So, Eqn. (42) leads to

$$n(\varrho,t_0)=\Phi_1(\varrho,t_0)T(\varrho)+\Phi_2(\varrho,t_0)\eta(\varrho)+\Phi_3(\varrho,t_0)\xi(\varrho)$$

Then one can find that

$$n(\varrho,t_0)=\Phi_1(\varrho,t_0)T(\varrho)+\left(\cosh\theta(\varrho)\ \Phi_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Phi_3(\varrho,t_0)\right)N(\varrho)+\left(\sinh\theta(\varrho)\ \Phi_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Phi_3(\varrho,t_0)\right)B(\varrho)$$

We know that $\omega(\varrho)$ is a geodesic curve if and only if

$$\begin{cases}\Phi_1(\varrho,t_0)=0\\ \cosh\theta(\varrho)\ \Phi_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Phi_3(\varrho,t_0)\neq 0\\ \sinh\theta(\varrho)\ \Phi_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Phi_3(\varrho,t_0)=0\end{cases} \tag{43}$$

Then from Eqns. (42) and (43), we obtain

$$\tanh\theta(\varrho)=-\frac{\kappa_1(\varrho)}{\kappa_2(\varrho)} \tag{44}$$

Using Eqns. (35) and (44), we get $\kappa_1(\varrho)=\kappa_2(\varrho)=0$, which gives a contradiction which complete the proof. □

Corollary 4.2. Let $\omega=\omega(\varrho)$ be a unit speed spacelike curve with moving Bishop frame in Minkowski 3-space $\mathcal{R}_1^3$. Then $T\xi$ -spacelike Smarandache curves of a spacelike curve $\omega(\varrho)$ is not geodesic on the surface $\Psi(\varrho,t)$.

Proof. Let $\gamma(s(\varrho))$ be $T\xi$ -spacelike Smarandache curve on the surface $\Psi(\varrho,t)$. Thus, from (29), we have

$$\Psi(\varrho,t)=\frac{1}{\sqrt{2}}[T(\varrho)+\xi(\varrho)]+[f(\varrho,t)T(\varrho)+\mathrm{g}(\varrho,t)\,\eta(\varrho)+h(\varrho,t)\xi(\varrho)],$$

If the $T\xi$ -spacelike Smarandache curve is a parametric curve on this surface, $\Psi(\varrho,t_0)=\frac{1}{\sqrt{2}}[T(\varrho)+\xi(\varrho)]$ that is,

$$f(\varrho,t_0)=\mathrm{g}(\varrho,t_0)=h(\varrho,t_0)=0 \tag{45}$$

Then

$$n(\varrho,t)=\left[\frac{\partial h(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)\,f(\varrho,t)+\frac{\partial \mathrm{g}(\varrho,t)}{\partial\varrho}\right)+\frac{\partial \mathrm{g}(\varrho,t)}{\partial t}\left(\frac{\kappa_2(\varrho)}{\sqrt{2}}+\kappa_2(\varrho)f(\varrho,t)-\frac{\partial h(\varrho,t)}{\partial\varrho}\right)\right]T(\varrho)+\left[\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{-\kappa_2(\varrho)}{\sqrt{2}}-\kappa_2(\varrho)\,f(\varrho,t)+\frac{\partial h(\varrho,t)}{\partial\varrho}\right)+\right.$$

$$\frac{\partial h(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)\,g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\,\kappa_2(\varrho)}{\sqrt{2}}\right)\Big]\eta(\varrho)-\Big[\frac{\partial g(\varrho,t)}{\partial t}\Big(\varepsilon\kappa_1(\varrho)g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon\,\kappa_2(\varrho)}{\sqrt{2}}\Big)+\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{2}}+\kappa_1(\varrho)f(\varrho,t)+\frac{\partial g(\varrho,t)}{\partial\varrho}\right)\Big]\xi(\varrho) \tag{46}$$

If we write

$$\begin{cases}\Theta_1(\varrho,t_0)=\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial h(\varrho,t_0)}{\partial t}+\frac{\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial g(\varrho,t_0)}{\partial t}\\ \Theta_2(\varrho,t_0)=\frac{-\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial f(\varrho,t_0)}{\partial t}+\frac{\varepsilon\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial h(\varrho,t_0)}{\partial t}\\ \Theta_3(\varrho,t_0)=\frac{-\varepsilon\kappa_2(\varrho)}{\sqrt{2}}\frac{\partial g(\varrho,t_0)}{\partial t}-\frac{\kappa_1(\varrho)}{\sqrt{2}}\frac{\partial f(\varrho,t_0)}{\partial t}\end{cases} \tag{47}$$

we obtain

$$n(\varrho,t_0)=\Theta_1(\varrho,t_0)T(\varrho)+\Theta_2(\varrho,t_0)\eta(\varrho)+\Theta_3(\varrho,t_0)\xi(\varrho)$$

