Some Results on Prime and $k-$Prime Labeling

S. K. Vaidya (Corresponding author)

Department of Mathematics, Saurashtra University

Rajkot 360005, India

E-mail: samirkvaidya@yahoo.co.in

U. M. Prajapati

Department of Mathematics, St. Xavier's College

Ahmedabad 380009, India

E-mail: udayan64@yahoo.com

Abstract

A graph $G = (V, E)$ with n vertices is said to admit *prime labeling* if its vertices can be labeled with distinct positive integers not exceeding n such that the labels of each pair of adjacent vertices are relatively prime. A graph G which admits prime labeling is called a prime graph. In the present work we investigate some classes of graphs which admit prime labeling. We also introduce the concept of $k-$prime labeling and investigate some results concern to it. This work is a nice combination of graph theory and elementary number theory.

Keywords: Graph Labeling, Prime labeling, Prime graph, $k-$prime labeling, $k-$prime graph

1. Introduction

We begin with simple, finite, undirected and non-trivial graph $G = (V, E)$ with the vertex set V and the edge set E. The number of elements of V, denoted as $|V|$ is called the *order* of the graph G while the number of elements of E, denoted as $|E|$ is called the *size* of the graph G. In the present work C_n denotes the cycle with n vertices and P_n denotes the path of n vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to K_1 is called the *apex vertex* and the vertices corresponding to C_n are called the *rim vertices*. The graph $f_n = P_{n-1} + K_1$ is called a *fan* and the vertex corresponding to K_1 is called the *apex vertex* of the fan. For various graph theoretic notations and terminology we follow (Gross, J. & Yellen, J., 2004) whereas for number theory we follow (Burton, D. M.,1990). We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For latest survey on graph labeling we refer to (Gallian, J. A., 2009). Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. For any graph labeling problem following three features are really noteworthy:

- a set of numbers from which vertex labels are chosen;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

The present work is aimed to discuss one such labeling known as prime labeling.

Definition 1.2 *A prime labeling* of a graph G is an injective function $f : V \rightarrow \{1, 2, \cdots, |V|\}$ such that for every pair of adjacent vertices u and v, $\gcd(f(u), f(v)) = 1$. The graph which admits prime labeling is called a *prime graph*.

The notion of prime labeling was originated by Entringer and was discussed in (Tout, A., 1982, p. 365-368). Many researchers have studied prime graphs. It has been proved by (Fu, H. L., 1994, p. 181-186) P_n on n vertices is a prime graph. It has been proved by (Lee, S. M., 1988, p. 59-67) wheel graph W_n is a prime graph if and only if n is even. In (Deretsky, T. D., 1991, p. 359-369) cycle C_n is a prime graph.

Definition 1.3 The graph $G = \ < W_n : W_m >$ is the graph obtained by joining apex vertices of wheels W_n and W_m to a new vertex w.

Definition 1.4 A *t-ply* $P_t(u, v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v.

Definition 1.5 Two prime integers are said to be *twin primes* if they differ by 2.

2. Some Results on Prime Graphs

Theorem 2.1 Let n_1 and n_2 be two even positive integers such that $n_1 + n_2 + 3 = p$ where p and $p - 2$ are twin primes. Then the graph $G = \ < W_{n_1} : W_{n_2} >$ is a prime graph.

Proof: Let $u_1, u_2, \cdots, u_{n_1}$ be the consecutive rim vertices of W_{n_1} and $v_1, v_2, \cdots, v_{n_2}$ be the consecutive rim vertices of W_{n_2}. Let u_0 and v_0 be the apex vertices of W_{n_1} and W_{n_2} respectively which are adjacent to a new common vertex w.

Define $f : V \longrightarrow \{1, 2, 3, \cdots, |V|\}$ as follows:

$$f(u_i) = 1 + i, \forall i = 0, 1, 2, \cdots, n_1;$$

$$f(v_0) = p\ ;$$

$$f(v_j) = 1 + n_1 + j, \forall j = 1, 2, 3, \cdots, n_2 \text{ and}$$

$$f(w) = p - 1.$$

Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$.

To prove our claim the following cases are to be considered.

1. If e is an edge of W_{n_1} then we have the following possibilities:
   - if $e = u_j u_{j+1}$ for some $j \in \{1, 2, \cdots, n_1 - 1\}$ then $\gcd(f(u_j), f(u_{j+1})) = \gcd(1 + j, 1 + j + 1) = \gcd(j + 1, j + 2) = 1$ as $j + 1$ and $j + 2$ are consecutive positive integers;
   - if $e = u_{n_1} u_1$ then $\gcd(f(u_{n_1}), f(u_1)) = \gcd(n_1 + 1, 2) = 1$ as $n_1 + 1$ is an odd integer;
   - if $e = u_0 u_j$ for some $j \in \{1, 2, \cdots, n_1\}$ then $\gcd(f(u_0), f(u_j)) = \gcd(1, j + 1) = 1$.
2. If $e = u_0 w$ then $\gcd(f(u_0), f(w)) = \gcd(1, p - 1) = 1$.
3. If $e = v_0 w$ then $\gcd(f(v_0), f(w)) = \gcd(p, p - 1) = 1$ as p and $p - 1$ are consecutive positive integers.
4. If e is an edge of W_{n_2} then we have the following possibilities:
   - if $e = v_j v_{j+1}$ for some $j \in \{1, 2, 3, \cdots, n_2 - 1\}$ then $\gcd(f(v_j), f(v_{j+1})) = \gcd(1 + n_1 + j, 1 + n_1 + j + 1) = 1$ as $1 + n_1 + j$ and $1 + n_1 + j + 1$ are consecutive positive integers;
   - if $e = v_{n_2} v_1$ then $\gcd(f(v_{n_2}), f(v_1)) = \gcd(1 + n_1 + n_2, n_1 + 2) = \gcd(p - 2, n_1 + 2) = 1$ as $p - 2$ is a prime number greater than $n_1 + 2$;
   - if $e = v_0 v_j$ for some $j \in \{1, 2, \cdots, n_2\}$ then $\gcd(f(v_0), f(v_j)) = \gcd(p, 1 + n_1 + j) = 1$ as p is a prime number greater than $1 + n_1 + j$.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. i.e. G is a prime graph.

