

# The Marshall–Olkin Generalized Inverse Weibull Distribution: Properties and Application

Hamdy M. Salem<sup>1</sup>

<sup>1</sup> School Department of Statistics, Faculty of Commerce, Al-Azher University, Egypt & Community College in Buraidah, Qassim University, Saudi Arabia

Correspondence: Hamdy M. Salem, School Department of Statistics, Faculty of Commerce, Al-Azher University, Egypt & Community College in Buraidah, Qassim University, Saudi Arabia. E-mail: d.hamdysalm@yahoo.com

Received: Dec.6, 2018

Accepted: Dec.17, 2018

Online Published: January 3, 2019

doi:10.5539/mas.v13n2p54

URL: <https://doi.org/10.5539/mas.v13n2p54>

## Abstract

In this paper, a new distribution namely, The Marshall–Olkin Generalized Inverse Weibull Distribution is illustrated and studied. The new distribution is very flexible and contains sub-models such as inverse exponential, inverse Rayleigh, Weibull, inverse Weibull, Marshall–Olkin inverse Weibull and Fréchet distributions. Also, the hazard function of the new distribution can produce variety of forms: an increase, a decrease and an upside-down bathtub. Some properties such as hazard function, quintile function, entropy, moment generating function and order statistics are obtained. Different estimation approaches namely, maximum likelihood estimators, interval estimators, least square estimators, fisher information matrix and asymptotic confidence intervals are described. To illustrate the superior performance of the proposed distribution, a simulation study and a real data analysis are investigated against other models.

**Keywords:** Marshall–Olkin family, Inverse Weibull Distribution, Maximum likelihood estimator, Hazard function, Order Statistics

## 1. Introduction

Recently, many authors have been interested in the Weibull distribution which is extensively used over the past decades for reliability studies and is adequate for modeling monotone data. In some studies, such as breast cancer data and mortality of lung, many authors showed that the Inverse Weibull IW distribution is an appropriate model for this kind of data because its hazard function is non-monotone or unimodal (see for example (Bennett 1983), (Langlands 1979), (Kundu and Howladu 2010)). Several authors studied this distribution in different fields as (Drapella 1993), (Mudholkar and Kollia 1994).

By adding a new shape parameter to the distribution function of the IW distribution, the new distribution is called the Generalized Inverse Weibull GIW distribution. (Felipe et al. 2011) discussed the properties of the GIW distribution including hazard function, survival function, central and non-central moments and moment generating function. They provided maximum likelihood estimators, asymptotic confidence intervals and fisher information matrix as well. The cumulative density function cdf and probability density function pdf of the GIW distribution respectively are

$$F(x) = e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta} \quad ; \quad x, \alpha, \beta, \lambda > 0 \quad (1)$$

And

$$f(x) = \lambda \beta \alpha^\beta x^{-(\beta+1)} e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta} \quad ; \quad x, \alpha, \beta, \lambda > 0 \quad (2)$$

Where  $\beta$  and  $\lambda$  are the shape parameters.

In the recent past, to analyze complex data and obtaining a better fit, researchers were interested in generating new continuous distributions. Also, comprehensive studies used Marshall–Olkin method in generating new distributions. Some statistical properties of these new distributions were illustrated. For example, Marshall–Olkin logistic processes has been introduced by (Alice and Jose 2005), (Gui 2013) studied Marshall–Olkin power lognormal distribution. Marshall–Olkin extended Weibull distribution was studied by (Cordeiro and Lemonte

2013). (Ghitany et al. 2007) introduced Marshall–Olkin Extended Lomax distribution.

(Marshall and Olkin 1997) have added a parameter to the base line cdf in exponential and Weibull families as follows

$$G(x) = \frac{F(x)}{\theta - (\theta - 1)F(x)} \quad ; \quad 0 < x < \infty, \theta > 0 \quad (3)$$

This study aims at providing a new extension of weibull distribution namely, The Marshall–Olkin Generalized Inverse Weibull (MOGIW for short) distribution. It contains more significant fits than some distributions such as Marshall–Olkin extended inverse Weibull, inverse Weibull, Fréchet, inverse Rayleigh and inverse exponential being sub-models from the proposed distribution.

The motivations to introduce the MOGIW distribution are: (i) some known distributions are sub-models from it; (ii) in case the hazard function is non-monotone or unimodal, the MOGIW distribution is a superior fit to other common lifetime distributions; (iii) the MOGIW distribution can use in different areas such as breast cancer and wear-out periods in medical and physical fields respectively; (iv) to illustrate the applicability of the MOGIW distribution by conducting study of a numerical and an actual data.

