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Heisenberg-Ivanenko Nonlinear Spinor Field Equation: Spherical Symmetric Soliton-Like Solutions in Gravitational Theory

Adebayo Louis Essoun[1], M. Abel Konnon[2], Jonas Edou[1] & A. Adomou[2]

[1] Department of Physics, Lboratory of Theoretical Physics and Physics Mathematics, University of Abomey-Calavi, Abomey-Calavi, Benin

[2] National Higher Institute of Industrial Technology, National University of Sciences Technologies, Engineering and Mathematics of Abomey, Abomey, Benin

Correspondence: Jonas Edou, Lboratory of Theoretical Physics and Physics Mathematics, University of Abomey-Calavi, Abomey-Calavi, Benin. Tel: 229- 95-094-449. E-mail: edjonas131988@gmail.com

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Abstract

This research work deals with the concept of soliton as regular localized stable solutions of nonlinear differential equations. In this context, exact static, spherically symmetric solutions to Heisenberg-Ivanenko nonlinear spinor field equation have been obtained in General Relativity. We opted to the static spherical symmetric metric defined in the pseudo-riemannian varieties. It has been shown that the obtained solutions are regular with localized energy density and a finite total energy. In addition, the total charge and the total spin are bounded. Therefore the obtained solutions of Heisenberg-Ivanenko nonlinear spinor field equation are soliton-like configurations. Note that the effect of gravitational field on the properties of regular localized solutions significantly depends on the symmetry of the system.

Keywords: Lagrangian, self-interaction, pseudo-riemannian varieties

1. Introduction

The theory of soliton is an important branch of nonlinear science. One of the fields to apply the soliton concept is the elementary particles physics, where the soliton solutions of nonlinear differential field equations are used as simplest models of extended particles (Perring & Skyrme, 1962; Rybakov, 1982). With the development of general relativity and quantun field theory the description of elementary particles by considering the nonlinear phenomena and taking into account the proper gravitational field of elementary particles became very important. Indeed, the theory which considers elementary particles as material points has shortcomings. With this theory, it is impossible to obtain a finite value of mass, charge and spin of elementary particles. In this approach, elementary particles are modeled by soliton-like solutions of corresponding to nonlinear equations. By doing so, the basics of soliton concept in General Relativity have been investigated (Shikin, 1995). In sequel, exact plane-symmetric solutions to the spinor field equations with nonlinear terms, which are arbitrary function of the invariant $S = \bar{\psi}\psi$ are widely studied by (Adomou et al, 1998). Five years later, B. Saha and G.N. Shikin studied the system of nonlinear spinor and scalar fields with minimal coupling in general relativity (Saha et al, 2003). It should be emphasize that, in 2012, V. Adanhoumè, A. Adomou, F.P. Codo and M.N. Hounkonnou extended the research work [Gravitation and Cosmology, Vol.4, 1998, pp.107C113] to exact spherical symmetric soliton-like solutions (Adanhoumè et al., 2012). In a series of remarkable papers appeared in 2019, Alain Adomou, Jonas Edou and Siaka Massou obtained soliton-like solutions of the nonlinear spinor field equations in plane-symmetric metric and spherical symmetric metric (Adomou et al, 2019). In all these cases, the solutions are regular, the energy density is localized and the total energy is finite. In addition, the charge density and total spin are not bounded in the plane-symmetric metric but they are localized in the spherical symmetric metric. The geometrical symmetries of the space-time are very important in the gravitational theory. Let us emphasize that, the role of symmetries in general theory of relativity has been introduced by Katzin, Lavine and Davis in a series of papers (Katzin et al, 1969).

The aim of this paper is to describe the configuration of elementary particles by the soliton model in general relativity. In addition, the paper deals with the role of the proper gravitational field and the nonlinear terms in the formation of field configuations with limited total energy, spin and charge.

The paper is organized as follows. In section 2, we established the basics equations and concepts using the variational principle and usual algebraic manipulations. Section 3 addresses the general analytical fundamental solutions. In section 4, we analyzed and discussed in detail the principal results. Finally, the conclusion and outlook are outlined in section 5.

2. Fundamental Equations

We shall study a self-consistent system of nonlinear spinor and Einstein gravitational fields. Note that these two fields are be codetermined by the following action

$$S(g, \psi, \bar{\psi}) = \int L \sqrt{-g} d\Omega \tag{1}$$

According to (1), we defined the nonlinear generalization of the lagrangian of the two fields in the following way:

$$L = \frac{R}{2æ} + L_{Sp}. \tag{2}$$

The spinor field part of the Lagrangian in (2) has the form:

$$L_{Sp} = \frac{i}{2}(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi) - m\bar{\psi}\psi + L_N. \tag{3}$$

$L_N = F(S)$ is some arbirary functions characterizing the self-interaction of the spinor field. In the context of Heisenberg-Ivanenko equation investigate, it takes the form $F(S) = \lambda S^2$, λ is the parameter of nonlinearity and the invariant function corresponding to the real bilinear forms is $S = \bar{\psi}\psi$. Then, $æ = \frac{8\Pi G}{C^4}$ is Einstein's gravitational constant, G is Newton's gravitational constant, c is the velocity of light in vacuum and $m\bar{\psi}\psi$ is Majorana's mass. ψ is the 4-components Dirac's spinor with $\bar{\psi}$ its conjugate.