Illustration 2.2 A prime labeling of $< W_{10} : W_6 >$ is shown in *Figure 1*.

Theorem 2.3 If $n_1 \geq 4$ is an even integer and $n_2 \in N$ then the disjoint union of the wheel W_{n_1} and the path graph P_{n_2} is a prime graph.

Proof: Let u_0 be the apex vertex, $u_1, u_2, \cdots, u_{n_1}$ be the consecutive rim vertices of W_{n_1} and $v_1, v_2, \cdots, v_{n_2}$ be the consecutive vertices of P_{n_2}. Let G be the disjoint union of W_{n_1} and P_{n_2}.

Define $f : V \rightarrow \{1, 2, 3, \cdots, |V|\}$ as follows:

$f(u_i) = 1 + i, \forall i = 0, 1, 2, \cdots, n_1$; and $f(v_j) = n_1 + 1 + j, \forall j = 1, 2, 3, \cdots, n_2$.

Then obviously f is an injection.

For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$ due to following reasons:

1. If e is an edge of W_{n_1} such that $e = u_0 u_i$ for some $i \in \{1, 2, 3, \cdots, n_1\}$ then $\gcd(f(u_0), f(u_i)) = \gcd(1, f(u_i)) = 1$;
2. If e is an edge of W_{n_1} such that $e = u_i u_{i+1}$ for some $i \in \{1, 2, 3, \cdots, n_1 - 1\}$ then $\gcd(f(u_i), f(u_{i+1})) = \gcd(1 + i, 1 + i + 1) = 1$ as every pair of consecutive positive integers are relatively prime;

3. If e is an edge of W_{n_1} such that $e = u_{n_1}u_1$ then
$\gcd(f(u_{n_1}), f(u_1)) = \gcd(n_1 + 1, 2) = 1$ because $n_1 + 1$ is an integer;

4. If e is an edge of P_{n_2} such that $e = v_jv_{j+1}$ for some $j \in \{1, 2, \cdots, n_2 - 1\}$. Then $\gcd(f(v_j), f(v_{j+1})) = \gcd(n_1 + 1 + j, n_1 + 1 + j + 1) = 1$ as $n_1 + 1 + j$ and $n_1 + 1 + j + 1$ are consecutive positive integers.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. i.e. G is a prime graph.
Illustration 2.4 A prime labeling of the disjoint union of W_8 and P_6 is shown in the *Figure 2*.

Theorem 2.5 If $n_1 \geq 4$ is an even integer and $n_2 \in N$ then the graph obtained by identifying any of the rim vertices of a wheel W_{n_1} with an end vertex of a path graph P_{n_2} is a prime graph.

Proof: Denote the apex vertex of W_{n_1} by u_0 and the consecutive rim vertices of W_{n_1} by $u_1, u_2, \cdots, u_{n_1}$. Let $v_1, v_2, \cdots, v_{n_2}$ be the consecutive vertices of P_{n_2}. Without loss of generality assume that the end vertex v_1 of P_{n_2} is identified with the rim vertex u_{n_1} of W_{n_1}. Define $f : V \to \{1, 2, 3, \cdots, |V|\}$ as follows: $f(u_i) = 1 + i$, $\forall i = 0, 1, 2, \cdots, n_1$; and $f(v_j) = n_1 + j$, $\forall j = 2, 3, 4, \cdots, n_2$. Then obviously f is an injection. For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$ due to following reasons:

1. If e is an edge of W_{n_1} such that $e = u_0u_i$ for some $i \in \{1, 2, 3, \cdots, n_1\}$ then $\gcd(f(u_0), f(u_i)) = \gcd(1, f(u_i)) = 1$;

2. If e is an edge of W_{n_1} such that $e = u_iu_{i+1}$ for some $i \in \{1, 2, 3, \cdots, n_1 - 1\}$ then for $i \neq n_1 - 1$, $\gcd(f(u_i), f(u_{i+1})) = \gcd(i + 1, i + 2) = 1$ as $i + 1$ and $i + 2$ are consecutive positive integers;

3. If e is an edge of W_{n_1} such that $e = u_{n_1}u_1$ then $\gcd(f(u_{n_1}), f(u_1)) = \gcd(n_1 + 1, 2) = 1$ as $n_1 + 1$ is an odd positive integer;

4. If e is an edge of P_{n_2} such that $e = v_jv_{j+1}$ for some $j \in \{1, 2, \cdots, n_2 - 1\}$. In this case $\gcd(f(v_j), f(v_{j+1})) = \gcd(n_1 + j, n_1 + j + 1) = 1$ as $n_1 + j$ and $n_1 + j + 1$ are consecutive positive integers.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. Which implies that G is a prime graph.

