Pure -Jump Lévy Processes and Self-decomposability in Financial Modeling

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Abstract

In this study, we review the connections between Lévy processes with jumps and self-decomposable laws. Self-decomposable laws constitute a subclass of infinitely divisible laws. Lévy processes additive processes and independent increments can be related using self-similarity property. Sato (1991) defined additive processes as a generalization of Lévy processes. In this way, additive processes are those processes with inhomogeneous (in general) and independent increments and Lévy processes correspond with the particular case in which the increments are time homogeneous. Hence Lévy processes are considerable as a particular type. Self-decomposable distributions occur as limit law an Ornstein-Uhlenbeck type process associated with a background driving Lévy process. Finally as an application, asset returns are representing by a normal inverse Gaussian process. Then to test applicability of this representation, we use the nonparametric threshold estimator of the quadratic variation, proposed by Cont and Mancini (2007).

Keywords: Pure jump Lévy process, Self-decomposability, Self-similar additive process, Ornstein-Uhlenbeck Process, Non parametric threshold estimator for quadratic variation

1. Introduction

The usual models of modern finance are based on the assumption of normality for asset returns. However, a remarkable number of empirical studies have shown that the assumption of normally distributed observations is a poor approximation for the real data. This is because the returns have features such as jumps, semi-heavy tails and asymmetry. In traditional diffusion models, price movements are very small in short period of times. But in real markets, prices may show big jumps in short time periods. When price process model includes the jumps, the perfect hedging is imposable. In this case, market participants can not hedge risks by using only underlying assets. For these reasons, diffusion models used in finance is not a sufficient model. A good model should consider discontinuities and jumps in price process. There exists an intensive literature proposing different models to overcome this deficiency. Some examples of this are jump diffusion model (Merton(1976)), Stochastic volatility models (Heston(1993)), pure jump Lévy processes (Barndorff-Nielsen (1998)), (Eberlein and Keller(1995)).

Lévy processes is a useful tool in financial modeling providing a good adjustment as can be seen in with reel data by (Carr and Wu (2004)), (Eberlein et.al. (1998)) or (Fajardo (2006)). Lévy process is a simple Markov process with jumps which allow us to capture asset returns without the necessity of introducing extreme parameter values by (Fajardo (2006, p.353)). Lévy models are not adequately fit implied volatility surfaces of equity options across both strike and maturity. The increments of additive process provide us more flexible models. These processes were studied by (Madan, Carr and Chang(1998)), (Carr et.al(2007)) being obtained from self-decomposable distributions.

A law in class of self-decomposable laws can be decomposable into the sum of a scaled down version themselves and an independent term. The class of self-decomposable distributions is obtained as a limit law of a sequence that independent and suitably normalized. The properties of the return distributions depend on length of return interval. Log returns are taken monthly can be reasonably represented by a normal distribution. If one is dealing with tick data, then return distributions may have heavy tails.

Aim of these paper is to review pure- jump Lévy process arising from self-decomposable distributions in financial modeling and to test the presence of a Brownian motion component and discriminating between finite or infinite variation jumps.

The paper is organized as follows. In section 2, we review fundamental properties of Lévy processes and pure jump Lévy process in financial modeling. In section 3, we review the self-decomposable distributions. In section 4, we give some information about the Ornstein – Uhlenbeck process. In section 5, we explain the properties of NIG process. In section 6, we review the modeling of returns. In section 7, we apply test statistics to reel financial return series. Finally we expound our conclusions in the last section.

2. Lévy Processes

In this section, we describe the probabilistic structure of a Lévy process, explaining its usefulness.

Definition 2.1 A cadlag stochastic process $X = (X_t)_{t\geq 0}$ is defined on a filtered probability space $\left(\Omega,\ \Im,\ (\Im_t)_{t\geq 0},\ P\right)$ is called a Lévy processes if the following conditions holds:

i) $X_0 = 0$ a.s.

ii) X has increments independent from the past evolution; i.e. $X_t - X_s$ independent from $\{X_u;\ u \leq s\}$ for $0 \leq s \leq t$

iii) X has stationary increments; i.e. $X_t - X_s$ has the same distribution with $X_{t-s},\ s \leq t$

iv) X_t is stochastically continuous, $\lim\limits_{k\to\infty} P\left(|X_{t+k} - X_t| \geq \varepsilon\right) = 0$ for $\forall\varepsilon > 0$

Lévy processes can be viewed as continuous time random walks. The Walks forming building blocks for both of Markov processes and semi-martingales. There exists a bijection between Lévy process and infinitely divisible distributions Kyprianou(2006).