The metric of space-time in spherical coordinates reads (Adanhoumè et al, 2012):

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}d\xi^2 - e^{2\beta}[d\theta^2 + \sin^2(\theta)d\varphi^2]. \tag{4}$$

Note that the speed of light c has been taken to be unity. As was mentioned in (Bronnikov, 1973), we assume that the spatial variable $\xi = \frac{1}{r}$, where r stands for the radial component of the spherical symmetric metric. It should be emphasized that α, β and γ are functions of ξ only and obey the harmonic coordinate condition given by the following expression (Bronnikov, 1973):

$$\alpha = 2\beta + \gamma. \tag{5}$$

The field equations for spinor and gravitational fields can be obtained from the variational principle. Variation of the Lagrangian with respect to the field functions ψ $(\bar{\psi})$ gives the nonlinear spinor field equations under the form (Bogoliubov et al, 1976):

$$i\gamma^{\mu}\nabla_{\mu}\psi - (m - 2\lambda S)\psi = 0, \tag{6}$$

$$i\nabla_{\mu}\bar{\psi}\gamma^{\mu} + (m - 2\lambda S)\bar{\psi} = 0. \tag{7}$$

Varying the lagrangian (2) with respect to the metric function $g_{\mu\nu}$ we get the Einstein's field equation:

$$G^{\nu}_{\mu} = R^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}R = -æT^{\nu}_{\mu}, \tag{8}$$

where G^{ν}_{μ} is the Einstein's tensor; R^{ν}_{μ} is the Ricci's tensor; $R = R_{\mu\nu}g_{\mu\nu}$ is the Ricci's scalar, δ^{ν}_{μ} is the Kronecker's symbol and T^{ν}_{μ} is the metric energy-momentum tensor of the spinor field.

Let us write the nontrivial components of Ricci and Riemann tensors as well as Christoffel symbols of the metric (4). The nontrivial Christoffel symbols of the metric (4) read

$$\Gamma^0_{01} = \Gamma^0_{10} = \gamma'.$$

$$\Gamma^1_{00} = \gamma' e^{-2\alpha+2\gamma}; \qquad \Gamma^1_{11} = \alpha'; \qquad \Gamma^1_{22} = -\beta' e^{-2\alpha+2\beta}.$$

$$\Gamma^1_{33} = -\beta' e^{-2\alpha+2\beta} \sin^2\theta$$

$$\Gamma^2_{12} = \beta'; \qquad \Gamma^2_{21} = \beta'; \qquad \Gamma^2_{33} = -e^{-2\alpha+2\beta}\sin\theta\cos\theta.$$

$$\Gamma^3_{13} = \beta'; \qquad \Gamma^3_{23} = \cot\theta; \qquad \Gamma^3_{31} = \beta'; \qquad \Gamma^3_{32} = \cot\theta.$$

The nontrivial components of the Ricci tensor are

$$\begin{aligned} R_{00} &= (-\gamma'' - \gamma' + 2\beta'\gamma')e^{-2\alpha+2\gamma}, \\ R_{11} &= 2\beta'' + \gamma'' - 4\beta'\gamma' - 2\beta'^2, \\ R_{22} &= (-1 + \beta'' + \beta'^2)e^{-2\alpha+2\beta}, \\ R_{33} &= (\beta'' + \beta'^2 - 1)e^{-2\alpha+2\beta}\sin^2\theta. \end{aligned}$$

Taking into account (8) and the Ricci tensors written above, one finds the components of the tensor G^ν_μ in the metric (4) under the coordinate condition (5) as follpows (Brill, 1957) :

$$G^0_0 = e^{-2\alpha}(2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -æT^0_0, \tag{9}$$

$$G^1_1 = e^{-2\alpha}(2\beta'\gamma' + \beta'^2) - e^{-2\beta} = -æT^1_1, \tag{10}$$

$$G^2_2 = e^{-2\alpha}(\beta'' + \gamma'' - 2\beta'\gamma' - \beta'^2) = -æT^2_2, \tag{11}$$

$$G^2_2 = G^3_3, \qquad T^2_2 = T^3_3, \tag{12}$$

where prime denotes differentiation with respect to the spatial variable ξ. The metric energy-momentum tensor of the material field T^ν_μ has the form:

$$T^\nu_\mu = \frac{i}{4}g^{\nu\rho}(\bar{\psi}\gamma_\mu\nabla_\nu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi) - \delta^\nu_\mu L_S\, p. \tag{13}$$