This paper is organized as follows: the model definition and its statistical properties including hazard function, quintile function, moments, moment generating function and order statistics are discussed in section 2. Then, different estimation approaches namely, maximum likelihood estimators, asymptotic confidence intervals based on asymptotic normality, fisher information matrix, interval estimators and least square estimators are described in section 3. Finally, in section 4, a simulation study and a real date set are conducted to illustrate the performance of the proposed distribution.

## 2. New Model and its Properties

in this section, the cdf and pdf for the proposed distribution and some of its statistical properties are discussed. Let  $X$  be a random variable with cdf of the GIW distribution which in (1) and by substitution in (3), then, the cdf of the MOGIW distribution is given as

$$G(x) = \frac{e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}} \quad ; \quad x, \alpha, \beta, \lambda, \theta > 0 \quad (4)$$

and the corresponding pdf is

$$g(x) = \frac{\lambda \beta \theta \alpha^\beta x^{-(\beta+1)} e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}}{\left[ \theta - (\theta - 1)e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta} \right]^2} \quad ; \quad x, \alpha, \beta, \lambda, \theta > 0 \quad (5)$$

Figure 1 and 2 describe different forms to the pdf and cdf respectively of the MOGIW distribution at different values for the parameters  $\alpha, \beta, \theta$  and  $\lambda$ .

The distribution which in (4) has special cases as the following

- (i) If  $\lambda = 1$ , then, the MOGIW distribution reduces to the Marshall–Olkin extended inverse Weibull distribution which studied by (Okasha et al 2017) with parameters  $\beta, \theta$  and  $\alpha$ .
- (ii) If  $\theta = \beta = 1$ , then, it reduces to the exponentiated Inverse Weibull distribution.
- (iii) If  $\lambda = \theta = 1$ , then, the MOGIW distribution reduces to the Inverse Weibull distribution.
- (iv) If  $\theta = 1, \beta = 2$  and  $\lambda = 1$ , then, it reduces to the Inverse Rayleigh distribution.
- (v) If  $\beta = \theta = 1$  and  $\lambda = 1$ , then, it reduces to the Inverse exponential distribution.

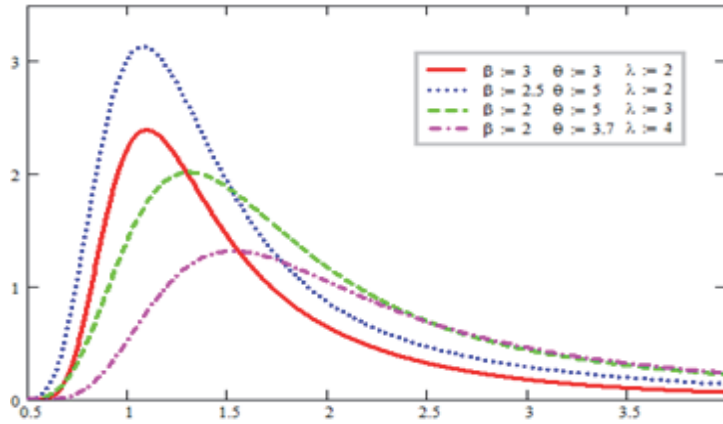


Figure 1. The pdf forms of the MOGIW distribution at  $\alpha = 1$  and different values of the parameters  $\beta, \theta$  and  $\lambda$

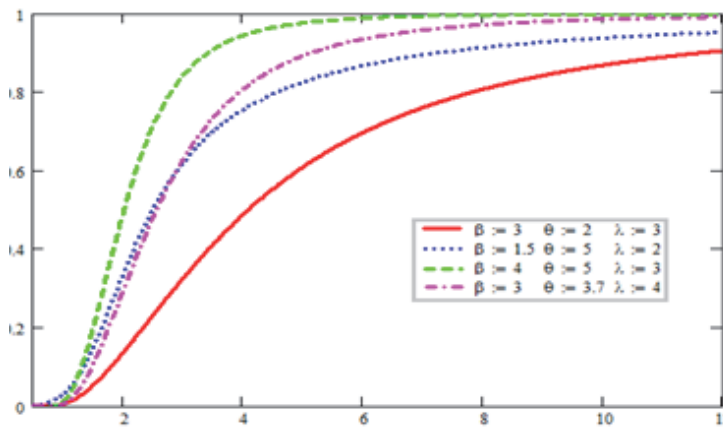


Figure 2. The cdf forms of the MOGIW distribution at  $\alpha = 1$  and different values of the parameters  $\beta, \theta$  and  $\lambda$

Using the expansion  $(1 - z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k + 1)z^i}{\Gamma(k)i!}$  where  $|z| < 1$ ,  $k > 0$  and  $\Gamma(\cdot)$  is a gamma function, then,

the equations (4) and (5) can be written respectively as

$$G(x) = \theta^2 \sum_{i=0}^{\infty} (i + 1)^{-1} \left( \frac{\theta}{\theta - 1} \right)^i e^{-\lambda(i+1)\left(\frac{\alpha}{x}\right)^\beta} \tag{6}$$

and

$$g(x) = \lambda \beta \theta^3 \alpha^\beta \sum_{i=0}^{\infty} (i + 1)^{-1} \left( \frac{\theta}{\theta - 1} \right)^i x^{-(\beta+1)} e^{-\lambda(i+1)\left(\frac{\alpha}{x}\right)^\beta} \tag{7}$$