From Eqn. (34), we get

$$n(\varrho,t_0)=\Theta_1(\varrho,t_0)T(\varrho)+\left(\cosh\theta(\varrho)\ \Theta_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Theta_3(\varrho,t_0)\right)N(\varrho)+\left(\sinh\theta(\varrho)\ \Theta_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Theta_3(\varrho,t_0)\right)B(\varrho)$$

If $\omega(\varrho)$ is a geodesic curve, then $\Theta_1(\varrho,t_0)=0$, and we get

$$\frac{\partial h(\varrho,t_0)}{\partial t}=-\frac{\kappa_2(\varrho)}{\kappa_1(\varrho)}\frac{\partial g(\varrho,t_0)}{\partial t} \tag{48}$$

and

$$\sinh\theta(\varrho)\ \Theta_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Theta_3(\varrho,t_0)=0$$

Then

$$\Theta_3(\varrho,t_0)=-\tanh\theta(\varrho)\ \Theta_2(\varrho,t_0) \tag{49}$$

So

$$\cosh\theta(\varrho)\ \Theta_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Theta_3(\varrho,t_0)\neq 0$$

using (49), we get

$$\frac{\partial f(\varrho,t_0)}{\partial t}\neq\varepsilon\frac{\partial h(\varrho,t_0)}{\partial t},\ \text{and}\ \ \kappa_2(\varrho)\neq 0 \tag{50}$$

In Eqn. (50) and using Eqns. (48), (49), together (35), we obtain

$$\frac{\partial f(\varrho,t)}{\partial t}=\varepsilon\frac{\partial h(\varrho,t)}{\partial t} \tag{51}$$

Combining Eqns. (50) and (51), the contradiction is obtained. □

Corollary 4.3. Let $\omega=\omega(\varrho)$ be a unit speed spacelike curve with moving Bishop frame in Minkowski 3-space $\mathcal{R}_1^3$. Then $T\eta\xi$ -spacelike Smarandache curves of the curve $\omega(\varrho)$ is not geodesic on the surface $\Psi(\varrho,t)$.

Proof. Let $\gamma(s(\varrho))$ be a spacelike $T\eta\xi$ -spacelike Smarandache curve on the surface $\Psi(\varrho,t)$. Then, from (29), $\Psi(\varrho,t)$ as follows:

$$\Psi(\varrho,t)=\frac{1}{\sqrt{3}}[T(\varrho)+\eta(\varrho)+\xi(\varrho)]+[f(\varrho,t)T(\varrho)+g(\varrho,t)\,\eta(\varrho)+h(\varrho,t)\xi(\varrho)]$$

If spacelike $T\eta\xi$-spacelike Smarandache curve is a parametric curve on this surface, then there exist a parameter $t=t_0$ such that $\Psi(\varrho,t_0)=\frac{1}{\sqrt{3}}[T(\varrho)+\eta(\varrho)+\xi(\varrho)]$ that is,

$$f(\varrho,t_0)=g(\varrho,t_0)=h(\varrho,t_0)=0 \tag{52}$$

Then

$$n(\varrho,t)=\Big[\frac{\partial h(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{3}}+\kappa_1(\varrho)f(\varrho,t)+\frac{\partial g(\varrho,t)}{\partial\varrho}\right)-\frac{\partial g(\varrho,t)}{\partial t}\left(\frac{\kappa_2(\varrho)}{\sqrt{3}}+\kappa_2(\varrho)f(\varrho,t)-\frac{\partial h(\varrho,t)}{\partial\varrho}\right)\Big]T(\varrho)+\Big[\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{-\kappa_2(\varrho)}{\sqrt{3}}-\kappa_2(\varrho)f(\varrho,t)+\frac{\partial h(\varrho,t)}{\partial\varrho}\right)+\frac{\partial h(\varrho,t)}{\partial t}\left(\varepsilon\kappa_1(\varrho)g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\sqrt{3}}\right)\Big]\eta(\varrho)-\Big[\frac{\partial g(\varrho,t)}{\partial t}\Big(\varepsilon\kappa_1(\varrho)g(\varrho,t)+\varepsilon\kappa_2(\varrho)h(\varrho,t)-\frac{\partial f(\varrho,t)}{\partial\varrho}+\frac{\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\sqrt{3}}\Big)+\frac{\partial f(\varrho,t)}{\partial t}\left(\frac{\kappa_1(\varrho)}{\sqrt{3}}+\kappa_1(\varrho)f(\varrho,t)+\frac{\partial g(\varrho,t)}{\partial\varrho}\right)\Big]\xi(\varrho) \tag{53}$$