Introducing (6) and (7) into (3), the spinor field Lagrangian $L_{S\,p}$ takes the form:

$$L_{S\,p} = -F(S) \tag{14}$$

Let us consider the spinors to be functions of ξ only. Thus we write the non null components of the metric energy-momentum tensor, which in our case read:

$$T^0_0 = T^2_2 = T^3_3 = -L_{S\,p} = F(S), \tag{15}$$

$$T^1_1 = \frac{i}{2}(\bar{\psi}\gamma^1\nabla_1\psi - \nabla_1\bar{\psi}\gamma^1\psi) + F(S). \tag{16}$$

Note that $\gamma^\mu(\xi)$ denotes Dirac's matrices of curved space-time which are linked with those of Minkowski's space-time as follows (Bogoliubov et al, 1976):

$$g_{\mu\nu}(\xi) = e^a_\mu(\xi)e^b_\nu(\xi)\eta_{ab}, \qquad \gamma_\mu(\xi) = e^a_\mu(\xi)\bar{\gamma}_a, \tag{17}$$

where $\eta_{ab} = diag(1, -1, -1, -1)$ is Minkowski's metric and $e^a_\mu(\xi)$ are a set of tetrad four-vectors.
The relation (17) leads

$$\gamma^0(\xi) = e^{-\gamma}\bar{\gamma}^0\ , \qquad \gamma^1(\xi) = e^{-\alpha}\bar{\gamma}^1\ , \qquad \gamma^2(\xi) = e^{-\beta}\bar{\gamma}^2\ , \qquad \gamma^3(\xi) = \tfrac{e^{-\beta}\bar{\gamma}^3}{\sin\theta},\ , \qquad \gamma^5(\xi) = \bar{\gamma}^5 \tag{18}$$

with

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \bar{\gamma}^5 = \gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where σ^i are the Hermitian matrices of Pauli:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the $\bar{\gamma}$, the γ and the σ matrices obey the following algebgra:

$$\bar{\gamma}^i\bar{\gamma}^j + \bar{\gamma}^j\bar{\gamma}^i = 2\eta^{ij}, \quad i, j = 0, 1, 2, 3$$

$$\bar{\gamma}^i\bar{\gamma}^5 + \bar{\gamma}^5\bar{\gamma}^i = 0, \quad (\bar{\gamma}^5)^2 = I, \quad i, j = 0, 1, 2, 3$$

$$\sigma^j\sigma^k = \delta_{jk} + i\varepsilon_{jkl}\sigma^l, \quad j, k, l = 1, 2, 3$$

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}$$

where $\eta_{ij} = diag(1, -1, -1, -1)$ is the diagonal matrix, δ_{jk} is the kronecker symbol and ε_{jkl} is the totally antisymmetric tensor with $\varepsilon_{123} = +1$.

In the expression above ∇_μ denotes the covariant derivative on the spinors. It takes the form (Brill et al., 1976):

$$\nabla_\mu\psi = \partial_\mu\psi - \Gamma_\mu\psi, \tag{19}$$

where Γ_μ is the spin connection. The spin affine connection matrices $\Gamma_\mu(\xi)$ are uniquely determined up an additive multiple of the unit matrix by the following equation (Zhelnorovich et al., 1976):

$$\partial_\mu\gamma_\nu - \Gamma^\alpha_{\nu\mu}\gamma_\alpha - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0, \tag{20}$$

having the solution:

$$\Gamma_\mu(\xi) = \frac{1}{4}g_{\rho\mu}(\partial_\mu e^b_\sigma e^\rho_a - \Gamma^\rho_{\mu\sigma})\gamma^\delta\gamma^\sigma. \tag{21}$$

From (21) and for the metric element (4) we obtain:

$$\Gamma_0 = -\frac{1}{2}e^{-2\beta}\bar{\gamma}^0\bar{\gamma}^1\gamma', \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2}e^{-\beta-\gamma}\bar{\gamma}^2\bar{\gamma}^1\beta', \quad \Gamma_3 = \frac{1}{2}(e^{-\beta-\gamma}\bar{\gamma}^3\bar{\gamma}^1\beta'\sin\theta + \bar{\gamma}^3\bar{\gamma}^2\cos\theta). \tag{22}$$

Using Einstein's convention, it is easy to show that

$$\gamma^\mu\Gamma_\mu = -\frac{1}{2}(e^{-\alpha}\alpha'\bar{\gamma}^1 + \bar{\gamma}^2 e^{-\beta}\cot\theta). \tag{23}$$