The survival function  $R(x)$  and the hazard rate function  $h(x)$  of the MOGIW distribution respectively are

$$R(x) = 1 - \frac{e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}} \tag{8}$$

and

$$h(x) = \frac{f(x)}{R(x)} = \frac{\lambda \beta \theta \alpha^\beta x^{-(\beta+1)} e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta}}{\left[ \theta \left( e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta} - \theta \right) - e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta} \right] \cdot \left[ e^{-\lambda \left(\frac{\alpha}{x}\right)^\beta} - 1 \right]} \tag{9}$$

2.1 Moments

In any statistical analysis, moments are essential, important and helpful in applications. The  $r^{th}$  moment  $\mu'_r$  of the MOGIW distribution is

$$\begin{aligned} E(X)^r &= \int_0^\infty x^r g(x) dx = \frac{\beta \alpha^\beta}{\theta} \sum_{i=0}^\infty (i+1) \left(\frac{\theta-1}{\theta}\right)^i \int_0^\infty x^{r-(\beta+1)} e^{-\lambda (i+1) \left(\frac{\alpha}{x}\right)^\beta} dx \\ &= \frac{1}{\theta} \alpha^{r+1} \sum_{i=0}^\infty (i+1) \left(\frac{\theta-1}{\theta}\right)^i [\lambda (i+1)]^{\frac{r}{\beta}+1} \Gamma\left(1 - \frac{r}{\beta}\right) \end{aligned} \tag{10}$$

When  $r$  equal one, two, three and four in (10), the  $J^{st}$  four moments for MOGIW distribution are given respectively. Also, by using the relation between the central and non-central moments which is

$$\mu_r = E(X - \mu)^r = \sum_{j=0}^r \binom{r}{j} (-1)^{j-1} \mu'_j \mu^{r-j}, \text{ then, the central moments } \mu_r \text{ are computed.}$$

2.2 Quantile Function and Median

By inverting (4) and suppose that  $Q(\eta), 0 < \eta < 1$  denote the quantile function of the MOGIW distribution, then,  $Q(\eta)$  is given by

$$x_u = \alpha \sqrt[\beta]{\frac{\ln\left(\frac{\eta\theta - \eta + 1}{\eta\theta}\right)}{\lambda}} \tag{11}$$

Setting  $\eta = (0.25, 0.5, 0.75)$  in (11), then, the 1st, 2nd and 3rd quartiles of the MOGIW distribution can be obtained. So, the median is

$$x_{0.5} = \alpha \sqrt[\beta]{\frac{\ln\left(\frac{0.5\theta + 0.5}{0.5\theta}\right)}{\lambda}} \tag{12}$$

2.3 Moment Generating Function

From (9), the moment generating function can be obtained as follows

$$\begin{aligned} M(t) &= E(e^{tx}) = \int_0^\infty e^{tx} g(x) dx \\ &= \frac{\beta \alpha^\beta}{\theta} \sum_{i=0}^\infty (i+1) \left(\frac{\theta-1}{\theta}\right)^i \int_0^\infty x^{-(\beta+1)} e^{tx - \lambda (i+1) \left(\frac{\alpha}{x}\right)^\beta} dx \end{aligned} \tag{13}$$

2.4 Order statistics

In statistical theory and application fields, the order statistics are appeared. In a random sample of size  $n$  from the MOGIW distribution, the density function  $f_{r:n}(x)$  of the  $j^{th}$  order statistic  $X_{r:n}$  can be written as

$$\begin{aligned}
 f_{r:n}(x) &= \frac{1}{\beta(r, n-r+1)} g(x) [G^{r-1}(x)][1-G(x)]^{n-r} \\
 &= \frac{1}{\beta(r, n-r+1)} \sum_{k=0}^{\infty} (-1)^k \binom{n-r}{k} g(x) [G(x)]^{k+r-1}
 \end{aligned}
 \tag{14}$$

### 3. Parameter Estimations

This section concerns with the estimation of parameters for MOGIW distribution using maximum likelihood estimator MLEs and least square estimator LSs methods. Also, fisher information matrix and asymptotic confidence intervals are discussed.