If we write

$$\begin{cases} \Upsilon_1(\varrho,t_0)=\frac{\kappa_1(\varrho)}{\sqrt{3}}\frac{\partial h(\varrho,t_0)}{\partial t}+\frac{\kappa_2(\varrho)}{\sqrt{3}}\frac{\partial g(\varrho,t_0)}{\partial t} \\ \Upsilon_2(\varrho,t_0)=\frac{-\kappa_2(\varrho)}{\sqrt{3}}\frac{\partial f(\varrho,t_0)}{\partial t}+\frac{\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\sqrt{3}}\frac{\partial h(\varrho,t_0)}{\partial t} \\ \Upsilon_3(\varrho,t_0)=\frac{-\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\sqrt{3}}\frac{\partial g(\varrho,t_0)}{\partial t}-\frac{\kappa_1(\varrho)}{\sqrt{3}}\frac{\partial f(\varrho,t_0)}{\partial t} \end{cases} \tag{54}$$

we obtain

$$n(\varrho,t_0)=\Upsilon_1(\varrho,t_0)T(\varrho)+\Upsilon_2(\varrho,t_0)\eta(\varrho)+\Upsilon_3(\varrho,t_0)\xi(\varrho)$$

So, from Eqn. (34), we get

$$n(\varrho,t_0)=\Upsilon_1(\varrho,t_0)T(\varrho)+\left(\cosh\theta(\varrho)\ \Upsilon_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Upsilon_3(\varrho,t_0)\right)N(\varrho) + \left(\sinh\theta(\varrho)\ \Upsilon_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Upsilon_3(\varrho,t_0)\right)B(\varrho)$$

We know that $\omega(\varrho)$ is a geodesic curve if and only if

$$\begin{cases} \Upsilon_1(\varrho,v_0)=0 \\ \cosh\theta(\varrho)\ \Upsilon_2(\varrho,t_0)+\sinh\theta(\varrho)\ \Upsilon_3(\varrho,t_0)\neq 0 \\ \sinh\theta(\varrho)\ \Upsilon_2(\varrho,t_0)+\cosh\theta(\varrho)\ \Upsilon_3(\varrho,t_0)=0 \end{cases} \tag{55}$$

In Eqn. (55) and using Eqn. (54), we obtain

$$\frac{\partial h(\varrho,t_0)}{\partial t}=-\frac{\kappa_2(\varrho)}{\kappa_1(\varrho)}\frac{\partial g(\varrho,t_0)}{\partial t},\kappa_1(\varrho)\neq 0 \tag{56}$$

$$\frac{\partial f(\varrho,t)}{\partial t}\neq\frac{\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\kappa_2(\varrho)}\frac{\partial h(\varrho,t_0)}{\partial t} \tag{57}$$

and

$$\frac{\partial f(\varrho,t)}{\partial t}=\frac{-\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\kappa_1(\varrho)}\frac{\partial g(\varrho,t_0)}{\partial t},\kappa_1(\varrho)\neq 0 \tag{58}$$

From Eqns. (56) and (58), we obtain

$$\frac{\partial f(\varrho,t)}{\partial t}=\frac{-\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\kappa_1(\varrho)}\frac{\partial h(\varrho,t_0)}{\partial t} \tag{59}$$

From Eqns. (56) and (59), we get

$$\frac{\partial f(\varrho,t)}{\partial t}\neq\frac{-\varepsilon(\kappa_1(\varrho)+\kappa_2(\varrho))}{\kappa_1(\varrho)}\frac{\partial g(\varrho,t_0)}{\partial t} \tag{60}$$

Combining Eqns. (58) and (60), we have a contradiction. □

Now let us consider other types of the marching-scale functions $f(\varrho,t)$, $g(\varrho,t)$ and $h(\varrho,t)$.

