Quasi-Stationary Distributions in Linear Birth and Death Processes

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Abstract

The quasi-stationary distributions $\{a_j\}$ for a linear birth and death process is determined by two methods. The first method obtains our desired results by computing directly while the second method bases on the relationship between $\{a_j\}$ and its limiting of probability generating function. In addition, we also obtain the stationary distribution for a linear birth, death and immigration process with the second method.

Keywords: Linear birth and death process, Quasi-stationary distributions, Probability generating function, Stationary distribution

1. Introduction

We are interested in the long-term behavior of absorbing Markov processes. The processes will be absorbed eventually, however, they often persist for extended periods of time and in fact appear to reach an equilibrium before reaching the absorbing state. The key to analyze the behavior of the processes before they die out is to simplify condition on their not having been absorbed, which leads us to consider quasi-stationary distributions and limiting conditional distributions, rather than the classical stationary and limiting distributions for irreducible processes. Quasi-stationary distributions and limiting conditional distributions for Markov processes have been studied by Vere-Jones (1969), Flaspohler (1974), Pollett (1988), Darlington and Pollett (2000) and Moler et.al (2000) in the general setting of absorbing continuous-time denumerable Markov chains.

As we all know, birth and death processes are the most important class of Markov processes and their relatively simple structure can make us study them with a rather extensive analysis. Quasi-stationary distribution for birth and death processes have been studied by Cavender (1978), Pollett (1988), Van Doorn (1991), Schoutens (2000) and Clancy and Pollett (2003). Again, the linear birth and death processes are the most important class of birth and death processes. In this paper we will study the quasi-stationary distributions in the setting of a linear birth and death process on a semi-infinite lattice of integers, the finite boundary point being an absorbing state which is reached with certainly.

In this article, we will calculate the quasi-stationary distributions for a linear birth and death process by two different methods. Meanwhile, we also obtain the stationary distribution for a linear birth, death and immigration process with our methods.

2. Preliminaries

Let E be the set {0,1,2,...} of non-negative integers, and let $\{\lambda_n, n \geq 0\}$ and $\{\mu_n, n \geq 0\}$ be sequences of non-negative numbers. A continuous-time Markov Chain $\{X(t), t \geq 0\}$ having state E and q-matrix given by

$$q_{ij} = \begin{cases} \lambda_i & \text{if } j = i+1, i \geq 0 \\ \mu_i & \text{if } j = i-1, i \geq 1 \\ -(\lambda_i + \mu_i) & \text{if } j = i, i \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

is called a birth and death process on E, with birth coefficients $\lambda_n,\ n \geq 0$, and death coefficient $\mu_n,\ n \geq 0$. Suppose $\lambda_0 = \mu_0 = 0,\ \lambda_n > 0,\ \mu_n > 0,\ n \geq 1$, then Q will be conservative and 0 is an absorbing state and C={1,2,...} is irreducible for the minimal Q-function, F, and hence for any Q-function.

As usual we define the potential coefficients $\pi = \{\pi_i, i \in C\}$ and

$$\pi_1 = 1, \quad \pi_n = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_2\mu_3\cdots\mu_n}, \qquad n \geq 2 \tag{2}$$

The transition probability $\{P_{ij}(t),\ i, j \in E,\ t \geq 0\}$ of a birth and death process satisfies the following conditions

$$\sum_j P_{ij}(t) \leq 1, \tag{3}$$

$$P_{ij}(t) \geq 0, \tag{4}$$

$$P_{ij}(0) = \delta_{ij}, \tag{5}$$

$$P_{ij}(s+t) = \sum_k P_{ik}(s)P_{kj}(t) \tag{6}$$

$$P'_{ij}(t) = \sum_k q_{ik}P_{kj}(t), \tag{7}$$

$$P'_{ij}(t) = \sum_k P_{ik}(t)q_{kj}, \tag{8}$$

for $i, j \in E$ and $t, s \geq 0$, where δ_{ij} is the Kronecker delta. The equations (6), (7) and (8) are called the Chapman-Kolmogorov equations, the backward equations(BE) and the forward equations(FE), respectively. We know, each Q-function, $P_{ij}(t)$, satisfies (7) since Q is conservative but might not satisfy (8).