Illustration 2.6 A prime labeling of the graph obtained by identifying an end vertex of P_6 with a rim vertex of W_8 is shown in the following *Figure 3*.

Theorem 2.7 If n_1 is even then the graph G obtained by identifying the apex vertex of a wheel graph W_{n_1} with an end vertex of P_{n_2} is a prime graph.

Proof: Let u_0 be the apex vertex, $u_1, u_2, \cdots, u_{n_1}$ be the consecutive rim vertices of W_{n_1} and $v_1, v_2, \cdots, v_{n_2}$ be the consecutive vertices of P_{n_2}. Without loss of generality assume that the end vertex v_1 of P_{n_2} is identified with the apex vertex u_0 of W_{n_1}. Define $f : V \to \{1, 2, 3, \cdots, |V|\}$ as follows: $f(u_i) = 1 + i$, $\forall i = 0, 1, 2, \cdots, n_1$; and $f(v_j) = n_1 + j$, $\forall j = 2, 3, 4, \cdots, n_2$. For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$. Following reasons prove the claim.

1. if e is an edge of W_{n_1} then the restriction of f on $\{1, 2, \cdots, |V(W_{n_1})|\}$ admits a prime labeling of W_{n_1} as reported in (Lee, S. M., 1988). Thus $\gcd(f(a), f(b)) = 1$ for this edge e.

2. if e is an edge of P_{n_2} such that $e = v_jv_{j+1}$ for some $j \in \{1, 2, \cdots, n_2\}$. In this case:

    - for $j \neq 1$, $\gcd(f(v_j), f(v_{j+1})) = \gcd(n_1 + j, n_1 + j + 1) = 1$ as $n_1 + j$ and $n_1 + j + 1$ are consecutive positive integers.
    - for $j = 1$, $\gcd(f(v_j), f(v_{j+1})) = \gcd(f(v_1), f(v_2)) = \gcd(1, n_1 + 2) = 1$.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. Which implies that G is a prime graph.

Illustration 2.8 A prime labeling of the graph defined by identifying an end vertex of P_6 with the apex vertex of W_8 is shown in the *Figure 4*.

Theorem 2.9 Let G_1 be a prime graph of order n_1 with a prime labeling f and having vertices u_1 and u_{n_1} with the labels 1 and n_1 respectively. Then the graph G obtained by identifying an end vertex of a path P_{n_2} with either u_1 or u_{n_1} of G_1 is a prime graph.

Proof: Let the vertices of a prime graph G_1 be $u_1, u_2, \cdots, u_{n_1}$ and the prime labeling f of G_1 be such that $f(u_1) = 1$ and $f(u_{n_1}) = n_1$. Let $v_1, v_2, \cdots, v_{n_2}$ be the consecutive vertices of P_{n_2}. We have the following two cases:

1. The graph G is obtained by identifying an end vertex v_1 of P_{n_2} with the vertex u_{n_1} of G_1. (The proof is similar if the other end vertex v_{n_2} of P_{n_2} is identified with the vertex u_{n_1} of G_1.) Define a labeling function g on G as follows:

   $g(u_i) = f(u_i),\ \forall i = 1, 2, 3, \cdots, n_1$; and $g(v_j) = n_1 + j - 1, \forall j = 2, 3, 4, \cdots, n_2$.

   Obviously g is an injection. Also g is an extension of the prime labeling function f on G, it is enough to prove the following cases:

   (a) $\gcd(g(u_{n_1}), g(v_2)) = 1$. To prove this we have $\gcd(g(u_{n_1}), g(v_2)) = \gcd(f(u_{n_1}), g(v_2)) = \gcd(n_1, n_1 + 1) = 1$ as n_1 and $n_1 + 1$ are consecutive integers.

   (b) For each $j \in \{2, 3, \cdots, n_2 - 1\}$, $\gcd(g(v_j), g(v_{j+1})) = 1$. To prove this we have $\gcd(g(v_j), g(v_{j+1})) = \gcd(n_1 + j - 1, n_1 + j) = 1$ as $n_1 + j - 1$ and $n_1 + j$ are consecutive integers.

2. The graph G is obtained by identifying the other end vertex v_1 of P_{n_2} with the vertex u_1 of G_1. (The proof is similar if the other end vertex v_{n_2} of P_{n_2} is identified with the vertex u_1 of G_1.)

   Define a labeling function g on G as follows:

   $g(u_i) = f(u_i),\ \forall i = 1, 2, 3, \cdots, n_1$; and $g(v_j) = n_1 + j - 1, \forall j = 2, 3, 4, \cdots, n_2$.