(i) If we choose

$$\begin{cases} f(\varrho,t)=\sum_{k=1}^{r}a_{1k}\,l(\varrho)^k x(t)^k \\ g(\varrho,t)=\sum_{k=1}^{r}a_{21k}\,m(\varrho)^k y(t)^k \\ h(\varrho,t)=\sum_{k=1}^{r}a_{3k}\,n(\varrho)^k z(t)^k \end{cases}$$

Then we can simply express the sufficient condition for which the spacelike curve $\omega(\varrho)$ is $\eta\xi$-spacelike Smarandache iso-geodesic curve on the surface $\Psi(\varrho,t)$ as

$$\begin{cases} x(t_0)=y(t_0)=z(t_0)=0 \\ a_{31}\neq 0, n(\varrho)\neq 0\ and\ \frac{dz(t_0)}{dt}\neq 0 \\ a_{21}m(\varrho)=\tanh\theta(\varrho)\,a_{31}\,n(\varrho)\,\frac{dz(t_0)}{dt} \end{cases} \tag{61}$$

where $l(\varrho)$, $m(\varrho)$, $n(\varrho)$, $x(t)$, $y(t)$ and $z(t)$ are functions of class C^1, $a_{ij}\in\mathbb{R}$, $i=1,2,3$, $1\leq j\leq r$.

(ii) If we choose

$$\begin{cases} f(\varrho,t) = u\left(\sum_{k=1}^{r} a_{1k}\, l(\varrho)^k x(t)^k\right) \\ \mathrm{g}(\varrho,t) = v\left(\sum_{k=1}^{r} a_{21k}\, m(\varrho)^k y(t)^k\right) \\ h(\varrho,t) = w\left(\sum_{k=1}^{r} a_{3k}\, n(\varrho)^k z(t)^k\right) \end{cases}$$

Then we can write the sufficient condition for which the spacelike curve $\omega(\varrho)$ is $\eta\xi$ -spacelike Smarandache isogeodesic curve on the surface $\Psi(\varrho,t)$ as

$$\begin{cases} x(t_0) = y(t_0) = z(t_0) = u(0) = v(0) = w(0) = 0 \\ a_{31} \neq 0, n(\varrho) \neq 0\,,\ \frac{dz(t_0)}{dt} \neq 0, and\ w'(0) \neq 0 \\ a_{21} m(\varrho)\frac{dy(t_0)}{dx} v'(0) = \tanh\theta(\varrho)\, a_{31}\, n(\varrho)\ \frac{dz(t_0)}{dt}\ w'(0) \end{cases} \tag{62}$$

where $l(\varrho)$, $m(\varrho)$, $n(\varrho)$, $x(t)$, $y(t)$ and $z(t)$are C^1 functions. Also, the conditions for different types of marching-scale functions can be obtained by using the Eqns. (32) and (37).

Example 4.1. Let $\omega(\varrho) = \left(\frac{\sqrt{3}}{2}\cosh\varrho, \frac{\sqrt{3}}{2}\sinh\varrho, \frac{\varrho}{2}\right)$ be a spacelike curve parametrized by arc-length with timelike principalnormal,$(\varepsilon = -1)$. Then it is easy to show that

$T(\varrho) = \left(\frac{\sqrt{3}}{2}\sinh\varrho, \frac{\sqrt{3}}{2}\cosh\varrho, \frac{1}{2}\right), \kappa = \frac{\sqrt{3}}{2} \neq 0,\ \tau = -\frac{1}{2} \neq 0$ and $\theta(\varrho) = -\frac{\varrho}{2} + c$, $c = constant$. Here, we can take $c = 0$. Then, we get $\kappa_1 = \frac{\sqrt{3}}{2}\cosh\frac{\varrho}{2}$, $\kappa_2 = \frac{\sqrt{3}}{2}\sinh\frac{\varrho}{2}$. From Eqn. (2), we get

$\eta(\varrho) = \int \kappa_1(\varrho)T(\varrho)\, d\varrho$, $\xi(\varrho) = \int \kappa_2(\varrho)T(\varrho)\, d\varrho$, then we have

$$\eta(\varrho) = \left(\frac{3}{4}\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right], \frac{3}{4}\left[\frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right) + \sinh\left(\frac{\varrho}{2}\right)\right], \frac{\sqrt{3}}{2}\sinh\left(\frac{\varrho}{2}\right)\right)$$

$$\xi(\varrho) = \left(\frac{3}{4}\left[\sinh\left(\frac{\varrho}{2}\right) - \frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right)\right], \frac{-3}{4}\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right], \frac{-\sqrt{3}}{2}\cosh\left(\frac{\varrho}{2}\right)\right)$$