We define

$$T = \inf\{t \geq 0 : X(t) = 0\} \tag{9}$$

the absorption (hitting) time at 0. We shall only be interested in processes for which $E_iT < \infty$ for all $i \geq 1$.

We will assume the eventual absorption at 0 is certain, which is equivalent to assuming

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} = \infty. \tag{10}$$

Imposing (10) implies that the condition

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=0}^{n} \pi_i = \infty \tag{11}$$

is satisfied, and as a consequence (see for example Reuter (1957), Kemperman (1962)), the transition probabilities $P_{ij}(t)$ constitute, under some obvious side conditions, the unique solution of the equation (7) with initial conditions (5).

Also, imposing (10) (and hence (11)) implies that the process is non-explosive and therefore honest (see for example Reuter (1957), Kemperman (1962)), that is

$$\sum_{j=0}^{\infty} P_{ij}(t) = 1,\ i \in E,\ t \geq 0. \tag{12}$$

Given a transition function $P_{ij}(t)$, a set $\{u_i, i \in E\}$ of non-negative numbers such that

$$\sum_{i \in E} u_i P_{ij}(t) = u_j \ \text{ for all } j \in E \text{ and } t \geq 0,$$

is called an invariant measure for $P_{ij}(t)$. If, furthermore, $\sum_{i\in E} u_i = 1$, then $u = \{u_i, i \in E\}$ is called a stationary distribution.

Suppose that $P_{ij}(t)$ is an irreducible transition function, then the limits $u_j = \lim_{t\to\infty} P_{ij}(t)$ exist and are independent of i for all $j \in E$. The set $\{u_i, i \in E\}$ of numbers is an invariant measure and either

$$\begin{aligned} &(a) \quad u_j = 0 \text{ for all } j \in E, \text{ or} \\ &(b) \quad u_j > 0 \text{ for all } j \in E \text{ and } \sum_{j\in E} u_j = 1. \end{aligned} \tag{13}$$

We call $A = \{a_j\}$ the limiting conditional distribution (LCD) if for each $j \geq 1$

$$a_j = \lim_{t\to\infty} P_i(X(t) = j|T > t) \text{ and } \sum a_j = 1 \tag{14}$$

provided the limits exists for some (and hence for all) i.

We say a measure $A = \{a_j\}$ on C is a quasi-stationary distribution (QSD) if $\sum a_i = 1$ and for each $j \geq 1$ and $t > 0$

$$a_j = P_A(X(t) = j|T > t). \tag{15}$$

For the above concepts on LCD and QSD, we can refer to Pakes (1995) or Ferrari et al. (1995). From Van Doorn (1991) and Vere-Jones (1969) we know that any QSD is a (proper) LCD and any (proper) LCD is a QSD. In this paper, we focus on the process which starts from an single state i and derive the corresponding results.

Our study will use three particular families of distributions: geometric, Pascal and Poisson. Their probabilities generating functions are respectively:

$$g(s) = \frac{(1-p)s}{1-sp}, \text{ for the geometric distribution with parameter } p \ (0 < p < 1) \tag{16}$$

$$g(s) = e^{\rho(s-1)}, \text{ for the Poisson distribution with parameter } \rho \tag{17}$$

$$g(s) = \left(\frac{1-p}{1-sp}\right)^r, \text{ for the Pascal distribution with parameters } r \text{ and } p \ (r > 0, 0 < p < 1) \tag{18}$$

3. Quasi-stationary distributions of birth and death processes

In this section we mainly study the quasi-stationary distributions for a linear birth and death process and its generator determined by

$$\lambda_i = i\lambda,\ \mu_i = i\mu,\ i \geq 0, \tag{19}$$

where $0 < \lambda < \mu$. The constants λ and μ are called the birth and death rates.