   Obviously g is an injection. Also g is an extension of the prime labeling function f on G, it is enough to prove the following cases:

   (a) $\gcd(g(u_1), g(v_2)) = 1$. To prove this we have $\gcd(g(u_1), g(v_2)) = \gcd(f(u_1), g(v_2)) = \gcd(1, n_1 + 1) = 1$.

   (b) For each $j \in \{2, 3, \cdots, n_2 - 1\}$ we need to show that $\gcd(g(v_j), g(v_{j+1})) = 1$. To prove this we have $\gcd(g(v_j), g(v_{j+1})) = \gcd(n_1 + j - 1, n_1 + j) = 1$ as $n_1 + j - 1$ and $n_1 + j$ are consecutive integers.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. Which implies that G is a prime graph.

Theorem 2.10 A graph G obtained by identifying all the apex vertices of m fans $f_{n_1}, f_{n_2}, \cdots, f_{n_m}$ (is called a multiple shell) is a prime graph.

Proof: A fan graph $f_n = P_{n-1} + K_1$ has n vertices and $2n - 3$ edges. Let the graph G is obtained by fusing all the apex vertices of $f_{n_1}, f_{n_2}, \cdots, f_{n_m}$. Let the common apex vertex of each of the fans f_{n_i} after fusing all the apex vertices of all the fans $f_{n_1}, f_{n_2}, \cdots, f_{n_m}$ be v_0. For each $i \in \{1, 2, 3, \cdots, m\}$, denote the remaining vertices of the fan f_{n_i} as $v_{i,1}, v_{i,2}, \cdots, v_{i,n_{i-1}}$ consecutively. Clearly $|V| = n_1 + n_2 + \cdots + n_m - m + 1$. Without loss of generality assume that $n_1 \leq n_2 \leq \cdots \leq n_m$. Define $f : V \to \{1, 2, \cdots, |V|\}$ as follows:

$f(v_0) = 1$ and $f(v_{i,j}) = \sum_{k=1}^{i-1} n_k - (i-2) + j, \forall i = 1, 2, \cdots, m$ and $\forall j = 1, 2, \cdots, n_i - 1$. Here we define $\sum_{x}^{y} a = 0$ if x and y are any positive integers with $y < x$. First we will show that f is an injection. It is easy to check that $f(v) = 1$ if and only if $v = v_0$. For $i \in \{1, 2, \cdots, m-1\}$, $j \in \{1, 2, \cdots, n_{i+1} - 1\}$ and $j' \in \{1, 2, \cdots, n_i - 1\}$, we get

$$\begin{aligned} f(v_{i+1,j}) - f(v_{i,j'}) &= \left(\sum_{k=1}^{i} n_k - (i + 1 - 2) + j\right) - \left(\sum_{k=1}^{i-1} n_k - (i-2) + j'\right) \\ &= (n_i - 1 - j') + j \\ &\geq j \quad \text{as} \quad 1 \leq j' \leq n_i - 1. \end{aligned}$$

Thus $f(v_{i+1,j}) - f(v_{i,j'}) > 0$. Thus if $i < i'$ with $i, i' \in \{1, 2, \cdots, m\}$ then $f(v_{i,j}) < f(v_{i',j'}), \forall j \in \{1, 2, \cdots, n_i - 1\}$ and $j' \in \{1, 2, \cdots, n_i' - 1\}$. If $i \neq i'$ then without loss of generality assume that $i < i'$. $i < i' \Rightarrow f(v_{i,j}) < f(v_{i,j'})$. Similarly we have $i' < i \Rightarrow f(v_{i',j'}) < f(v_{i,j})$. That is $f(v_{i',j'}) = f(v_{i,j}) \Rightarrow i' = i$. If $i = i'$ then $f(v_{i,j}) = f(v_{i',j'}) \Rightarrow f(v_{i,j}) = f(v_{i,j'})$. $\Rightarrow \sum_{k=1}^{i-1} n_k - (i-2) + j = \sum_{k=1}^{i-1} n_k - (i-2) + j' \Rightarrow j = j'$. Which shows that f is an injection. It is enough to show that f is a prime labeling. Let $e = uv$ be an edge of G. Then clearly it must be an edge of exactly one of the fans f_{n_i} for some $i = 1, 2, \cdots, m$. If one of the end vertices of e is v_0 say $u = v_0$ then $\gcd(f(u), f(v)) = \gcd(f(v_0), f(v)) = \gcd(1, f(v)) = 1$. If none of the end vertices of e is v_0 then clearly $\{u, v\} = \{v_{i,j}, v_{i,j+1}\}$ for some $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, n_i - 1\}$.

$$\begin{aligned} \text{That is} \quad \gcd(f(u), f(v)) &= \gcd\Big(f(v_{i,j}), f(v_{i,j+1})\Big) \\ &= \gcd\left(\sum_{k=1}^{i-1} n_k - (i-2) + j,\ \sum_{k=1}^{i-1} n_k - (i-2) + j + 1\right) \\ &= 1 \end{aligned}$$

as consecutive integers are relatively prime. Thus f admits a prime labeling for G. i.e. G is a prime graph.

Illustration 2.11 The graph G obtained by identifying all the apex vertices of three fans f_3, f_4, f_5 has prime labeling is shown in *Figure 5*.