If we take $f(\varrho,t) = 0$, $\mathrm{g}(\varrho,t) = \tanh\left(\frac{\varrho}{2}\right)\ \sinh(t)$ and $\mathrm{h}(\varrho,t) = \sinh(t)$, we obtain a family of the surfaces with common spacelike curve $\omega(\varrho)$ as

$$\Psi(\varrho,t) = \left(\frac{\sqrt{3}}{2}\cosh(\varrho) + \frac{3}{4}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right] + \frac{3}{4}\sinh(t)\left[\sinh\left(\frac{\varrho}{2}\right) - \frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right)\right], \frac{\sqrt{3}}{2}\sinh(\varrho) + \frac{3}{4}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)\left[\frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right) + \sinh\left(\frac{\varrho}{2}\right)\right] - \frac{3}{4}\sinh(t)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right], \frac{\varrho}{2} + \frac{\sqrt{3}}{2}\sinh\left(\frac{\varrho}{2}\right)\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right) - \frac{\sqrt{3}}{2}\cosh\left(\frac{\varrho}{2}\right)\right)$$

where $0 \leq \varrho \leq 2\pi, -1 \leq t \leq 1$ (see Figure 6).

If we take $f(\varrho,t) = 0$, $\mathrm{g}(\varrho,t) = \tanh\left(\frac{\varrho}{2}\right)\ \sinh(t)$ and $\mathrm{h}(\varrho,t) = \sinh(t)$, $t_0 = 0$ then the Eqns. (32) and (35) are satisfied.

Thus, we obtain a family of the surfaces with common spacelike $\eta\xi$ -spacelike Smarandache geodesic curves as

$$\Psi_1(\varrho,t) = \left(\frac{3}{4\sqrt{2}}\left[\frac{1}{3}\left(\cosh\left(\frac{3\varrho}{2}\right) - \sinh\left(\frac{3\varrho}{2}\right)\right) + \cosh\left(\frac{\varrho}{2}\right) + \sinh\left(\frac{\varrho}{2}\right)\right] + \frac{3}{4}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right] + \frac{3}{4}\sinh(t)\left[\sinh\left(\frac{\varrho}{2}\right) - \frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right)\right], \frac{3}{4\sqrt{2}}\left[\frac{1}{3}\left(\sinh\left(\frac{3\varrho}{2}\right) - \cosh\left(\frac{3\varrho}{2}\right)\right) + \sinh\left(\frac{\varrho}{2}\right) - \cosh\left(\frac{\varrho}{2}\right)\right] + \frac{3}{4}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)\left[\frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right) + \sinh\left(\frac{\varrho}{2}\right)\right] - \frac{3}{4}\sinh(t)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right) + \cosh\left(\frac{\varrho}{2}\right)\right], \frac{\sqrt{3}}{2\sqrt{2}}\left[\sinh\left(\frac{\varrho}{2}\right) - \cosh\left(\frac{\varrho}{2}\right)\right] + \frac{\sqrt{3}}{2}\sinh\left(\frac{\varrho}{2}\right)\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right) - \frac{\sqrt{3}}{2}\sinh(t)\cosh\left(\frac{\varrho}{2}\right)\right)$$

where $0 \leq \varrho \leq 2\pi, -1 \leq t \leq 1$ (see Figure 7).

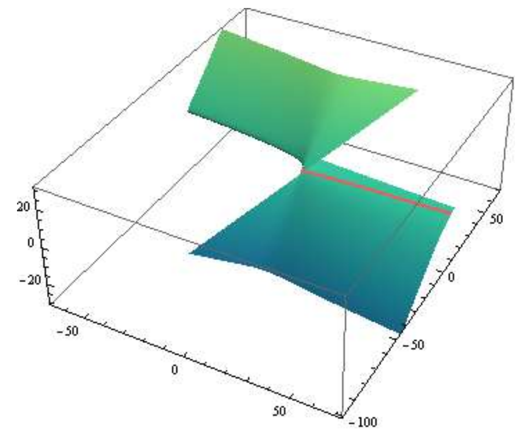

Figure 6. $\Psi(\varrho,t)$ as a family of surfaces and curve $\omega(\varrho)$.