Inserting (20) and (23) into (6) for the nonlinear spinor field, we have

$$ie^{-\alpha}\bar{\gamma}^1(\partial_\xi + \frac{1}{2}\alpha')\psi + \frac{i}{2}\bar{\gamma}^2 e^{-\beta}\psi\cot\theta - (m - 2\lambda S)\,\psi = 0, \tag{24}$$

Then for the components of the nonlinear spinor field from (24), one obtains:

$$V_4' + \frac{1}{2}\alpha' V_4 - \frac{i}{2}e^{\alpha-\beta}V_4\cot\theta + ie^\alpha\,(m - 2\lambda S)\,V_1 = 0, \tag{25}$$

$$V_3' + \frac{1}{2}\alpha' V_3 + \frac{i}{2}e^{\alpha-\beta}V_3\cot\theta + ie^\alpha\,(m - 2\lambda S)\,V_2 = 0, \tag{26}$$

$$V_2' + \frac{1}{2}\alpha' V_2 - \frac{i}{2}e^{\alpha-\beta}V_2\cot\theta - ie^\alpha\,(m - 2\lambda S)\,V_3 = 0, \tag{27}$$

$$V_1' + \frac{1}{2}\alpha' V_1 + \frac{i}{2} e^{\alpha-\beta} V_1 \cot\theta - i e^{\alpha} (m - 2\lambda S) V_4 = 0. \tag{28}$$

where $\psi(\xi) = V_\delta(\xi)$, $\delta = 1, 2, 3, 4$.

In the following section, we will solve analytically the fundamental field equations.

3. Analytical solutions

This section aims to carry out the exact analytical solutions to the basic equations established previously. For this, from the set of equations (25)-(28), we obtain the first-order differential equation satisfied by the invariant function:

$$S = \bar{\psi}\psi = V_1^* V_1 + V_2^* V_2 - V_3^* V_3 - V_4^* V_4$$

as follows

$$\frac{dS}{d\xi} + \alpha'(\xi) S = 0. \tag{29}$$

The solution of Eq. (29) is:

$$S(\xi) = C_0 \exp[-\alpha(\xi)], \qquad C_0 = const. \tag{30}$$

According to the expression (30), we deduce that the spinor field is more sensitive to gravitational field than the scalar or electromagnetic fields.This expression reflets also the natural link between the spinor and gravitational fields of elementary particles.

Before dealing with the Einstein equations let us combine the spinor field equations (6) and (7) to produce the relation between the component T_1^1 of the metric energy-momentum tensor, the spinor mass m, the invariant function S and the nonlinearity parameter λ:

$$T_1^1 = S(m - \lambda S). \tag{31}$$

This paragraph is devoted to solve the Einstein's equations. To this end, since $T_0^0 = T_2^2$, we have $G_0^0 - G_2^2 = 0$ that implies

$$\beta'' - \gamma'' = e^{2\beta+2\gamma}. \tag{32}$$

The equation (32) can be transformed into a Liouville equation with the solutions (G. N. Shikin, 1995)

$$\beta(\xi) = \frac{A}{4}(1 + \frac{2}{G}) \ln\left[\frac{A}{GT^2(k, \xi + \xi_1)}\right] = \left(1 + \frac{2}{G}\right)\gamma(\xi), \tag{33}$$

$$\gamma(\xi) = \frac{A}{4} \ln\left[\frac{A}{GT^2(k, \xi + \xi_1)}\right], \tag{34}$$

where A and G are integration constants and T is a function.

The function T has the following form:

$$T(k, \xi + \xi_1) = \begin{cases} \frac{1}{k}\sinh[k(\xi + \xi_1)], k > 0 \\ (\xi + \xi_1), k = 0 \\ \frac{1}{k}\sin[k(\xi + \xi_1)], k < 0 \end{cases} \tag{35}$$

where k and ξ_1 are integration constants.

In view of (33) and (34), from (5), one gets $\alpha(\xi)$ and the relation between the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$ in the following way:

$$\alpha(\xi) = \frac{A}{2}\left(\frac{3}{2} + \frac{2}{G}\right) \ln\left[\frac{A}{GT^2(k, \xi + \xi_1)}\right]. \tag{36}$$

$$\beta(\xi) = \left(\frac{2+G}{4+3G}\right)\alpha(\xi); \qquad \gamma(\xi) = \left(\frac{G}{4+3G}\right)\alpha(\xi). \tag{37}$$

Equation (10) looks like the first integral of the equatons (9) and (11). It is also a first order differential equation. Then, introducing (31) and (33) into (10), we have

$$(\alpha')^2 = \frac{(4+3G)^2}{3G^2+8G+4}e^{2\alpha}\left[e^{\frac{-4-2G}{4+3G}\alpha} - æS\,(m-\lambda S)\right]. \tag{38}$$

Taking into account $\alpha' = -\frac{1}{S}\frac{dS}{d\xi}$ and the fact that $S(\xi) = C_0 e^{-\alpha(\xi)}$, from (38), we obtain

$$\frac{dS}{d\xi} = \pm\frac{C_0(4+3G)}{\sqrt{3G^2+8G+4}}\sqrt{\left[\left(\frac{S}{C_0}\right)^{u_0} - æS\,(m-\lambda S)\right]} \tag{39}$$

with $u_0 = \frac{4+2G}{4+3G} \cong 1$ (V. Adanhoumè et al, 2012).