Theorem 2.12 A graph G obtained by identifying all the apex vertices of m wheels $W_{n_1}, W_{n_2}, \cdots, W_{n_m}$ is a prime graph if each $n_i \geq 4$ is an even integer for each $i \in \{1, 2, \cdots, m\}$ and $n_i - 1$ is relatively prime with $2 + \sum_{k=1}^{i-1} n_k$ for each $i \in \{2, 3, \cdots, m\}$.

Proof: Let the common apex vertex of G be u_0 and the consecutive rim vertices of each of the wheels W_{n_i} be $u_{i,1}, u_{i,2}, \cdots, u_{i,n_i}$ for each $i \in \{1, 2, \cdots, m\}$. Here we define $\sum_{x}^{y} a = 0$ if x and y are any positive integers with $y < x$.

Define $f : \{u_0\} \bigcup \left(\bigcup_{i=1}^{m} \{u_{i,1}, u_{i,2}, u_{i,3}, \cdots, u_{i,n_i}\} \right) \longrightarrow \{1, 2, \cdots, 1 + \sum_{i=1}^{m} n_i\}$ as

$$f(x) = \begin{cases} 1 & , \text{if } x = u_0 \\ 1 + j + \sum_{k=1}^{i-1} n_k & , \text{if } x = u_{i,j} \text{ for some } j \in \{1, 2, \cdots, n_i\},\ i \in \{1, 2, \cdots, m\} \end{cases}$$

To prove f is injective it is enough to prove that f is surjective as the cardinality of the domain and codomain are same. For each $y \in \{1, 2, \cdots, 1 + \sum_{i=1}^{m} n_i\}$ either $y = 1 = f(u_0)$ or there exists $i \in \{1, 2, \cdots, m\}$ such that $1 + \sum_{k=1}^{i-1} n_k < y \leq 1 + \sum_{k=1}^{i} n_k$.

$$\begin{aligned} \text{For letter case } j &= y - \left(1 + \sum_{k=1}^{i-1} n_k\right), \text{ then } 1 \leq j \leq n_i \\ \text{and } f(u_{i,j}) &= 1 + j + \sum_{k=1}^{i-1} n_k \\ &= 1 + y - \left(1 + \sum_{k=1}^{i-1} n_k\right) + \sum_{k=1}^{i-1} n_k \\ &= y. \end{aligned}$$

It shows that f is surjective. Let $e = xy$ be an edge of G then it must be an edge of one of the wheels W_{n_i} for some $i \in \{1, 2, \cdots, m\}$. we have the following two possibilities:

1. If one of the end vertices of e is the apex vertex u_0 with $x = u_0$ then $\gcd(f(x), f(y)) = \gcd(f(u_0), f(y)) = \gcd(1, f(y)) = 1$.
2. If none of the end vertices of e is the apex vertex u_0 and
   - if $\{x, y\} = \{u_{i,j-1}, u_{i,j}\}$ for some $j \in \{1, 2, \cdots, n_i\}$ then

$$\begin{aligned} \gcd(f(x), f(y)) &= \gcd\left(f(u_{i,j-1}), f(u_{i,j})\right) \\ &= \gcd\left(j + \sum_{k=1}^{i-1} n_k,\ 1 + j + \sum_{k=1}^{i-1} n_k\right) \\ &= 1 \end{aligned}$$

   as $j + \sum_{k=1}^{i-1} n_k$ and $1 + j + \sum_{k=1}^{i-1} n_k$ are consecutive integers so they necessarily be relatively prime.
   - if $\{x, y\} = \{u_{i,1}, u_{i,n_i}\}$ then

$$\begin{aligned} \gcd(f(x), f(y)) &= \gcd(f(u_{i,1}), f(u_{i,n_i})) \\ &= \gcd\left(1 + 1 + \sum_{k=1}^{i-1} n_k,\ 1 + n_i + \sum_{k=1}^{i-1} n_k\right) \\ &= \gcd\left(2 + \sum_{k=1}^{i-1} n_k,\ n_i - 1\right) \\ &= 1 \end{aligned}$$

as it is mentioned that for each $i \in \{2, 3, \cdots, m\}$, $n_i - 1$ is relatively prime with $2 + \sum_{k=1}^{i-1} n_k$.

Which shows that f admits a prime labeling i.e. G is a prime graph.

Illustration 2.13 The graph G obtained by identifying the apex vertices of two wheels W_6 and W_8 has prime labeling is shown in *Figure 6*.

Theorem 2.14 A t-ply graph $P_t(u, v)$ is a prime graph if the order of $P_t(u, v)$ is a prime number.