Theorem 2.1 (Cariboni(2007), p.36)). Let $X = (X_t)_{t\geq 0}$ be a Lévy process. Then X_t has an infinitely divisible distribution$F$for every t. Conversely if$F$is an infinitely divisible distribution, there exist a $X = (X_t)_{t\geq 0}$Lévy process, such that distribution of$X_1$is given F.

The distribution of a Lévy process $X = (X_t)_{t\geq 0}$ is completely determined by any of its marginal distributions (Eberlein (2007, p.4))

2.1 Infinite divisibility

Let $\varphi(u)$ be the characteristic function of random variable X. The law of a random variable X is said to be infinitely divisible if there exist another a characteristic function $\varphi_n(u)$,

$$\varphi(u) = \left[\varphi_n(u)\right]^n \tag{1}$$

If $X \sim \varphi(u)$ is infinitely divisible, then for all $n \in$ N there exist i.i.d. random variables $X_1^{(1/n)}, ..., X_n^{(1/n)}$ all distributed as $\varphi_n(u)$ such that

$$X \overset{d}{=} X_1^{(1/n)} + ... + X_n^{(1/n)} \tag{2}$$

In the other words, a random variable$X$ is always decomposable into the sum of an arbitrary finite number of i.i.d. random variables (Papapentoleon(2006))

2.2 Jump diffusion model

Jump diffusion models are those presenting jumps and a random evolution between the jump times

$$X_t = \mu\, t + \sigma\, W_t + \sum_{i=1}^{N_t} Y_i \tag{3}$$

where, W standard Brownian motion, μ is media and sigma the deviation,$\sum\limits_{i=1}^{N_t} Y_i$ compound Poisson process Now we consider its characteristic function,

$$\varphi_{X_t} = \exp\left\{t\left(iu\mu - \frac{u^2\sigma^2}{2} + \lambda \int_{-\infty}^{\infty}\left(e^{iux} - 1\right)F(dx)\right)\right\} \tag{4}$$

Now, we can consider infinite activity Lévy processes, i.e. they have infinitely many jumps in finite interval.

2.3 The probabilistic properties of Lévy processes

Let X_t be a Lévy process, we consider following characteristic function,

$$\varphi_{X_t}(u) = \exp(iuX_t)$$

For time interval$[0, t]$,$\Delta t = t_i - t_{i-1} = t/n$, by the assumption of independent and stationary

Increments,

$$X_t = (X_{t_1} - X_{t_0}) + ... + (X_{t_n} - X_{t_{n-1}}) \overset{d}{=} n\, X_{t/n} \tag{5}$$

where $\overset{d}{=}$ denotes equality in distribution.

$$\varphi_{X_t}(u) = E\left(\exp\ (iuX_t)\right) = E\left(\exp\ (iunX_{t/n})\right) = \left(E \exp\ (iuX_{t/n})\right)^n$$

$$\varphi_{X_t}(u) = \left(E \exp\ (iuX_1)\right)^{\,t} = e^{t\,\psi(u)} \quad \text{for } n = t, \tag{6}$$

$$\psi(u) = \log E \exp(iuX_1) \tag{7}$$

The every Lévy process can be represented in the following form

$$X_t = \mu\, t + \sigma\, W_t + Z_t \tag{8}$$

Where Z_t is a jump process with infinitely many jumps. The characteristic function of a Lévy process $X = (X_t)_{t \geq o}$ is given by the, *Lévy- Khintchine formula*,

$$E\ e^{iuX_t} = \exp\left\{\left(iub - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - iux1_{\{|x|\leq 1\}}\right)\nu(dx)\right)\right\} \tag{9}$$

Where $\left(b, \sigma^2, \nu\right)$ is called generating triplet, ν does not have mass on 0, $\nu(\{0\}) = 0$ and satisfies the following integrability condition;

$$\int_{\Re} \min\left(|x|^2, 1\right)\nu(dx) < \infty \tag{10}$$

Lévy-Ito representation describes the path structure of a Lévy process. A Lévy process can be represented in the following way,

$$X_t = \mu\, t + \sigma\, W_t + \sum_{s\leq t}\Delta X_s 1_{\{|\Delta X_s| \geq 1\}} + \lim_{\varepsilon\to 0}\left(\sum_{s\leq t}\Delta X_s 1_{\{\varepsilon\leq|\Delta X_s|\leq 1\}} - t\int x1_{\{\varepsilon\leq|x|\leq 1\}}\,\nu(dx)\right) \tag{11}$$

where $b \geq 0$, $\sigma \geq 0$, $(W_t)_{t\geq 0}$ is a standard Brownian motion.