Without the loss of generality, we may assume that $u_0 = 1$. Then, the solution of (39) can be written in quadrature in the following way after some mathematical manipulations:

$$S(\xi) = \left(\frac{1-C_0æm}{2C_0æ\lambda}\right)\left[\cosh\left[\sqrt{æ\lambda}\frac{4+3G}{\sqrt{3G^2+8G+4}}(\xi+\xi_0)\right] - 1\right] \tag{40}$$

Knowing the expression of the invariant function $S(\xi)$ in view of (30), we deduce the expression of the metric function $\alpha(\xi)$. Then, taking into account (37) one obtains the expressions of the functions $\beta(\xi)$ and $\gamma(\xi)$. One remarks that the invariant S and the functions $g_{00} = e^{2\gamma(\xi)}$, $g_{11} = -e^{2\alpha(\xi)}$, $g_{22} = -e^{2\beta(\xi)}$, $g_{33} = -e^{2\beta(\xi)}\sin^2(\theta)$ are regular and localized for $\xi \in [0, \xi_c]$, ξ_c being the center of the field configuration.

The energy density is explicitly defined by the following expression:

$$T_0^0(\xi) = \lambda\left(\frac{1-C_0æm}{2C_0æ\lambda}\right)^2\left[\cosh\left[\sqrt{æ\lambda}\frac{4+3G}{\sqrt{3G^2+8G+4}}(\xi+\xi_0)\right] - 1\right]^2 \tag{41}$$

Let us emphasize that the energy density is bounded when $\xi \in [0, \xi_c]$.

In virtue of (41), the energy density per unit invariant volume $f(\xi) = T_0^0(\xi)e^{2\alpha-\gamma}\sin\theta$ is defined in the following way:

$$f(\xi) = \lambda\left(\frac{1-C_0æm}{2C_0æ\lambda}\right)^2\left[\cosh\left[\sqrt{æ\lambda}\frac{4+3G}{\sqrt{3G^2+8G+4}}(\xi+\xi_0)\right] - 1\right]^2\left[\frac{A}{GT^2(k,\xi+\xi_1)}\right]^{\frac{A(8+5G)}{4G}}\sin\theta \tag{42}$$

From (42) the energy density per unit invariant volume of Heisenberg-Ivanenko equation of a nonlinear spinor field is localized when $\xi \in [0, \xi_c]$. Moreover, the total energy $E = \int_0^{\xi_c} f(\xi)d\xi$ is finite.

We can get a concrete form of the functions $V_\delta(\xi)$ by solving the set of equations (25)-(28) in more compact form if we pass to the functions $W_\delta(\xi) = e^{\frac{\alpha}{2}}V_\delta(\xi)$, with δ = 1, 2, 3, 4. Now, For this purpose, we have find the following set of equations:

$$W_4' - \frac{i}{2}e^{\alpha-\beta}W_4\cot\theta + ie^{\alpha}(m-2\lambda S)W_1 = 0, \tag{43}$$

$$W_3' + \frac{i}{2}e^{\alpha-\beta}W_3\cot\theta + ie^{\alpha}(m-2\lambda S)W_2 = 0, \tag{44}$$

$$W_2' - \frac{i}{2}e^{\alpha-\beta}W_2\cot\theta - ie^{\alpha}(m-2\lambda S)W_3 = 0, \tag{45}$$

$$W_1' + \frac{i}{2}e^{\alpha-\beta}W_1\cot\theta - ie^{\alpha}(m-2\lambda S)W_4 = 0, \tag{46}$$

where

$$W'_\rho = (V'_\rho + \frac{1}{2}\alpha' V_\rho)e^{\frac{1}{2}\alpha}. \quad (47)$$