Proof: Suppose a t-ply $P(u, v)$ is obtained from t distinct paths P_i, for each $i = 1, 2, \cdots, t$, each of length n_i, such that the vertices of P_i are $v_{i,0}, v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}$ consecutively. Identifying all the vertices $v_{1,0}, v_{2,0}, v_{3,0}, \cdots, v_{t,0}$ into a single vertex u and identifying all the vertices $v_{1,n_1}, v_{2,n_2}, v_{3,n_3}, \cdots, v_{t,n_t}$ into a single vertex v. The number of vertices of $P_t(u, v)$ is a prime number p with

$$\begin{aligned} p &= |V(P_t(u, v))| \\ &= \sum_{i=1}^{t} (|V(P_i)| - 2) + 2 \\ &= \sum_{i=1}^{t} (n_i + 1) - 2t + 2 \\ &= \sum_{i=1}^{t} n_i - t + 2. \end{aligned}$$

Define $f : V(P_t(u, v)) \rightarrow \{1, 2, 3, \cdots, p\}$ as follows: $f(u) = 1$ and $f(v_{i,j}) = j + \sum_{k=1}^{i-1}(n_k - 1)$, $\forall i = 1, 2, \cdots, t$ and $\forall j = 1, 2, \cdots, n_i - 1$ and $f(v) = p$ where p is a prime number. Here we define $\sum_{x}^{y} a = 0$ if x and y are any positive integers with $y < x$. First we will show that f is an injection. It is easy to check that $f(w) = 1$ if and only if $w = u$ as well as $f(w) = p$ if and only if $w = v$. Suppose $f(v_{i,j}) = f(v_{i',j'})$ for some positive integers i, j, i', j' with $i \in \{1, 2, \cdots, t\}$, $i' \in \{1, 2, \cdots, t\}$ and $j \in \{1, 2, \cdots, n_i - 1\}$ and $j' \in \{1, 2, \cdots, n'_i - 1\}$. If $i \neq i'$ then without loss of generality assume that $i < i'$, so $i \leq i' - 1$.

$$\begin{aligned} \text{Then } f(v_{i,j}) &= j + \sum_{k=1}^{i-1}(n_k - 1) \\ &\leq n_i - 1 + \sum_{k=1}^{i-1}(n_k - 1) \\ &= \sum_{k=1}^{i}(n_k - 1) \\ &\leq \sum_{k=1}^{i'-1}(n_k - 1) \\ &< j' + \sum_{k=1}^{i'-1}(n_k - 1) \\ &= f(v_{i',j'}). \end{aligned}$$

Thus we have $i < i' \Rightarrow f(v_{i,j}) < f(v_{i,j'})$ and similarly $i' < i \Rightarrow f(v_{i',j'}) < f(v_{i,j})$.

$$\begin{aligned} \text{That is } f(v_{i,j}) = f(v_{i',j'}) &\Rightarrow i = i' \\ &\Rightarrow f(v_{i,j}) = f(v_{i,j'}) \\ &\Rightarrow j + \sum_{k=1}^{i-1}(n_k - 1) = j' + \sum_{k=1}^{i-1}(n_k - 1) \\ &\Rightarrow j = j'. \end{aligned}$$

Thus $f(v_{i,j}) = f(v_{i',j'}) \Rightarrow i = i', j = j'$, which shows that f is injective.

It is enough to show that $\gcd(f(x), f(y)) = 1$, for every pair of adjacent vertices x and y. Let $e = xy$ be an edge of G. Then clearly it must be an edge of exactly one of the paths P_{n_i} for some $i = 1, 2, \cdots, t$.

1. If one of the end vertices of e is u say $x = u$ then $\gcd(f(x), f(y)) = \gcd(f(u), f(y)) = \gcd(1, f(y)) = 1$.
2. If one of the end vertices of e is v say $x = v$ then $\gcd(f(x), f(y)) = \gcd(f(v), f(y)) = \gcd(p, f(y)) = 1$.
3. If none of the end vertices of e is u and v then clearly $\{u, v\} = \{v_{i,j}, v_{i,j+1}\}$ for some $j \in \{1, 2, \cdots, n_i - 2\}$.

$$\begin{aligned} \text{i.e. } \gcd(f(x), f(y)) &= \gcd(f(v_{i,j}), f(v_{i,j+1})) \\ &= \gcd\left(j + \sum_{k=1}^{i-1}(n_k - 1),\ j + 1 + \sum_{k=1}^{i-1}(n_k - 1)\right) \\ &= 1 \end{aligned}$$

as any two consecutive integers are relatively prime.

Thus f admits a prime labeling for G. That is G is a prime graph.

Illustration 2.15 The 5-ply graph obtained by taking five paths of lengths 6, 6, 4, 6 and 10 respectively has prime labeling is shown in *Figure 7*.

3. $k-$Prime Labeling- a New Concept

Definition 3.1 *A $k-$prime labeling* of a graph G is an injective function $f : V \to \{k, k+1, k+2, k+3, \cdots, k+|V|-1\}$ for some positive integer k that induces a function $f^+ : E(G) \to N$ of the edges of G defined by $f^+(uv) = \gcd(f(u), f(v))$, $\forall e = uv \in E(G)$ such that $\gcd(f(u), f(v)) = 1$, $\forall e = uv \in E(G)$. The graph which admits a $k-$prime labeling is called a $k-$prime graph.

One note that every prime graph is a $k-$prime graph for $k = 1$.

Lemma 3.2 For each positive integer m the path graph P_m is a $k-$prime graph for each positive integer k.