$\Delta X_s = X_s - X_{s^-}$: denote the jump at times. ν is called Lévy measure of $\{X_t\}$. Thus,$\nu(dx)$ is the intensity of jumps of size$x$.We assumed that the paths of Lévy process is defined over a finite intervals $[0, t]$. As a consequence the sum of the jumps in time interval $[0, t]$ with absolute jump size bigger than 1 is a finite sum for each path by Eberlein(2007).

$$\nu(A) = E\left(card\,\{s \in [0, 1]\ : \Delta X_s \neq 0\,, \Delta X_s \in A\}\right), \quad A \in B(R) \tag{12}$$

In the other words,$\nu(A)$ is the average number of jumps of process X in time interval $[0, 1]$ whose sizes fall in$A$.In general in a Lévy process, the frequency of the big jumps determines existence of moments of process. The fine structure of the paths of the process can be read of the frequency the small jumps by Eberlein(2007).

2.4 Finite activity

A stochastic process has finite activity if almost all paths have only a finite number of jumps along any time interval of finite length,

$$\nu(R) = \int_R \nu(dx) < \infty \tag{13}$$

In case almost all paths have infinitely many jumps along any time interval of finite length, we say the processes has infinite activity Sato(1999).

2.5 Finite variation

Let $X = (X_t)_{t\geq 0}$ be a Lévy process.

i) If $\sigma^2 = 0$ and $\int 1_{\{|x|\leq 1\}} |x| \ \nu(dx) < \infty$ and X process have finite variation i.e., $\sum_{s\leq t} |\Delta X_s| < \infty$ if and only if $\int_{\Re} \min(|x|\ , 1)\ \nu(dx) < \infty$

ii) If $\sigma^2 \neq 0$ or $\int 1_{\{|x|\leq 1\}} |x| \ \nu(dx) = \infty$, then process X have infinite variation Eberlein(2007).

3. Self-Decomposable Laws

There is a very close connection the laws of class L and self-decomposable laws. For these reason, first time, we describe self-decomposable laws and related process.

3.1 Laws of class L

The name of class$L$first time used by Lévy(1937) and Khintchine(1938). Any random variable in class L is infinitely divisible by Sato((1999), theorem, 9.3), Pardo(2007). The infinitely divisible laws are the limit laws of triangular arrays, where arrays of independent random variables which individually negligible. Infinite divisibility is preserved under affine transformations. Let $(Y_n\ ; n = 1, 2, ...)$ is a sequence of independent random variables and $S_n = \sum_{i=1}^{n} Y_i$ denotes their sum. Suppose that, there exist centering constants $a_n \in R$ and scaling constants $b_n > 0$, such that the distribution of $b_n S_n + a_n$ converges to the distribution of some random variable X. Then we say that, random variable X is a member of class L. As explained above, we shortly can say that, if a random variable X has same a distribution the limit of some sequence of normalized sums of independent random variables, random variable X has a distribution of class$L$by Carr, Geman, Madan and Yor (2007).

Definition 3.1 (Self-decomposability). We suppose that $\varphi(u)$ is the characteristic function of a law. We say that this law is self-decomposable when for $c \in (0, 1)$, we can a find another characteristic function $\varphi_c(u)$ such that,

$$\varphi(u) = \varphi(cu)\,\varphi_c(u) \tag{14}$$

with $\varphi_c(u)$ again a characteristic function. $\varphi_c(u)$ is uniquely determined. For example normal case, $\varphi_c(u) = \exp[-(1/2)(1-c)u]$. We can restate this definition for random variables as follows, the distribution of a random variable$X$is self-decomposable, for any constant $c \in (0, 1)$ we can a find independent random variable X_c such that,