#### 3.1 Maximum likelihood estimators

Let  $x_1, x_2, \dots, x_n$  be a random sample from MOGIW distribution  $X \approx MOGIW(\underline{\tau})$ , then, the likelihood and log-likelihood functions for  $\underline{\tau} = (\lambda, \beta, \theta, \alpha)$  can be written respectively as

$$L = \frac{(\lambda \beta \theta \alpha^\beta)^n \prod_{j=1}^n x_j^{-(\beta+1)} \prod_{j=1}^n e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta}}{\prod_{j=1}^n \left[ \theta - (\theta - 1) e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta} \right]^2}
 \tag{15}$$

and

$$\begin{aligned}
 \ell &= n \ln(\lambda \beta \theta \alpha^\beta) - (\beta + 1) \sum_{j=1}^n \ln(x_j) - \lambda \left(\frac{\alpha}{x_j}\right)^\beta \\
 &\quad - 2 \sum_{j=1}^n \ln \left( \theta - (\theta - 1) e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta} \right)
 \end{aligned}
 \tag{16}$$

The first derivative of  $\ell$  for the parameters  $\lambda, \beta, \theta$  and  $\alpha$  respectively are

$$\begin{aligned}
 \frac{\partial \ell}{\partial \alpha} &= n \lambda \beta^2 \theta \alpha^{\beta-1} - \frac{\lambda \beta}{\alpha} \left(\frac{\alpha}{x_j}\right)^\beta + \frac{2(\theta-1)\lambda \beta}{\alpha} \left(\frac{\alpha}{x_j}\right)^\beta \\
 &\quad \times \sum_{j=1}^n s
 \end{aligned}
 \tag{17}$$

$$\begin{aligned}
 \frac{\partial \ell}{\partial \beta} &= n \lambda \beta \theta \alpha^\beta \ln(\alpha) + n \lambda \theta \alpha^\beta - \sum_{j=1}^n \ln(x_j) - \lambda \left(\frac{\alpha}{x_j}\right)^\beta \ln\left(\frac{\alpha}{x_j}\right) \\
 &\quad + 2 \lambda (\theta - 1) \lambda \sum_{j=1}^n \left(\frac{\alpha}{x_j}\right)^\beta \ln\left(\frac{\alpha}{x_j}\right) s
 \end{aligned}
 \tag{18}$$

$$\frac{\partial \ell}{\partial \theta} = n \lambda \beta \alpha^\beta - 2 \sum_{j=1}^n s
 \tag{19}$$

$$\frac{\partial \ell}{\partial \lambda} = n \beta \theta \alpha^\beta - \left(\frac{\alpha}{x_j}\right)^\beta - 2(\theta - 1) \left(\frac{\alpha}{x_j}\right)^\beta \sum_{j=1}^n s \quad (20)$$

$$\text{Where } s = \frac{e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta}}{\left(\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta}\right)}$$

By solving the last 4 nonlinear equations iteratively after equal them by zero, then, the MLEs of the parameters are found.

Now, to construct the elements of asymptotic Fisher information matrix, the second derivative of  $\ell$  for the parameters  $\lambda, \beta, \theta$  and  $\alpha$  respectively are obtained and inverting the observed information matrix.