Proof: Denote the vertices of P_m as $v_1, v_2, \cdots, v_m$ in the order. For each positive integer k, define $f : V \to \{k, k+1, k+2, k+3, \cdots, k+|V|-1\}$ as $f(v_i) = k + i - 1$ for each $v_i \in V$. For $\forall e = uv \in E(G)$, f induces a function $f^+ : E(G) \to N$ defined by $f^+(uv) = \gcd(f(u), f(v))$. $\forall e = v_i v_{i+1} \in E(G)$ it is easy to deduce that $\gcd(f(v_i), f(v_{i+1})) = \gcd(k+i-1, k+i) = 1$. Thus the path graph P_m is a $k-$prime graph.

Theorem 3.3 The graph G obtained by disjoint union of a prime graph G_1 of order n_1 and a $(n_1+1)-$prime graph G_2 is a prime graph.

Proof: Let $f_1 : \{u_1, u_2, \cdots, u_{n_1}\} \longrightarrow \{1, 2, \cdots, n_1\}$ be a prime labeling of a prime graph G_1. Let $f_2 : \{v_1, v_2, \cdots, v_{n_2}\} \longrightarrow \{n_1+1, n_1+2, n_1+3, \cdots, n_1+n_2\}$ be a $(n_1+1)-$prime labeling of a (n_1+1)-prime graph G_2. Let the graph G be obtained by disjoint union of G_1 and G_2.

Define $f : \{u_1, u_2, \cdots, u_{n_1}\} \cup \{v_1, v_2, \cdots, v_{n_2}\} \longrightarrow \{1, 2, \cdots, n_1+n_2\}$ as

$$f(x) = \begin{cases} f_1(u_i) & \text{, if } x = u_i \text{ for some } i \in \{1, 2, \cdots, n_1\} \\ f_2(v_j) & \text{, if } x = v_j \text{ for some } j \in \{1, 2, \cdots, n_2\} \end{cases}$$

Obviously f is an injection. Let $e = xy$ be an arbitrary edge of G. Then either $e \in E(G_1)$ or $e \in E(G_2)$.

1. If $e \in E(G_1)$ then $\gcd(f(x), f(y)) = \gcd(f_1(x), f_1(y)) = 1$, as f_1 is a prime labeling of G_1.
2. If $e \in E(G_2)$ then $\gcd(f(x), f(y)) = \gcd(f_2(x), f_2(y)) = 1$, as f_2 is a prime labeling of G_2.

Thus f admits a prime labeling of G and consequently G is a prime graph.

Theorem 3.4 Let G_1 be a prime graph of order n_1 with a prime labeling f_1 and having vertices u_1 and u_{n_1} with $f_1(u_1) = 1$ and $f_1(u_{n_1}) = n_1$. Let G_2 be a n_1-prime graph of order n_2 with a n_1-prime labeling f_2 having a vertex v_1 with $f_2(v_1) = n_1$. Then the graph G obtained by identifying the vertex v_1 of G_2 with either to u_1 or to u_{n_1} of G_1 is a prime graph.

Proof: Let $f_1 : \{u_1, u_2, \cdots, u_{n_1}\} \longrightarrow \{1, 2, \cdots, n_1\}$ be a prime labeling of a prime graph G_1. Let $f_2 : \{v_1, v_2, \cdots, v_{n_2}\} \longrightarrow \{n_1, n_1+1, n_1+2, \cdots, n_1+n_2-1\}$ be a n_1-prime labeling of a n_1-prime graph G_2.

1. Consider the graph G obtained by identifying the vertex v_1 of G_2 to u_1 of G_1.

Define $f : \{u_1, u_2, \cdots, u_{n_1}, v_2, v_3, \cdots, v_{n_2}\} \longrightarrow \{1, 2, \cdots, n_1 + n_2\}$ as

$$f(x) = \begin{cases} f_1(u_i) & \text{, if } x = u_i \text{ for some } i \in \{1, 2, \cdots, n_1\} \\ f_2(v_j) & \text{, if } x = v_j \text{ for some } j \in \{2, 3, \cdots, n_2\} \end{cases}$$

We claim that f is an injection because

- f_1 is an injection from $\{u_1, u_2, \cdots, u_{n_1}\}$ with the range $\{1, 2, \cdots, n_1\}$.
- f_2 is an injection from $\{v_1, v_2, \cdots, v_{n_2}\}$ with the range $\{n_1,\ n_1 + 1,\ \cdots\ \cdots\ , n_1 + n_2 - 1\}$ and $f_2(v_1) = n_1$ then the restriction of f_2 on $\{v_2, v_3, \cdots, v_{n_2}\}$ is also an injection with the range $\{n_1 + 1, n_1 + 2, \cdots, n_1 + n_2 - 1\}$.

Let $e = xy$ be an arbitrary edge of G. Then either $e \in E(G_1)$ or $e \in E(G_2)$.

(a) If $e \in E(G_1)$ then $\gcd(f(x), f(y)) = \gcd(f_1(x), f_1(y)) = 1$, as f_1 is a prime labeling of G_1.

(b) If $e \in E(G_2)$ then $\gcd(f(x), f(y)) = \gcd(f_2(x), f_2(y)) = 1$ as f_2 is a prime labeling of G_2.