$$X \overset{d}{=} c\,X + X_c \tag{15}$$

where variables on the right are independent. A random variable X has a distribution of class L if and only if the law of the X is self-decomposable. The class of self-decomposable distribution is a subclass of infinitely divisible distributions. Self-decomposable laws arise as marginal laws in autoregressive time series models,

$$X_t = c\,X_{t-1} + \varepsilon_t$$

The Lévy measure of the self-decomposable laws is absolutely continuous with following density form,

$$\nu(dx) = \frac{k(x)}{x}\,dx \tag{16}$$

where, $k(x)$ is increasing for $(-\infty, x)$ and decreasing for (x, ∞). The density of self-decomposable distributions is unimodal. Let be a Lévy process $X = (X_t : t \geq 0)$. (X_1) is self-decomposable if and only if (X_t) is self-decomposable for every $t > 0$ (Carr, Geman, Madan and Yor (2007)). The characteristic function of self-decomposable laws have following form,

$$E\left[e^{iuX}\right] = \exp\left\{ iu\mu - \frac{u^2\sigma^2}{2} + \int_{\Re} \left(e^{iux} - 1 - iux\,1_{\{|x|<1\}}\right) \frac{k(x)}{x}\,dx \right\} \tag{17}$$

Where $\mu \in R$, $k(x) \geq 0$ and $\int_{\Re} \min\left(1, |x|^2\right) \frac{k(x)}{x}\,dx < \infty$.

(Sato(1999), p.95, Corollary15.11), (Carr et.al(2007), p.34). A self-decomposable random variable X is the value at unit time of some pure jump Lévy processes which sample paths have bounded variation. When the levy density integrates $|x|$ in the region $|x| < 1$, for$\mu = \int_{|x|<1} x\ (k(x)/x)\,dx$. The characteristic function of the processes$X$,

$$E\left[e^{iuX}\right] = \exp\left[\int_{-\infty}^{\infty}\left(e^{iux_1}\right)\frac{k(x)}{x}\,dx\right] \tag{18}$$

We can consider that, the returns are the sum of a suitable number of approximately independent random variables. Furthermore the return distribution is a limit distribution. Self-decomposable distributions can be consider as candidate for the unit period distribution of asset returns. Halgreen (1977) is shown that, the hyperbolic distributions are self-decomposable.

3.2 Self-similarity and self-decomposability

A stochastic process $X = (X(t) : t \geq 0)$ is called self-similar for any given $c \geq 0$,

$$(X(ct) : t \geq 0) \stackrel{d}{=} \left(c^H X(t) : t \geq 0\right) \tag{19}$$

Where,$H > 0$Hurst exponent Petroni ((2008), p.1882). In the other words, we say that one stochastic process is self-similar such that, the change an in time scale can be compensated by a corresponding change in the spaces scale. The connection between self-decomposable laws and self-similar additive process is given by Sato(1991). Law is self-decomposable if and only if it is the law at unit time of a self-similar additive process. Let $\varphi(u)$ be a characteristic function of a law, then, it can take a characteristic function of Lévy process as follow,

$$\varphi_t(u) = [\varphi(u)]^{t/N} \tag{20}$$

Where, N is time scale, if $\varphi(u)$ is infinite divisible, $\varphi_t(u)$ is a characteristic function. Now we describe a new function as follows,

$$\psi_{k,H}(\theta) = \varphi\left[\left(\frac{k}{N}\right)^H \theta\right]\varphi\left[\left(\frac{h}{N}\right)^H \theta\right]^{-1} \tag{21}$$

It is a characteristic function if and only if $\varphi(u)$ is self-decomposable by Sato ((1999),p.99), Petroni((2008),p.1884).The stationary process and the self-similar process are related by using first Lamperti representation.

Proposition 3.1 Let $X = (X_t)_{t\in R}$be a stationary process, the a new process defined by

$$Z_t = t^H X_{\log t}\,, t > 0,\ Z_0 = 0\ \ and\ \ H > 0 \tag{22}$$

is a self-similar process with index $H > 0$. Conversely if the process Z is self-similar with $H > 0$, then process defined by

$$(X_t)_{t\in R} = \left(e^{-tH} Z_{e^t}\right)_{t\in R} \tag{23}$$

is a stationary process((For proof, Pardo(2007), p.9). In general a Lévy process $L = (L_t : t \geq 0)$ may be constructed from the H-self-similar process Z_t in accordance with