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \alpha^2} &= n \lambda \beta^2 (\beta - 1) \theta \alpha^{\beta-2} - \frac{\lambda \beta (\beta - 1)}{\alpha^2} \left(\frac{\alpha}{x_j}\right)^\beta + \frac{2(\theta - 2) \lambda \beta}{\alpha^2} \\ &\times \sum_{j=1}^n \left(\frac{\alpha}{x_j}\right)^\beta s \left(\beta - 1 - \beta \lambda \left(\frac{\alpha}{x_j}\right)^\beta (1 - s)\right) \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \beta^2} &= n \alpha^\beta \ln(\alpha) \theta \lambda (\beta \ln(\alpha) + 2) + \lambda \left(\frac{\alpha}{x_j}\right)^\beta \ln\left(\frac{\alpha}{x_j}\right)^2 \\ &- \sum_{j=1}^n t s \left(1 - \lambda \left(\frac{\alpha}{x_j}\right)^\beta - \lambda (\theta - 1) s \left(\frac{\alpha}{x_j}\right)^\beta\right) \end{aligned} \quad (22)$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = 2 \sum_{j=1}^n \left( \frac{1 - e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta}}{\left(\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_j}\right)^\beta}\right)} \right)^2 \quad (23)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = 2 \sum_{j=1}^n \frac{t s + t s^2}{\lambda \ln\left(\frac{\alpha}{x_j}\right)^2} \quad (24)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = n \alpha^{\beta-1} \beta \lambda (\ln(\beta) \beta + 2) - \lambda \left( \frac{\alpha}{x_j} \right)^\beta \ln \left( \frac{\alpha}{x_j} \right) - \frac{\lambda \left( \frac{\alpha}{x_j} \right)^\beta}{\alpha} - \frac{\beta}{\alpha} \sum_{j=1}^n \left[ \frac{t s}{\ln \left( \frac{\alpha}{x_j} \right)} + \frac{t s}{\ln \left( \frac{\alpha}{x_j} \right)^2} - \frac{t s}{\ln \left( \frac{\alpha}{x_j} \right)} + \frac{t s}{\ln \left( \frac{\alpha}{x_j} \right)^2} \right] \tag{25}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = n \alpha^{\beta-1} \beta^2 \lambda - \frac{\beta}{\alpha} \sum_{j=1}^n \left[ 2 \lambda \left( \frac{\alpha}{x_j} \right)^\beta s + \frac{t s e^{-\lambda \left( \frac{\alpha}{x_j} \right)^\beta}}{\ln \left( \left( \frac{\alpha}{x_j} \right) \right)} \left( 1 - e^{-\lambda \left( \frac{\alpha}{x_j} \right)^\beta} \right) \right] \tag{26}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{\alpha^\beta \beta^2}{\alpha} (n \theta - \beta) - \frac{\beta}{\alpha} \sum_{j=1}^n \frac{1}{\ln \left( \frac{\alpha}{x_j} \right)^2} \left( \frac{t s}{\lambda} - t s - t s^2 (\theta - 1) \right) \tag{27}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \theta} = n \alpha^\beta \beta \lambda - 2 \sum_{j=1}^n s \tag{28}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \lambda} = n \alpha^\beta \beta \theta - \left( \frac{\alpha}{x_j} \right)^\beta - 2 \sum_{j=1}^n \left( \frac{\alpha}{x_j} \right)^\beta (\theta - 1) s \tag{29}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = n \alpha^\beta \beta - \sum_{j=1}^n \left( \frac{\alpha}{x_j} \right)^\beta \left[ 2 s - (2 \theta - 2) \frac{e^{-\lambda \left( \frac{\alpha}{x_j} \right)^\beta} \left( 1 - e^{-\lambda \left( \frac{\alpha}{x_j} \right)^\beta} \right)}{\left( \theta - (\theta - 1) e^{-\lambda \left( \frac{\alpha}{x_j} \right)^\beta} \right)^2} \right] \tag{30}$$

$$\text{Where } s = \frac{e^{-\lambda\left(\frac{\alpha}{x_j}\right)^\beta}}{\left(\theta - (\theta - 1)e^{-\lambda\left(\frac{\alpha}{x_j}\right)^\beta}\right)} \text{ and } t = \lambda(2\theta - 2)\left(\frac{\alpha}{x_j}\right)^\beta \ln\left(\frac{\alpha}{x_j}\right)^2$$

By solving the inverse dispersion matrix analytically, then, the asymptotic variance and covariance matrix of the MLs  $\hat{\lambda}, \hat{\beta}, \hat{\theta}$  and  $\hat{\alpha}$  for the parameters  $\lambda, \beta, \theta$  and  $\alpha$  respectively is yielded.

### 3.2 Asymptotic Confidence Intervals

To construct approximate confidence intervals for the parameters, the expected information matrix is one of the mean requirement for derive the confidence interval estimates. (Lindgren 1976) showed that for the large samples and under certain regularity conditions, the MLs say,  $\hat{\tau} = (\hat{\lambda}, \hat{\beta}, \hat{\theta}, \hat{\alpha})$  for the parameters  $\tau = (\lambda, \beta, \theta, \alpha)$  are asymptotically normally distributed with mean  $\hat{\tau}$  and variance  $\sigma^2(\hat{\tau})$  .. Now, let  $x_1, x_2, \dots, x_n$  be a random sample from MOGIW distribution  $X \approx MOGIW(\tau)$ , then, the lower and upper confidence intervals for the parameters with approximate  $100\%(1 - \gamma)$  percentage are

$$\begin{aligned} \hat{\alpha} \pm Z_{\gamma/2} \sigma(\hat{\alpha}), & \quad \hat{\beta} \pm Z_{\gamma/2} \sigma(\hat{\beta}), \\ \hat{\theta} \pm Z_{\gamma/2} \sigma(\hat{\theta}), & \quad \hat{\lambda} \pm Z_{\gamma/2} \sigma(\hat{\lambda}) \end{aligned} \tag{31}$$

### 3.3 Least Square Estimators

Swain et al. (1988) [20] used an originally least-square method to parameter estimations for Beta distributions. This method depends on minimizing the distance between the vector of order statistics and its corresponding expected values. Let  $x_1, x_2, \dots, x_n$  be a random sample of size n from the MOGIW distribution function