With the set of equations (43)-(46) where $W = W_\delta(\xi)$ let us pass to the system of equations depending on functions of the argument $S(\xi)$, i.e. $W_\delta(S) = W_\delta(\xi)$, $S(\xi) = C_0 e^{-\alpha(\xi)}$. We obtain for the functions $W_\delta(S)$ the set of equations as follows:

$$\frac{dW_4}{dS} - i\Phi(S)W_4 + i\Omega(S)W_1 = 0, \quad (48)$$

$$\frac{dW_3}{dS} + i\Phi(S)W_3 + i\Omega(S)W_2 = 0, \quad (49)$$

$$\frac{dW_2}{dS} - i\Phi(S)W_2 - i\Omega(S)W_3 = 0, \quad (50)$$

$$\frac{dW_1}{dS} + i\Phi(S)W_1 - i\Omega(S)W_4 = 0, \quad (51)$$

where

$$\Phi(S) = \frac{1}{2}\frac{\left(\frac{C_0}{S}\right)^{\frac{2+2G}{4+3G}}\cot\theta}{\frac{dS}{d\xi}}; \qquad \Omega(S) = \frac{\left(\frac{C_0}{S}\right)(m - 2\lambda S)}{\frac{dS}{d\xi}} \quad (52)$$

with $\frac{dS}{d\xi}$ is determined by (39).

Differentiating now equations (48)-(51) and inserting (48) and (51) into the result, one obtains second-differential equations satisfied by the functions $W_1(S)$ and $W_4(S)$

$$W''_4 - \frac{\Omega'(S)}{\Omega(S)}W'_4 + \left[\Phi^2(S) - \Omega^2(S) + i\frac{\Omega'(S)\Phi(S) - \Omega(S)\Phi'(S)}{\Omega(S)}\right]W_4 = 0. \quad (53)$$

$$W''_1 - \frac{\Omega'(S)}{\Omega(S)}W'_1 + \left[\Phi^2(S) - \Omega^2(S) + i\frac{\Omega(S)\Phi'(S) - \Omega'(S)\Phi(S)}{\Omega(S)}\right]W_1 = 0. \quad (54)$$

Summing (53)-(54) and considering $U = W_1 + W_4$, we obtain the following second-order differential equations of the function $U(S)$:

$$U''(S) - \frac{\Omega'(S)}{\Omega(S)}U'(S) + 2\left[\Phi^2(S) - \Omega^2(S)\right]U(S) = 0. \quad (55)$$

The equation (55) may be transformed to:

$$\frac{1}{\Omega(S)\sqrt{2\varepsilon}}\frac{d}{dS}\left[\frac{U'(S)}{\Omega(S)\sqrt{2\varepsilon}}\right] - U(S) = 0 \quad (56)$$

under the condition $\Phi^2(S) = (1-\varepsilon)\Omega^2(S)$ with $0 < \varepsilon \leq 1$ (V. Adanhoumè et al, 2012).

The equation (56) has the first integral

$$U'(S) = \pm\sqrt{U^2(S) + C_1}\,\Omega(S)\sqrt{2\varepsilon}, \qquad C_1 = const. \quad (57)$$

Taking the first integral one gets:

$$W_1 + W_4 = a_1 \sinh N_1(S), \quad (58)$$

where

$$N_1(\xi) = -2\lambda\sqrt{2\varepsilon}(\xi + \xi_0) + M\tanh\left[\frac{m æ C_0(4+3G)}{2\sqrt{(1+\lambda æ C_0)(3G^2+8G+4)}}(\xi + \xi_0)\right] + R_1 \quad (59)$$

with

$$M = \frac{2\sqrt{2\varepsilon}(1+\lambda æ C_0)^{\frac{3}{2}}\sqrt{3G^2+8G+4}}{m æ^2 C_0(4+3G)} = const.$$

Substracting equations (48) and (51) and taking into account (58), we have:

$$W_1 - W_4 = -ia_1\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1(\xi), \tag{60}$$

It then, according to (58) and (60), it follows that:

$$W_1(\xi) = a_0\left[\sinh N_1(\xi) - i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1(\xi)\right], \tag{61}$$

$$W_4(\xi) = a_0\left[\sinh N_1(\xi) + i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1(\xi)\right], \tag{62}$$

with $a_0 = \frac{a_1}{2}$

Doing the same operating on the equations (49) to (50) leads to:

$$W_2(\xi) = b_0\left[\cosh N_2(\xi) + i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2(\xi)\right] \tag{63}$$

$$W_3(\xi) = b_0\left[\cosh N_2(\xi) - i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2(\xi)\right] \tag{64}$$

$b_0 = const.$ The function $N_2(\xi)$ has the same expression as $N_1(\xi)$, only now the constant R_1 is replaced by R_2. Let us emphasize that substituting $N_{1,2}(\xi)$ from (59) into (61) to (64) we get in an explicit form of the functions $W_\delta(\xi)$. Then, one passes to the initial functions $V_\delta(\xi)$ by multiplying the expressions $W_\delta(\xi)$ by $e^{-\frac{1}{2}\alpha(\xi)}$.