$$L_t = \int_t^{e^t} \left(1\big/ s^H\right) dL_s \tag{24}$$

4. Ornstein-Uhlenbeck Processes

An Ornstein-Uhlenbeck process (OU) $X = (X_t)_{t\geq 0}$ satisfies following differential equation,

$$dX_t = -\lambda X_t dt + \sigma dL_t \tag{25}$$

Where, $\lambda > 0$, $\sigma > 0$ and $L = (L_t : t \geq 0)$ is a Lévy process. Homogenous Lévy process L_t has that property $E\left[\log(1 + |L_1|)\right] < \infty$. If L_t is non-Gaussian Lévy process, above differential equation has a unique solution. It has following form,

$$X_t = e^{-\lambda t} X_0 + \sigma \int_0^t e^{-\lambda(t-s)} dL_s \tag{26}$$

Where, $\lambda \in R$ and X_0 is initial state. Let v_0 be Lévy measure such that satisfies log- integrable condition,

$$\int_{|x|\geq 1} \log\, |x|\; v_0\,(dx)\; < \infty \tag{27}$$

If a Lévy process satisfies (27) condition then it has a self-decomposable distribution. If an OU process $X = (X_t)_{t\geq 0}$ is stationary, it's characteristic function has following form,

$$\varphi\,(u) = \varphi\left(u\,e^{-\lambda\,t}\right)\,\varphi_c\,(u) \tag{28}$$

This denotes that, the marginal distribution of (X_t) is self-decomposable.

$$\varphi_c\,(u) = \exp\left(\kappa\,(u) - \kappa\left(u\,e^{-\lambda\,t}\right)\right) \tag{29}$$

Where $\kappa\,(u)$ is cumulant of (X_t) and denoted by as $\kappa\,(u) = \log\,\varphi\,(u)$by Barndorff-Nielsen O.E.,(1998), lemma 3.1). Let L be the background driving Lévy process of OU processes $X$and $\int_{|x|\geq 1} \log\,|x|\; v_0\,(dx)\; < \infty$. Where v is Lévy measure of the process$L$. Then the law of X_t converge towards ζ a self-decomposable law as $u \to \infty$. Characteristic function of this process is given by

$$\zeta\,(u) = \exp\left\{\int_0^{\infty} \phi\left(u\,e^{-\lambda\,s}\right)\,ds\right\} \tag{30}$$

ϕ is the characteristic exponent of L_1 process. (For details Sato (1999), theorem 17.5)

Finally, we can say that the limit distribution ζ of an OU process is self-decomposable. The distribution of a random variable X_1 is self-decomposable if

$$X_1 \overset{d}{=} \int_0^{\infty} e^{-s} dL_s \tag{31}$$

An OU process X_t has a stationary distribution. It's characteristic function $\varphi\,(u) = \exp\,(\psi\,(\theta))$ is self-decomposable and

$$\psi_{X_1}\,(\theta) = \int_0^{\infty} \psi_{L_1}\,(\theta\,e^{-as})ds \tag{32}$$

5. Normal inverse Gaussian distribution

Normal Inverse Gaussian Distribution was introduced by Barndorff-Nielsen(1997). This distribution is used in Bolviken and Benth(2000), Prause(1999), Rydberg(1997) to model equity returns. NIG distribution has semi-heavy tails and a special case with $\lambda = -1/2$ parameter in the class of Generalized Hyperbolic distributions (GH). GH class is infinitely divisible and self-decomposable. The probability density function of the $NIG\,(\alpha,\beta,\mu,\delta)$ is defined by as follows,

$$f_{NIG}\,(x,\alpha,\beta,\mu,\delta) = \frac{\alpha\,\delta}{\pi}\,\exp\left[\delta\,\sqrt{\alpha^2-\beta^2} + \beta\,(x-\mu)\right].\,\frac{K_1\left(\alpha\,\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}} \tag{33}$$

with parameters, $\delta > 0$, $\mu \in (-\infty,\infty)$ *and* $0 < |\beta| < \infty$. The function K_ν is modified Bessel function of third kind with index ν.

$$K_\nu\,(x) = \frac{1}{2}\int_0^{\infty} u^{\nu-1}\exp\left\{-\frac{1}{2}\,x\left(u+\frac{1}{u}\right)\right\}du \qquad (x>0)$$