$G(X_i)$  and  $x_{(1)}, x_{(2)}, \dots, x_{(n)}$  are order statistics of the observed sample  $E[G(X_{(i)})] = \frac{i}{n+1}$ , then, the least-square estimator  $\tilde{\tau} = (\tilde{\lambda}, \tilde{\beta}, \tilde{\theta}, \tilde{\alpha})$  for the parameters  $\tau = (\lambda, \beta, \theta, \alpha)$  can obtained by differentiate the following equation with respect to  $\lambda, \beta, \theta$  and  $\alpha$  :

$$\sum_{i=1}^n \left( \frac{e^{-\lambda\left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\left(\theta - (\theta - 1)e^{-\lambda\left(\frac{\alpha}{x_{(i)}}\right)^\beta}\right)} - \frac{i}{n+1} \right)^2 = 0 \tag{32}$$

So, (6) gives



$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^n 2 \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} - \frac{i}{n+1} \right) \cdot \left( -\frac{\lambda \beta}{\alpha} \left(\frac{\alpha}{x_{(i)}}\right)^\beta \right) \times \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} + (\theta - 1) \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\left[ \theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \right]^2} \right)^2 \tag{33}$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^n 2 \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} - \frac{i}{n+1} \right) \cdot \left( -\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta \ln \left(\frac{\alpha}{x_{(i)}}\right) \right) \times \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} + (\theta - 1) \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\left[ \theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \right]^2} \right)^2 \tag{34}$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^n -2 \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} - \frac{i}{n+1} \right) \cdot \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \left( 1 - e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \right)}{\left[ \theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \right]^2} \tag{35}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^n 2 \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} - \frac{i}{n+1} \right) \cdot \left( -\left(\frac{\alpha}{x_{(i)}}\right)^\beta \right) \times \left( \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}} + (\theta - 1) \frac{e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta}}{\left[ \theta - (\theta - 1)e^{-\lambda \left(\frac{\alpha}{x_{(i)}}\right)^\beta} \right]^2} \right)^2 \tag{36}$$

By solving the last four system equations in  $\lambda, \beta, \theta$  :and  $\alpha$  numerically, then, the least square estimators can be obtained.

**4. Data Analysis with Numerical Comparisons and Empirical Application**

In this section, a simulation study and real date set are conducted to illustrate the performance of the proposed distribution. To obtain a numerical illustration for the last theoretical results, MATHCAD software is used. 1000 samples generated from MOGIW distribution with parameters  $\lambda, \beta, \theta$  and  $\alpha$  at various size samples. MLEs, Relative bias (R Bias) and mean square error (MSE) are evaluated for the MOGIW distribution parameters.

Where R Bias =  $\left| \frac{\hat{\epsilon} - \epsilon}{\epsilon} \right|$  and MSE =  $E(\hat{\epsilon} - \epsilon)^2$ . They are included in table (1) and them values are decreasing

by increasing the sample size n.

Table 1. MLES, R BIAS and MSE of parameters estimates

$\alpha$	$\beta$	$\theta$	$\lambda$	n	$\hat{\alpha}$			$\hat{\beta}$			$\hat{\theta}$			$\hat{\lambda}$		
					MLE	R Bias	MSE	MLE	R Bias	MSE	MLE	R Bias	MSE	MLE	R Bias	MSE
0.5	1	0.5	2	25	0.487	-0.010	0.010	0.990	-0.010	0.010	0.433	-0.010	0.010	1.987	-0.013	0.014
				40	0.491	-0.010	0.010	0.990	-0.010	0.010	0.469	-0.011	0.010	1.991	-0.013	0.014
				100	0.490	-0.010	0.010	0.991	-0.009	0.010	0.482	-0.018	0.009	1.990	-0.013	0.014
				200	0.490	-0.009	0.010	0.993	-0.007	0.010	0.489	-0.031	0.008	1.990	-0.012	0.014
				400	0.490	-0.013	0.006	0.997	-0.003	0.010	0.490	-0.067	0.000	1.990	-0.016	0.010
				1000	0.490	-0.010	0.010	0.996	-0.004	0.010	0.878	-0.031	0.009	1.490	-0.013	0.014
0.5	1	1	1.5	25	0.489	-0.011	0.010	0.998	-0.002	0.010	0.904	-0.036	0.009	1.489	-0.014	0.014
				40	0.488	-0.012	0.009	1.006	0.006	0.010	0.938	-0.062	0.006	1.488	-0.015	0.013
				100	0.489	-0.011	0.008	1.015	0.015	0.009	0.964	-0.096	0.005	1.489	-0.014	0.012
				200	0.490	-0.007	0.004	1.022	0.022	0.008	0.969	-0.122	0.010	1.493	-0.010	0.008
				400	0.490	-0.010	0.010	2.990	-0.010	0.010	0.925	-0.011	0.010	1.490	-0.013	0.014
				1000	0.490	-0.010	0.010	2.990	-0.010	0.010	0.962	-0.012	0.010	1.490	-0.013	0.014
0.5	3	1	1.5	100	0.489	-0.011	0.010	2.991	-0.009	0.010	0.976	-0.024	0.009	1.489	-0.014	0.014
				200	0.490	-0.008	0.009	2.992	-0.008	0.010	0.988	-0.038	0.008	1.492	-0.011	0.013
				400	0.490	-0.008	0.004	2.994	-0.006	0.010	0.989	-0.075	0.001	1.492	-0.011	0.008
				1000	0.489	-0.011	0.010	2.993	-0.007	0.010	2.901	-0.025	0.010	1.989	-0.014	0.014
				2000	0.490	-0.009	0.010	2.994	-0.006	0.010	2.927	-0.029	0.010	1.990	-0.013	0.014
				5000	0.488	-0.011	0.009	2.997	-0.003	0.010	2.962	-0.038	0.009	1.989	-0.014	0.013
0.5	3	3	2	100	0.490	-0.005	0.008	3.006	0.006	0.009	2.971	-0.073	0.001	1.995	-0.009	0.012
				200	0.490	-0.004	0.003	3.005	0.005	0.010	2.975	-0.099	0.029	1.996	-0.007	0.007
				400	0.490	-0.004	0.003	3.005	0.005	0.010	2.975	-0.099	0.029	1.996	-0.007	0.007
				1000	0.489	-0.011	0.010	2.993	-0.007	0.010	2.901	-0.025	0.010	1.989	-0.014	0.014
				2000	0.490	-0.009	0.010	2.994	-0.006	0.010	2.927	-0.029	0.010	1.990	-0.013	0.014
				5000	0.488	-0.011	0.009	2.997	-0.003	0.010	2.962	-0.038	0.009	1.989	-0.014	0.013