If we take $f(\varrho,t)=0$, $g(\varrho,t)=\sum_{k=1}^{3}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)^{k}\sin^{k}(t)$ and $h(\varrho,t)=\sum_{k=1}^{3}\cosh^{k}\left(\frac{\varrho}{2}\right)\sinh^{k}(t)$, and $t_0=0$, then the Eqn.(60) is satisfied. Thus, we obtain a family of the surfaces with common spacelike $\eta\xi$-spacelike Smarandache geodesic curves as

$$\Psi_2(\varrho,t)=\Bigg(\frac{3}{4\sqrt{2}}\left[\frac{1}{3}\left(\cosh\left(\frac{3\varrho}{2}\right)-\sinh\left(\frac{3\varrho}{2}\right)\right)+\cosh\left(\frac{\varrho}{2}\right)+\sinh\left(\frac{\varrho}{2}\right)\right]+\frac{3}{4}\left(\sum_{k=1}^{3}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)^{k}\sin^{k}(t)\right)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right)+\cosh\left(\frac{\varrho}{2}\right)\right]+\frac{3}{4}\left(\sum_{k=1}^{3}\cosh^{k}\left(\frac{\varrho}{2}\right)\sinh^{k}(t)\right)\left[\sinh\left(\frac{\varrho}{2}\right)-\frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right)\right],\frac{3}{4\sqrt{2}}\left[\frac{1}{3}\left(\sinh\left(\frac{3\varrho}{2}\right)-\cosh\left(\frac{3\varrho}{2}\right)\right)+\sinh\left(\frac{\varrho}{2}\right)-\cosh\left(\frac{\varrho}{2}\right)\right]+\frac{3}{4}\left(\sum_{k=1}^{3}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)^{k}\sin^{k}(t)\right)\left[\frac{1}{3}\sinh\left(\frac{3\varrho}{2}\right)+\sinh\left(\frac{\varrho}{2}\right)\right]-\frac{3}{4}\left(\sum_{k=1}^{3}\cosh^{k}\left(\frac{\varrho}{2}\right)\sinh^{k}(t)\right)\left[\frac{1}{3}\cosh\left(\frac{3\varrho}{2}\right)+\cosh\left(\frac{\varrho}{2}\right)\right],\frac{\sqrt{3}}{2\sqrt{2}}\left[\sinh\left(\frac{\varrho}{2}\right)-\cosh\left(\frac{\varrho}{2}\right)\right]+\frac{\sqrt{3}}{2}\sinh\left(\frac{\varrho}{2}\right)\left(\sum_{k=1}^{3}\left(\tanh\left(\frac{\varrho}{2}\right)\sinh(t)\right)^{k}\sin^{k}(t)\right)-\frac{\sqrt{3}}{2}\left(\sum_{k=1}^{3}\cosh^{k}\left(\frac{\varrho}{2}\right)\sinh^{k}(t)\right)\cosh\left(\frac{\varrho}{2}\right)\Bigg)$$

where $0\le\varrho\le 2\pi, -1\le t\le 1$ (see Figure 7).

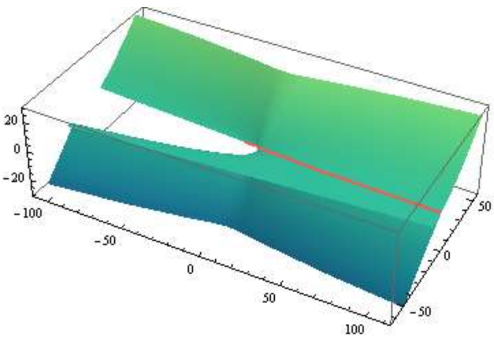

Figure 7. $\Psi_1(\varrho,t)$ as a family of surfaces and its $\eta\xi$ -spacelike Smarandache geodesic curves.

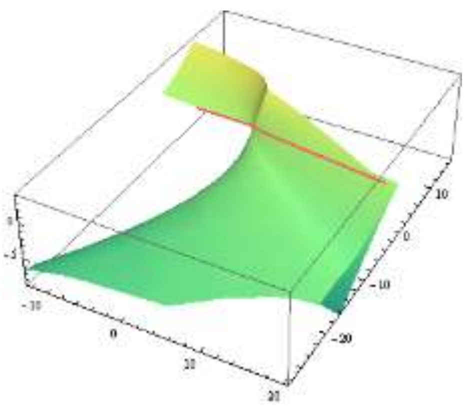

Figure 8. $\Psi_2(\varrho,t)$ as a family of surfaces and its $\eta\xi$ -spacelike Smarandache geodesic curves.