According to Blaesild(1981),

$$K_{\mp\frac{1}{2}}\,(x) = \sqrt{\frac{\pi}{2x}}.e^{-x}\text{and}K_\nu\,(x) = K_{-\nu}\,(x)$$

The characteristic function of NIG distribution,

$$\varphi_{NIG}(u) = \exp\left\{\delta\sqrt{\alpha^2-\beta^2} - \delta\sqrt{\alpha^2-(\beta+iu)^2} + iu\mu\right\} \tag{34}$$

The normal Inverse Gaussian distribution has semi-heavy tails i.e.

$$f_{NIG}(x) \sim const\ |x|^{-3/2}\, e^{(-\alpha\ |x|+\beta\, x)} \qquad as \quad x \to \mp\infty \tag{35}$$

The central moments of a random variable $X \sim NIG(\alpha, \beta, \mu, \delta)$ are

$$E[X_1] = \mu + \delta\frac{\beta}{\sqrt{\alpha^2-\beta^2}} \qquad Var[X_1] = \frac{\delta}{\sqrt{\alpha^2-\beta^2}} + \frac{\delta\beta^2}{\left(\sqrt{\alpha^2-\beta^2}\right)^3} = \delta\frac{\alpha^2}{\gamma^3}$$

$$Skew[X_1] = 3\frac{\beta}{\alpha\sqrt{\delta\sqrt{\alpha^2-\beta^2}}} \qquad Kurtosis[X_1] = 3 + 3\left[1 + 4\left(\frac{\beta}{\alpha}\right)^2\right]\frac{1}{\delta\sqrt{\alpha^2-\beta^2}}$$

The Lévy measure of NIG has following form,

$$\nu_{NIG}(dx) = \frac{\alpha}{\pi}.\frac{\delta}{|x|}\ \exp\{\beta\, x\}\ K_1(\alpha\ |x|)\ dx \tag{36}$$

6. The Modeling of Returns

In this study, we consider log price changes,

$$X_t = \ln S_{t+\Delta} - \ln S_t \tag{37}$$

Where, X process reflects the multiplicative character of price changes. When we use daily price data, Δ typically will have the value 1. In generally, the properties of return distribution depend on the length of the return interval Δ. For long Δ as a result of central limit theorem, the returns can be described with a Gaussian distribution. In general, the normal distribution is not suitable model returns. In this case, as a model the Generalized hyperbolic distributions or it's a subset can be used. These distributions are infinitely divisible and self-decomposable. In this study we used a non-parametric threshold technique which is proposed to test integrated variance which based on discrete observed prices by Cont and Mancini(2007). We can determine whether jump type process is suitable or not as a model for price process, using with this technique. We assume that $X = (X_t : t \geq 0)$ is Lévy process generated by the normal inverse Gaussian distribution that is fitted to the real data. In this case, the increments of price process a long time interval 1, $X_{t+1} - X_t$ are distributed according to the NIG distribution. The distribution of X_1 is the fitted to NIG distribution.

6.1 Test statistics

In this section we consider Blumenthal-Getoor index is described as follows,

$$\alpha = \inf\left\{p \geq 0\,,\ \int_{\{|x|\leq 1\}} |x|^p\ \nu(dx)\ < \infty\right\} < 2 \tag{38}$$

This index measures activities of small jumps of Lévy process. For jump type Lévy process, $\alpha \in [0, 2]$. For example Normal inverse Gaussian motion has infinitely variation and $\alpha = 1$ by Cont and Mancini(2007).

6.1.1 Test for the presence of a continuous martingale component of a Lévy model

For choice a coefficient β we suggest that it must be near 1 but different to 1. $\beta \in [0.5, 1]$

We choose a threshold $r(k) = k^\beta$ and

$$\Delta_i Y = \Delta_i X + \sigma\ \sqrt{k}\, Z_i$$

Where $Z_i \sim N(0, 1)$ as $n = T/k$ or $k \to 0$,time horizon $T = nxk$, T is measured by as an annually, $\Delta_i X$ denotes log-returns

Null Hypothesis $H_0 :\ \sigma \equiv 0$ Test statistics as follow,

$$\hat{V}_k = \sum\nolimits_{i=1}^{n} (\Delta_i Y)^2 \, I\left\{(\Delta_i Y)^2 \le r(k)\right\} \quad , \quad \hat{Q}_k = (1/3k) \sum\nolimits_{i=1}^{n} (\Delta_i Y)^4 \, I\left\{(\Delta_i Y)^2 \le r(k)\right\}$$

$$T_k^{\alpha} = \left[\left(\hat{V}_k - \sigma^2 n.k\right) \Big/ \sqrt{2k\,\hat{Q}_k}\right] \to N(0,1)$$

If $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\}$ is near 0.05, we accept null hypothesis.