As one sees, we conclude that the equation (24) has soliton-like solutions. Here, the existence of the soliton-like configurations with localized energy density, finite total energy in Heisenberg-Ivanenko type nonlinear equation is a revolutionary and an interesting result.

4. Total Charge and Total Spin

Let us now determine the total charge and total spin of the configuration obtained. In doing so, using the concrete form of the functions $V_\delta(\xi)$, we can determine the components of the spinor current vector $j^\mu = \bar{\psi}\gamma^\mu\psi$ under the general form

$$j^0 = \left(V_1^*V_1 + V_2^*V_2 + V_3^*V_3 + V_4^*V_4\right)e^{-(\alpha+\gamma)}, \tag{65}$$

$$j^1 = \left(V_1^*V_4 + V_2^*V_3 + V_3^*V_2 + V_4^*V_1\right)e^{-2\alpha}, \tag{66}$$

$$j^2 = -i\left(V_1^*V_4 - V_2^*V_3 + V_3^*V_2 - V_4^*V_1\right)e^{-(\alpha+\beta)}, \tag{67}$$

$$j^3 = \left(V_1^*V_3 - V_2^*V_4 + V_3^*V_1 - V_4^*V_1\right)e^{-(\alpha+\beta)}. \tag{68}$$

In the case of Heisenberg-Ivanenko type nonlinear equation, the components of the spinor current vector may be rewritten in the following way

$$j^0 = 2e^{-\alpha-\gamma}\left\{a_0^2\left[\sinh^2 N_1(\xi) + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\cosh^2 N_1(\xi)\right] + b_0^2\left[\cosh^2 N_2(\xi) + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\sinh^2 N_2(\xi)\right]\right\} \tag{69}$$

$$j^1 = 2e^{-2\alpha}\left\{a_0^2\left[\sinh^2 N_1(\xi) - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\cosh^2 N_1(\xi)\right] + b_0^2\left[\cosh^2 N_2(\xi) - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\sinh^2 N_2(\xi)\right]\right\} \tag{70}$$

$$j^2 = 4e^{-\alpha-\beta}\left[a_0^2\left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)\cosh N_1(\xi)\sinh N_1(\xi) - b_0^2\left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)\cosh N_2(\xi)\sinh N_2(\xi)\right] \tag{71}$$

$$j^3 = 0 \tag{72}$$

As in this study the configuration is static, the components j^1, j^2 and j^3 are evident. But only the component j^0 is nonzero. This assumption gives additional relation between the constant of integration. Thus, we have $a_0 = b_0 = a$, $R_1 = R_2 = R$, $N_1(\xi) = N_2(\xi) = N(\xi)$ and $\varepsilon = 1$. The component j^0 defines the charge density of the spinor field that has the following chronometric invariant form:

$$\varrho(\xi) = \left(j_0 j^0\right)^{\frac{1}{2}} = 4a^2 e^{-\alpha(\xi)}\cosh 2N(\xi) \tag{73}$$

where $N(\xi)$ is defined by the expression (59) and

$$e^{-\alpha(\xi)} = \frac{S}{C_0} = \left(\frac{1-C_0æm}{2C_0^2æ\lambda}\right)\left\{\cosh\left[\sqrt{æ\lambda}\frac{4+3G}{\sqrt{3G^2+8G+4}}(\xi+\xi_0)\right]-1\right\} \tag{74}$$

The charge density is localized when $\xi \in [0, \xi_c]$. The total charge of the spinor field in the Heisenberg-Ivanenko type nonlinear equation is:

$$Q = \int_0^{\xi_c} \rho\sqrt{-3_g}d\xi = 4a^2\int_0^{\xi_c}\cosh 2N(\xi)e^{\alpha-\gamma}\sin\theta d\xi < \infty, \tag{75}$$

ξ_C being the center of the field configuration and

$$e^{\alpha-\gamma} = \left(\frac{2C_0^2æ\lambda}{(1-C_0æm)\left\{\cosh\left[\sqrt{æ\lambda}\frac{4+3G}{\sqrt{3G^2+8G+4}}(\xi+\xi_0)\right]-1\right\}}\right)^{\frac{4+2G}{4+3G}} \tag{76}$$

From (75) the total charge is finite when $\xi \in [0, \xi_c]$.