Also, a real data set is used to discuss that the MOGIW can be a good ness of fit model to data comparing with several known distributions. These distributions are Marshall–Olkin extended inverse Weibull (MOEIW), inverse Weibull (IW), Fréchet (F), inverse Rayleigh (IR) and inverse exponential (IE). Using the MATHCAD software, the parameters of the MOGIW distribution are estimated via MLE method as follows: First, the MLEs and the corresponding standard errors of the parameters are reported. Second, the formal goodness-of-fit tests including Akaike information criterion ( $AIC = -2 \ln L + 2p$ ), consistent Akaike information criterion

$$\left( CAIC = -2 \ln L + \frac{2pn}{n-p-1} \right), \text{ Bayesian information criterion } (BIC = -2 \ln L + p \ln n) \text{ and Hannan-Quinn}$$

information criterion ( $HQIC = -2 \ln L + p \ln(\ln n)$ ) are applied to verify which distribution fits better the real data set. where p is the number of the model parameters and n is the sample size. In general, the smaller values of these statistics is the better fit to data. The real data set was originally reported by (Bjerkedal 1960), which represents two treatments of guinea pigs. Each treatment divided into 6 groups of 72 guinea pigs of animals which observed for two years. Table (2) gives a descriptive summary for these data.

Table 2. 72 observations for guinea pigs of animals.

12	15	22	24	24	32	32	33	34	38	38	43	44	48	52	53	54	54
55	56	57	58	58	59	60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91	95	96	98	99	109	110
121	127	129	131	143	146	146	175	175	211	233	258	258	263	297	341	341	376

Table (3) shows that the MLEs estimates for the parameters for the MOGIW distribution and their sub-models by using equations, (17) - (20). We notice that, the proposed model presents the best model among all others to fit this data because the MOGIW presents the smallest values of the statistics AIC, BIC, CAIC and HQIC. Also, some statistics such Kolmogorov-Smirnov (K-S), Anderson-Darling (A) and Cramér-von Mises (W) are used. The values of these tests lead to that the proposed model presents the best model among all others because the proposed model is the smallest values of all statistics. While, the p-value of the MOGIW distribution is the greater value than others.

Table 3. (K-S), (A), (W), AIC, BIC, CAIC and HQIC statistics for data.

Distribution	K-S	p-value	A	W	AIC	BIC	CAIC	HQIC
MOGIW	0.07112	0.8575	0.6081	0.10376	111.714	110.71	112.766	113.952
MOEIW	0.07627	0.8445	0.6341	0.10522	153.598	152.595	154.651	154.471
IW	0.1318	0.1381	1.5177	0.2538	289.691	288.687	290.743	291.375
F	0.6398	8.95*10 <sup>-27</sup>	44.042	9.6086	121.867	132.583	122.920	123.892
IR	0.2396	4.88*10 <sup>-4</sup>	6.5639	1.2661	120.594	129.166	121.283	121.885
IE	0.1864	0.0128	4.6425	0.8379	118.700	125.130	119.107	120.225

### 5. Conclusion

The well-known inverse Weibull distribution, is extended by introducing one extra shape parameters, thus defining the Marshal Olikin Generalized Inverse Weibull (MOGIW) distribution having a broader class of hazard rate and density functions. This is achieved by taking the Inverse Weibull cdf as the baseline cumulative distribution of the Marshal Olikin family. A detailed study on the statistical properties of the new model is presented. The MOGIW distribution includes as special sub-models as Marshal Olikin Inverse Weibull (MOIW), exponentiated inverse Weibull (EIW), inverse Weibull (IW), Fréchet (F), inverse Rayleigh (IR) and inverse exponential (IE) distributions. Estimation of MOGIW distribution parameters is obtained by maximum likelihood and least square methods as well as the observed information matrix and asymptotic confidence intervals are discussed. An application to a real data set indicates that the new model fits data than the principal sub-models.