Under the alternative hypothesis$H_1 : \sigma \neq 0$, above test statistic diverges to $+\infty$

Remark: For a given constant σ value, the above procedure again apply

$$T_{k(j)}^{\alpha} = \left[\left(\hat{V}_k - \sigma^2 n.k\right) \Big/ \sqrt{2k\,\hat{Q}_k}\right] \quad , \; j = 1, 2, 3, ..., M$$

Then we calculate rate of $T_{k(j)}^{\alpha}$ statistic which satisfies condition $\left|T_{k(j)}^{\alpha}\right| > 1.96$.

$$\hat{V}_k = \sum\nolimits_{i=1}^{n} (\Delta_i Y)^2 \, I\left\{(\Delta_i Y)^2 \le r(k)\right\} \quad , \quad \hat{Q}_k = (1/3k) \sum\nolimits_{i=1}^{n} (\Delta_i Y)^4 \, I\left\{(\Delta_i Y)^2 \le r(k)\right\}$$

If $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\} >> 0.05$, we reject the null hypothesis. Where $\Delta_i X = X_{t_i} - X_{t_{i-1}}$ and σ^2 is a know variance.

6.1.2 The test whether the jump component has finite variation

Null Hypothesis $H_0 : \alpha < 1$

$$\Delta_i \hat{M} = \Delta_i X \; I\left\{(\Delta_i X)^2 > r(k)\right\} + \sigma \sqrt{k}\, Z_i \qquad , \; Z_i \sim N(0,1)$$

$$\hat{V}_k = \sum\nolimits_{i=1}^{n} \left(\Delta_i \hat{M}\right)^2 I\left\{\left(\Delta_i \hat{M}\right)^2 \le r(k)\right\} \quad , \quad \hat{Q}_k = (1/3k) \sum\nolimits_{i=1}^{n} \left(\Delta_i \hat{M}\right)^4 I\left\{\left(\Delta_i \hat{M}\right)^2 \le r(k)\right\}$$

$$T_k^{\alpha} = \left[\left(\hat{V}_k - \alpha^2 T\right) \Big/ \sqrt{2k\,\hat{Q}_k}\right] \to N(0,1)$$

If $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\}$ is near 0.05, we accept null hypothesis.

6.1.3 The test of presence jumps

The variance of a Lévy processes is estimated as following form, $S_X = \sum\limits_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$

Where each t_i is a division of interval $[0, t]$ for each $n \in$ N. $\left|t_i - t_{t_{i-1}}\right| = (t/n) = \Delta t_i$ and $t = n.\Delta_{t_i}$

$$S_X^2 = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})\,(X_{t_i} - X_{t_{i-1}}) \quad , \quad S_X^4 = \left[\frac{n}{t}\right] \sum_{i=4}^{n} \left|\Delta_n X_{t_{i-1}}\right| \; ... \; \left|\Delta_n X_{t_{i-4}}\right|$$

Null hypothesis $H_0 : \; X_t$ is a continuous Lévy process

$$Z_n = \left[\left(S_X - \frac{S_X^2}{K^2}\right) \Bigg/ \left(\sqrt{\frac{t}{n}} \; \sqrt{\frac{S_X^4}{K^4}}\right)\right] \quad , \quad K = (2/\pi)^{0.5}$$

If $Z_n > z_\alpha$, we reject the null hypothesis , where z_α is chosen as a quantile (Petr(2007,p.2008)).

7. Application to Real Data

In this section we consider the time series of NIKKEI 225 in Japan daily returns from 04.01.1982 to 30.09.2005 and ISE Compound 100 index in Turkey daily returns from 06.02.2002 to 15.02.2007 as an application for above presented model. We use $k = 1/252 = 0,003968$ and $r(k) = k^{0,75}$. First time we divided return that each has 500 daily return for NIKKEI 225 index, similarly ISE Composite 100 index returns divided sub groups which include 300 daily return.