It is clear that (10) and (11) are all satisfied. The properties of this process have been established in detail in Karlin and McGregor (1958). We know that Van Doorn (1991) obtained the quasi-stationary distributions by Karlin and McGregor's spectral representation of the transition probabilities of the process. However, the smallest point in the support of ψ is a slight difficult to be calculated. Here, we will calculate the quasi-stationary distributions with two other ways. The two methods base on the specificity that we have know the transition function of the process. The explicit form for $P_{ij}(t)$ follows from Anderson (1991) or Karlin and McGregor (1958):

$$P_{ij}(t) = \gamma^j \frac{(1-\frac{\sigma}{\gamma})^{i+j}}{(1-\sigma)^{i+j}} \sum_{k=0}^{i\wedge j} \binom{i}{k} (-1)^k \left(\frac{1-\frac{\sigma}{\gamma^2}}{1-\frac{\sigma}{\gamma}}\right)^k \left(\frac{1-\sigma}{1-\frac{\sigma}{\gamma}}\right)^k \frac{(i)_{j-k}}{(j-k)!}, \tag{20}$$

where $i \wedge j = \min(i, j)$ and

$$\sigma = \gamma e^{-(\mu-\lambda)t},\ \gamma = \frac{\lambda}{\mu},\ (a)_k = \begin{cases} a(a+1)\cdots(a+k) & k \geq 1 \\ 1 & k = 0 \end{cases},$$

and its probability generating function

$$G_i(z,t) = \sum_{j=0}^{\infty} P_{ij}(t) z^j = \left(\frac{r\mu - 1}{r\lambda - 1}\right)^i, \quad i \geq 1, |z| \leq 1, \tag{21}$$

where

$$r = e^{-(\mu-\lambda)t}\left(\frac{1-z}{\mu-\lambda z}\right).$$

3.1 The first method

In this section we obtain QSD (15) by computing LCD (14). Obviously, when the process starts from an single state i, (15) and (14) are equivalent. Aslo (14) can be written as

$$a_j = \lim_{t\to\infty} \frac{P_{ij}(t)}{P_i(T > t)}. \tag{22}$$

First, we introduce the first method, that is, the quasi-stationary distributions of the process is obtained directly by taking the limit for (22).

Let $\beta = e^{-(\mu-\lambda)t}$, then the equation (20) becomes

$$P_{ij}(t) = \gamma^j \frac{(1-\beta)^{i+j}}{(1-\gamma\beta)^{i+j}} \sum_{k=0}^{i\wedge j} \binom{i}{k} (-1)^k \left(\frac{\gamma - \gamma^2\beta - \beta + \gamma\beta^2}{\gamma(1-\beta)^2}\right)^k \frac{(i)_{j-k}}{(j-k)!}. \tag{23}$$

Again, since T is absorption time and hence

$$P_i(T > t) = 1 - P_i(T \leq t) = 1 - P_{i0}(t) = 1 - \frac{(1-\beta)^i}{(1-\gamma\beta)^i}. \tag{24}$$

Then

$$\frac{P_{ij}(t)}{1-P_{i0}(t)} = \frac{1}{(1-\gamma\beta)^i - (1-\beta)^i} \left\{ \gamma^j \frac{(1-\beta)^{i+j}}{(1-\gamma\beta)^j} \sum_{k=0}^{i\wedge j} \binom{i}{k} (-1)^k \left(\frac{\gamma - \gamma^2\beta - \beta + \gamma\beta^2}{\gamma(1-\beta)^2}\right)^k \frac{(i)_{j-k}}{(j-k)!} \right\}. \tag{25}$$

Obviously, the expressions is $\frac{0}{0}$ type when take the limit, we use L'Hôspital's Rule to tackle it. After a series of operations, we have

$$(25) \quad \rightarrow \quad \frac{1}{i(1-\gamma)}\left\{\gamma^{j}(i+j-j\gamma)\sum_{k=0}^{i\wedge j}\binom{i}{k}(-1)^{k}\frac{(i)_{j-k}}{(j-k)!}+\gamma^{j-1}(\gamma-1)^{2}\sum_{k=0}^{i\wedge j}\binom{i}{k}(-1)^{k}\frac{(i)_{j-k}}{(j-k)!}k\right\}$$

$$= \frac{\gamma^{j-1}}{i(\gamma-1)}\sum_{k=0}^{i\wedge j}\binom{i}{k}(-1)^{k}\frac{(i)_{j-k}}{(j-k)!}\left[(i+j-j\gamma)\gamma+k(\gamma-1)^{2}\right] \quad \text{as } t\rightarrow\infty.$$