5. Conclusion

In this article, we found the necessary and sufficient conditions that make a special spacelike Smarandache curve is Iso-asymptotic, and the necessary and sufficient conditions that make it was Iso-geodesic by using the Bishop frame in Minkowski 3-space E_1^3. In the future, we need to discuss whether it is possible or impossible to create the necessary conditions that make this curve is an Iso-asymptotic or Iso-geodesic in Minkowski 7-space.

References

Akyildiz, F. T., Kasap, E. & Orbay, K. (2008). A generalization of surfaces family with common spatial geodesic. *Applied Mathematics and Computation,* (201), 30-36. https://doi.org/10.1016/j.amc.2008.01.016

Ali, A. T. (2010). Special Smarandache Curves in Euclidean Space. *Journal of Mathematical Combinatorics, 2,* 30-36.

Bayram, E., Güler, F. & Kasap, E. (2012). Parametric representation of a surface pencil with a common asymptotic curve. *Comput. Aided Des*. https://doi.org/10.1016/j.cad.2012.02.007

Bektas, Ö. & Yüce, S. (2013). Special Smarandache curves according to Darboux frame in Euclidean 3- Space. *Romanian Journal of Mathematics and Computer Science,* (3), 48-59. https://www.researchgate.net/publication/268368549

Bükcü, B. & Karacan, M. K. (2010). Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space. Selçuk. *J. Appl. Math, 11*(1), 15-25.

Cetin, M. & Kocayiğit, H. (2013). On the quaternionic Smarandache curves in Euclidean 3-space. Int. J. Contemp. *Math. Sciences, 8*(3), 139-150. http://www.oalib.com/paper/2660649.XTVWeo4zbIU. https://doi.org/10.12988/ijcms.2013.13014

Cetin, M., Tunçer, Y. & Karacan, M. K. (2014). Smarandache curves according to Bishop frame in Euclidean 3-space. *General Mathematics Notes, 20*(2), 50-66. http://www.geman.in/volumes/all volumes 2014.

Karakus, F. & Yayli, Y. (2016). The Fermi-Walker Derivative on the Spherical Indicatrix of Spacelike Curve in Minkowski 3-Space. *Adv. Appl. Cliff. Algebr, 26*, 625-644. https://doi.org/10.1007/s00006-015-0635-9

Li, C. Y. Wang, R. H. & Zhu, C. G. (2011). Parametric representation of a surface pencil with a common line of curvature. *Comput. Aided Des, 9*(43), 1110-1117. https://doi.org/ 10.1016/j.cad.2011.05.001.

López, R. (2014). Diffierential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, *International Electronic Journal of Geometry, 7*(1), 44-107. arXiv:0810.3351 [math. DG].

Okuyucu O. Z., Değirmen, C. & Yldz, Ö. G. (2019). Smarandache Curves in three dimensional Lie groups. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68*(1), 1175-1185. https://doi.org/10.31801/cfsuasmas.511677

O'Neill, B. (1983). Semi-Riemannian Geometry, Academic Press, New York.

O'Neill, B. (1966). Elementary Differential Geometry, Academic Press Inc., New York. https://doi.org/10.1016/B978-1-4832-3170-9.50011-7

Tai, C. L., Tang, K. & Wang, G. J. (2004). Parametric representation of a surface pencil with a common spatial geodesic, *Comput. Aided Des, 5*(36), 447-459. https://doi.org/10. 1016/S0010-4485(03)00117-9

Tasköprü, K. & Tosun, M. (2012). Smarandache Curves According to Sabban Frame on S 2, *Boletim da Sociedade Paraneanse de Matematica, 32*(1), 51-59. http:// doi.org/10.5281/zenodo.32333

Turgut, M. & Yilmaz, S. (2008). Smarandache curves in Minkowski space-time. *International Journal of Mathematical Combinatorics, 3,* 51-55.

Yilmaz, B.et al. (2016). Extended Rectifying Curves in Minkowski 3-Space. *Adv. Appl. Cli_. Algebr, 26*, 861-872. https://doi.org/10.1007/s00006-015-0637-724