Let us deal with the spin tensor of the nonlinear spinor field. Its general form is:

$$S^{\mu\nu,\lambda} = \frac{1}{4}\bar{\psi}\left\{\gamma^\lambda\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda\right\}\psi. \tag{77}$$

Using (77), the spatial density of the spin tensor $S^{ik,0}$, $i, k = 1; 2; 3$ is:

$$S^{ik,0} = \frac{1}{4}\bar{\psi}\left\{\gamma^0\sigma^{ik} + \sigma^{ik}\gamma^0\right\}\psi = \frac{1}{2}\bar{\psi}\gamma^0\sigma^{ik}\psi. \tag{78}$$

Thus, we have

$$S^{12,0} = S^{13,0} = 0. \tag{79}$$

$$S^{23,0} = 2a^2\cosh 2N(\xi)e^{-\alpha}. \tag{80}$$

The relation (80) leads to the definition of the chronometric invariant of the spatial density as follows:

$$S^{23,0}_{chI} = \left(S_{23,0}S^{23,0}\right)^{\frac{1}{2}} = 2a^2\cosh 2N(\xi)e^{-\alpha}. \tag{81}$$

Thus, the projection of the spin vector on the radial axis has the form:

$$S_1 = \int_0^{\xi_c} S^{23,0}_{chI}\sqrt{-3_g}d\xi = 2a^2\sin\theta\int_0^{\xi_c}\cosh 2N(\xi)e^{\alpha-\gamma}d\xi. \tag{82}$$

Note that the spin tensor of the spinor field has a finite value and positive because the integrand is positive.

We can conclude that the Heisenberg-Ivanenko type nonlinear equation possesses soliton-like configuration with finite value of the total charge and the total spin. In addition, the metrics functions are stationnary and regular. Therefore, these solutions must be used to describe the configuration of elementary particles with mass.

5. Concluding Remarks

In this manuscript, taking into account the proper gravitational field of elementary particles, we obtained the general solutions of Einstein and nonlinear spinor field equations. We analyzed in particular the Heisenberg-Ivanenko type nonlinear spinor field equations. We note that the solutions of Heisenberg-Ivanenko equation are regular and possess a bounded energy density and limited total energy. Similarly, the metric functions are stationary. The total charge and the total spin are finite quantities as well. We demonstrated that the soliton-like solutions exist in flat space-time and absent in linear case. The nonlinearity of the spinor field vanishes in the space-time without gravitation. Therefore, we note that, the gravitational field is nonlinear by nature and its nonlinearity induces the nonlinearity of the spinor field. In order to extend the present analysis, the forthcoming paper will address to Nonlinear Spinor Field Equation of the Bilinear Pauli-Fierz Invariant $I_v = S^2 + P^2$: Exact Spherical Symmetric Soliton-Like Solutions In General Relativity.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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References

Adanhoumè, A., Adomou, A., Codo F. P., & Hounkonnou, M. N. (2012). Nonlinear Spinor Field Equations in Gravitational Theory: Spherical Symmetric Soliton-Like Solutions. *Journal of Modern Physics, 3*, 935. https://doi.org/10.4236/jmp.2012.39122

Adomou, A., & Shikin, G. N. (1998). *Gravitation and Cosmology, 4*(2), 107-113.

Adomou, A., Edou, J., & Massou, S. (2019). Plane Symmetric Solutions to the Nonlinear Spinor Field Equations in General Relativity Theory. *Journal of Modern Physics, 10, 1222-1234* . https://doi.org/10.4236/jmp.2019.1010081

Adomou A., Edou, J., & Massou, S. (2019). *Journal of Applied Mathematics and Physics, 7, 2018-2835* .

Bogoliubov, N. N., & Shirkov, D. V. (1976). *Introduction to the theory of Quantized Fields*. Moscow: Nauka.

Bronnikov, K. A. (1973). Scalar-Tensor Theory and Scalar Charge. *Acta Phys. Pol. B, 4*, 251 , https://www.actaphys.uj.edu.pl/R/4/3/251

Brill, D., & Wheeler, J. (1957). *Rev. Mod. Phys., 29*, 465.

Heisenberg, W. (1966). *Introduction to Unified Field Theory of Elementary Particles.* London: Interscience Publishers.

Katzin, G. H., Livine, J., & Davis, W.R. (1969). Curvature collineation: A fundamental symmetry property of the space-times of the general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor. *Journal of Mathematical Physics, 10*, 617-620.

Massou S., Adomou A. & Edou J. (2019). *International Journal of Applied Mathematics and Theoretical Physics, 5*(4), 118-128. https://doi.org/10.11648/j.ijamtp.20190504.14

Perring J. K., & Skyrme T.H. R. (1962). *Nuclear Physics, 31* , 550.

Rybakov, Yu. P.(1982). *Particle Structure in Nonlinear Field Theory*. Moscow: PFU.

Saha, B., & Shikin, G. N. (2003), *Czechoslovak Journal of Physics, 54*, 597-620. https://doi.org/10.1023/B:CJOP.0000029690.61308.a

Shikin, G. N. (1995). *Theory of Solitons in General Relativity*. URSS, Moscow.

Zhelnorovich, V. A. (1982). *Spinor Theory and Its applications in Physics and Mechanics*. Moscow: Nauka.