Let $i \geq j$, then, setting

$$A_j = \frac{\gamma^{j-1}}{i(\gamma-1)}\sum_{k=0}^{j}\binom{i}{k}(-1)^{k}\frac{(i)_{j-k}}{(j-k)!}\left[(i+j-j\gamma)\gamma+k(\gamma-1)^{2}\right]. \tag{26}$$

By (26), it is not hard to find $A_1 = 1-\gamma,\ A_2 = (1-\gamma)\gamma,\ A_3 = (1-\gamma)\gamma^2$, recursively, we conjecture $A_j = (1-\gamma)\gamma^{j-1}$, $j \geq 1$. Below, we will use the method of mathematical induction to testify. Suppose $A_j = (1-\gamma)\gamma^{j-1}$, then, $A_{j+1} = (1-\gamma)\gamma^{j}$. Actually, from (26) and assumption, we see easily

$$\sum_{k=0}^{j}\binom{i}{k}(-1)^{k}\frac{(i)_{j-k}}{(j-k)!}\left[(i+j-j\gamma)\gamma+k(\gamma-1)^{2}\right] = -i(1-\gamma)^{2} \tag{27}$$

Obviously, the left of (27) is independent of j, and so

$$A_{j+1} = \frac{\gamma^{j}}{i(\gamma-1)}\sum_{k=0}^{j+1}\binom{i}{k}(-1)^{k}\frac{(i)_{j+1-k}}{(j+1-k)!}\left[(i+j+1-(j+1)\gamma)\gamma+k(\gamma-1)^{2}\right] = \frac{\gamma^{j}}{i(\gamma-1)}(-i)(1-\gamma)^{2} = (1-\gamma)\gamma^{j} \tag{28}$$

Therefore, we obtain

$$a_j = (1-\gamma)\gamma^{j-1} = (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{j-1},\ j \geq 1. \tag{29}$$

The result is in keeping with Van Doorn (1991).

Remark: The above proof course only considers the situation $i \geq j$, actually, because of the feature of $\frac{(i)_{j-k}}{(j-k)!}$, we know the result is invariable when $i \leq j$.

3.2 The second method

In this section we will introduce the second method and the idea partly refers to Karlin and Iavaré (1982). From (21), we see that

$$G_i(z,t) = \left[\frac{1-\beta+(\beta-\gamma)z}{1-\beta\gamma-z\gamma(1-\beta)}\right]^{i},\ i \geq 1,\ |z| \leq 1, \tag{30}$$

where $\beta = e^{-(\mu-\lambda)t}, \gamma = \frac{\lambda}{\mu}$.

To yield our result more conveniently,we induct the following lemma.

Lemma 3.1.

$$\lim_{t\rightarrow\infty}\beta^{-1}(G_i(z,t)-1) = iA(z), \quad i \geq 1,$$

where

$$A(z) = \frac{(1-\gamma)(z-1)}{1-z\gamma}, \quad 0 \leq z \leq 1.$$

Proof: From (30), we have

$$G_i(z,t) = \left[1+\frac{(1-\gamma)(z-1)\beta}{1-\beta\gamma-z\gamma+\beta z\gamma}\right]^{i} = \sum_{k=0}^{i}\binom{i}{k}\left[\frac{(1-\gamma)(z-1)\beta}{1-\beta\gamma-z\gamma+\beta z\gamma}\right]^{k}$$

and so,

$$G_i(z,t)-1 = \sum_{k=1}^{i}\binom{i}{k}\left[\frac{(1-\gamma)(z-1)\beta}{1-\beta\gamma-z\gamma+\beta z\gamma}\right]^{k},$$

and thus,

$$\lim_{t\rightarrow\infty}\beta^{-1}(G_i(z,t)-1) = \lim_{t\rightarrow\infty}\frac{i(1-\gamma)(z-1)}{1-\beta\gamma-z\gamma+\beta z\gamma} = \frac{i(1-\gamma)(z-1)}{1-z\gamma} = iA(z).$$