For ISE 100 index returns, the probability $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\}$ is not near to 0.05. For this reason, we don't accept H_0 : $\sigma \equiv 0$ null hypothesis. In this case, ISE Composite 100 index price process has continuous time martingale component (i.e. Brownian motion component) Table 1.

In the testing finite variation of jump component of ISE Composite 100 return process, we found that different value for k values, $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\} >> 0.05$ thus we don't accept H_0 : $\alpha < 1$ null hypothesis. This result said that to us, ISE Composite 100 return process can be modeled by Levy process specially we can use NIG distribution as a model for ISE Composite 100 return process. Because Blumenthal-Getoor index α can be $\alpha \geq 1$.This mean is that the return process has infinitely variation thus we can use hyperbolic models in this case.Table2.

For NIKKEI 225 index, $P\left\{ \left|T_k^{\alpha}\right| > 1.96 \right\} >> 0.05$ so we can accept that this index include the Brownian component and infinite variation of jump component.Table3 and Table4.

Both of the ISE 100 and NIKKEI 225 indexes, the values of Z_n statistics are very high. For this reason, we reject that H_0 : X_t is a continuous Lévy process, null hypothesis. Because the both index include Brownian motion part and jump component together. You can look Table5.

8. Conclusion

The laws of self-decomposable distributions class are be constitute as a limit laws of Lévy – driven Ornstein-Uhlenbeck process. These laws can be related with general additive process. The Lévy processes has flexible structure to model in real phenomena in finance such as heavy-tails, jumps and volatility smile. In this study, we focus on Levy process and it's increments laws. The family of self-decomposable laws always has both of self-similarity and stationary of increments. We tested whether the pure jump Levy process is a suitable model or not for financial return series. We used a threshold estimator of quadratic variation by proposed by Cont and Mancini. We applied this test statistics to two international index return series as ISE composite 100 and NIKKEI 225. The empirical results on the ISE Composite 100 and NIKKEI 225 indexes indicated that jumps are present in the data. Both of them have presence of infinity activity jumps and a continuous component. The short-time behavior of the pure jump component can be quantized its Blumenthal-Getoor index. When the value of this index approach to 2, the small jumps near to zero increase. In this case process likes diffusions. We do not claim on the basis of this limited evidence that empirical results are a general way for all market. Because the empirical results can depend on the considering assets the time period the type of the data.

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Table 1. The testing for the presence of a Brownian component for ISE Composite 100 index price process

ISE 100	1	2	3	4	5
$\hat{V}_k$	0,029252	0,190011	0,130958	0,166943	0,027818
$\hat{Q}_k$	0,586629	0,12095	0,015813	0,040404	0,003087
$T_{k(j)}$	0,413469	6,116505	11,56244	9,301056	5,570338

Table 2. The testing finite variation of jump component of ISE Composite 100 index price process

ISE 100	1	2	3	4
$\hat{V}_k$	0,280954	0,067261	0,001438	0,023769
$\hat{Q}_k$	0,525418	0,093732	1,89E-06	0,018
$T_{k(j)}$	4,334696	2,447545	0,073143	1,956218

Table 3. The testing for the presence of a Brownian component for NIKKEI 225 Index

NIKKEI 225	1	2	3	4	5	6
$\hat{V}_k$	0,08143	0,223476	0,559763	0,378513	0,525433	0,27848
$\hat{Q}_k$	0,002493	0,207762	0,183874	0,064663	0,1103662	0,025918
$T_{k(j)}$	18,26515	5,491945	14,62282	16,67741	17,72089	19,37838

Table 4. The testing finite variation of jump component of NIKKEI 225 Index returns

NIKKEI 225	1	2	3	4	5	6
$\hat{V}_k$	0,000185	0,059225	0,098538	0,0498	0,067396	0,000549
$\hat{Q}_k$	9,09E-09	0,187877	0,119161	0,030012	0,0472258	9,39E-08
$T_{k(j)}$	0,336161	1,521648	3,166666	3,181129	3,427891	-0,01413

Table 5. The testing the presence of jumps under the hypothesis that price process is continuous Lévy process

Index	Z_n
NIKKIE 225	179.1
ISE Composite 100	113.7

Table 6. Estimated parameters of the normal inverse Gaussian distribution

	NIG Parameters			
Index	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\delta}$
NIKKEI 225	0,261649	0,000218	0,000096	0,000045
ISE 100	0,525443	0,007822	0,001017	0,000238