### References

Alice, T., & Jose, K. (2005). Marshall-Olkin logistic processes. *STARS Int. J.*, 6, 1-11.

Bennete, S. (1983). Log-Logistic Regression Models for Survival Data. *Journal of the Royal Statistical Society Applied Statistics*, 32, 165-171. Retrieved from <https://www.jstor.org/stable/2347295>

Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *Am. J. Hyg.*, 72, 130-148. <https://doi.org/10.1093/oxfordjournals.aje.a120129>

Cordeiro G., & Lemonte A. (2013). On the Marshall-Olkin extended weibull distribution. *Stat. Papers*, 54, 333-

353. <https://doi.org/10.1007/s00362-012-0431-8>.

Drapella, A. (1993). Complementary Weibull Distribution: Unknown or Just Forgotten. *Quality and Reliability Engineering International*, 9, 383-385. <https://doi.org/10.1002/qre.4680090426>

Felipe, R., Edwin, M., & Gauss, M. (2011). The generalized inverse Weibull distribution. *Statistical Papers*, 52, 591- 619. <https://doi.org/10.1007/s00362-009-0271-3>

Flaih, A., Elsalloukh, H, Mendi, E., & Milanova, M. (2012). The exponentiated inverse Weibull distribution. *Appl Math Inf. Sci.*, 6, 167-171.

Ghitany, M. (2005). Marshall–Olkin extended pareto and its application. *Int. J. Appl. Math.*, 18, 17-32.

Ghitany, M. Al-Awadhi, F., & L. Alkhalaf, L (2007). Marshall–Olkin extended lomax distribution and its application to censored data. *Commun. Stat. Theory Methods*, 36, 1855-1866. <https://doi.org/10.1080/03610920601126571>

Ghitany, M., Al-Hussaini, E., & Al-Jarallah, R. (2005). Marshall–Olkin extended Weibull distribution and its application to censored data. *J. Appl. Stat.*, 32, 1025-1034. <https://doi.org/10.1080/02664760500165008>

Gui, W. (2013). Marshall–Olkin power log-normal distribution and its applications to survival data. *Int. J. Stat. Probab.*, 2, 63-72. <http://dx.doi.org/10.5539/ijsp.v2n1p63>

Jose, K., & Krishna, E. (2011). Marshall–Olkin extended uniform distribution. *Probab. Stat. Optim.*, 4, 78-88.

Kundu, D., & Howlader, H. (2010). Bayesian Inference and Prediction of the Inverse Weibull Distribution for Type-II Censored Data. *Computational Statistics and Data Analysis*, 54, 1547-1558. <https://doi.org/10.1016/j.csda.2010.01.003>

Langlands, A., Pocock, S., Kerr, G., & Gore, M. (1979). Long Term Survival of Patients with Breast Cancer: A Study of Curability of the Disease. *British Medical Journal*, 1247-1251. <https://www.ncbi.nlm.nih.gov/pmc/articles/PMC1596940/>

Marshall, A., & Olkin I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika*, 84, 641-652. <https://doi.org/10.1093/biomet/84.3.641>

Mudholkar, G., & Kollia, G. (1994). Generalized Weibull Family: A Structural Analysis. *Communications in Statistics - Theory and Methods*, 23, 1149-1171. <https://doi.org/10.1080/03610929408831309>

Okasha, H., & Kayid, M. (2016). A new family of Marshall–Olkin extended generalized linear exponential distribution. *J. Comput. Appl. Math.*, 296, 576-592. <https://doi.org/10.1016/j.cam.2015.10.017>

Okasha, H., El-Baz, A., Tarabia, A., & Basheer, A. (2017). Extended inverse Weibull distribution with reliability application. *Journal of the Egyptian Mathematical Society*, 25, 343-349. <https://doi.org/10.1016/j.joems.2017.02.006>

Pararai, M., Warahena, G., & Oluyede, B. (2014). A new class of generalized inverse Weibull distribution with applications. *J. Appl. Math.*, 4, 17-35.

Swain, J., Venkatraman, S., & Wilson, J. (1988). Least squares estimation of distribution function in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29, 271-297. <https://doi.org/10.1080/00949658808811068>

## Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (<http://creativecommons.org/licenses/by/4.0/>).