Moreover, from (12), we have

$$P_i(T > t) = \sum_{j=1}^{\infty} P_i(X(t) = j) = G_i(1,t) - G_i(0,t).$$

To establish (14), we use lemma 3.1 to see that for $0 \le z \le 1$,

$$\begin{aligned}\sum_{j=1}^{\infty} P_i(X(t) = j|T > t)z^j &= \frac{G_i(z,t) - G_i(0,t)}{G_i(1,t) - G_i(0,t)} = \frac{(G_i(z,t)-1)\beta^{-1} - (G_i(0,t)-1)\beta^{-1}}{(G_i(1,t)-1)\beta^{-1} - (G_i(0,t)-1)\beta^{-1}} \\ &\to \frac{A(z) - A(0)}{A(1) - A(0)} = \frac{z - z\gamma}{1 - z\gamma} \quad \text{as } t \to \infty,\end{aligned}$$

and hence, from (16) we see that $a_j = \lim_{t\to\infty} P_i(X(t) = j|T > t)$ is geometric distribution with parameter $\gamma = \frac{\lambda}{\mu}$, so we obtain

$$a_j = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{j-1}, \ j \ge 1. \tag{31}$$

Obviously, (31) is in accord with (29) and Van Doorn (1991). Certainly, some authors led to the same conclusion by other techniques, we may refer to Cavender (1978), Pollett (1988) and Van Doorn (1991).

4. Stationary distribution of linear birth,death and immigration processes

In this section, we will adopt the method of section (3.2) to compute stationary distribution for a linear birth, death and immigration process defined by

$$\lambda_i = a + i\lambda, \ \mu_i = i\mu, \ i \ge 0, \tag{32}$$

where $a > 0, \lambda \ge 0, \mu \ge 0$. The constants a, λ, μ are called the immigration, birth, and death rates.

Obviously, the (10) and (11) are all satisfied. And therefore

$$P_i(T > t) = \sum_{j=0}^{\infty} P_i(X(t) = j) = G_i(1,t) = 1.$$

We discuss the following several situations to calculate the stationary distribution of the processes.

Case 1. $0 < \lambda < \mu$.

The probability generating function

$$G_i(z,t) = \left[\frac{1 - \beta - z(\gamma - \beta)}{1 - \beta\gamma - \gamma z(1 - \beta)}\right]^i \left[\frac{1 - \gamma}{1 - \beta\gamma - \gamma z(1 - \beta)}\right]^\delta, \ i \ge 0, \tag{33}$$

where

$$\beta = e^{-(\mu-\lambda)t}, \ \gamma = \frac{\lambda}{\mu}, \ \delta = \frac{a}{\lambda}.$$

From (33), we have

$$G_i(z,t) \to \left(\frac{1-\gamma}{1-\delta z}\right)^\delta \quad \text{as } t \to \infty,$$

and from (18), we know $u_j = \lim_{t\to\infty} P_{ij}(t)$ is Pascal distribution with parameters γ and δ, hence

$$u_j = C_{j+\delta-1}^{\delta-1} \gamma^j (1-\gamma)^\delta = (1-\gamma)^\delta \gamma^j \frac{(\delta)_j}{j!}, \ j \ge 0. \tag{34}$$

Case 2. $0 = \lambda < \mu$.

The probability generating function is

$$G_i(z,t) = (1 - \beta + z\beta)^i e^{\delta(1-\beta)(z-1)}, \tag{35}$$

where

$$\beta = e^{-\mu t}, \ \delta = \frac{a}{\mu}.$$

So,

$$G_i(1,t) \to e^{\delta(z-1)} \quad \text{as } t \to \infty. \tag{36}$$

From (36), we know that u_j is Poisson distribution with parameter δ, and so

$$u_j = \frac{1}{j!}\delta^j e^{-\delta}, \ j \geq 0.$$

Case 3. $0 < \lambda < \mu, \ 0 = \mu < \lambda, \ 0 < \lambda = \mu.$

For these three conditions, we all have

$$G_i(z, t) \to 0, \tag{37}$$

and so,

$$u_j = \lim_{t\to\infty} P_{ij}(t) = 0. \tag{38}$$

From (38) and (13a), we know u_j is invariant measure.

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