2. Consider the graph G obtained by identifying the vertex v_1 of G_2 to u_{n_1} of G_1.

Define $f : \{u_1, u_2, \cdots, u_{n_1}, v_2, v_3, \cdots, v_{n_2}\} \longrightarrow \{1, 2, \cdots, n_1 + n_2\}$ as

$$f(x) = \begin{cases} f_1(u_i) & \text{, if } x = u_i \text{ for some } i \in \{1, 2, \cdots, n_1\} \\ f_2(v_j) & \text{, if } x = v_j \text{ for some } j \in \{2, 3, \cdots, n_2\} \end{cases}$$

We claim that f is an injection because

- f_1 is an injection from $\{u_1, u_2, \cdots, u_{n_1}\}$ with the range $\{1, 2, \cdots, n_1\}$.
- f_2 is an injection from $\{v_1, v_2, \cdots, v_{n_2}\}$ with the range $\{n_1,\ n_1 + 1,\ \cdots\ \cdots\ , n_1 + n_2 - 1\}$ and $f_2(v_1) = n_1$ then the restriction of f_2 on $\{v_2, v_3, \cdots, v_{n_2}\}$ is also an injection with the range $\{n_1 + 1, n_1 + 2, \cdots, n_1 + n_2 - 1\}$.

Let $e = xy$ be an arbitrary edge of G. Then either $e \in E(G_1)$ or $e \in E(G_2)$.

(a) If $e \in E(G_1)$ then $\gcd(f(x), f(y)) = \gcd(f_1(x), f_1(y)) = 1$ as f_1 is a prime labeling of G_1.

(b) If $e \in E(G_2)$ then $\gcd(f(x), f(y)) = \gcd(f_2(x), f_2(y)) = 1$ as f_2 is a prime labeling of G_2.

Thus in each of the possibilities f admits a prime labeling of G consequently G is a prime graph.

Corollary 3.5 A tadpole (graph obtained by identifying a vertex of a cycle to an end vertex of a path) is a prime graph.

Proof: As we know that every cycle is a prime graph. According to Lemma 3.2 every path is $k-$prime graph for every positive integer k. Then using Theorem 3.4 a tadpole is a prime graph.

4. Conclusion

Here we investigate eight results corresponding to prime labeling. We introduce a new concept of $k-$prime labeling and derive four results. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.

References

Burton, D. M. (1990). *Elementary Number Theory*. 2nd ed.

Deretsky, T.D., Lee, S.M., & Mitchem, J. (1991). On vertex prime labellings of graphs, Graph Theory, Combinatorics and Applications, Vol. 1, (Ed. J. Alvi, G. Chartrand, O. Oellerman, A. Schwenk), *Proceedings of the 6th International Conference Theory and Applications of Graphs* (pp. 359-369)

Fu, H. L. & Huang, K. C. (1994). On Prime Labelings. *Discrete Mathematics*, North Holland, 127, 181-186

Gallian, J. A. (2009). A Dynamic Survey of Graph Labeling *The Electronic Journal of Combinatorics*, 16, #DS6.

Gross, J. & Yellen, J. (2004). *Handbook of Graph Theory*, *CRC Press*.

Lee, S. M., Wui, I. & Yeh, J. (1988). On the amalgamation of prime graphs. *Bull. Malaysian Math. Soc.*(Second Series), **11**, pp 59-67.

Tout, A., Dabboucy, A. N. & and Howalla, K. (1982). Prime Labeling of Graphs. *Nat. Acad. Sci. Letters*, 11, pp. 365–368.

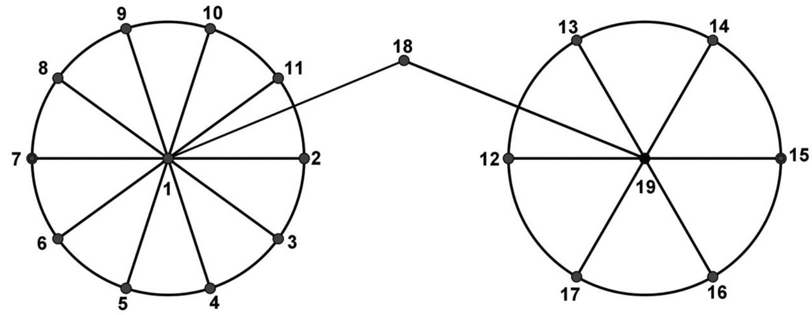

Figure 1. Prime labeling of $< W_{10} : W_6 >$ is shown.

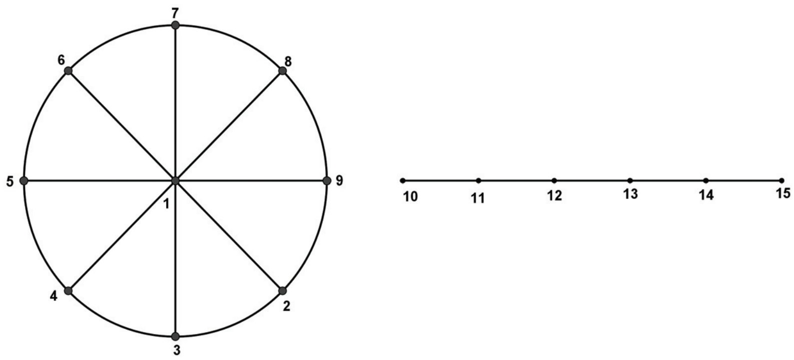

Figure 2. The disjoint union of W_8 and P_6 and its prime labeling

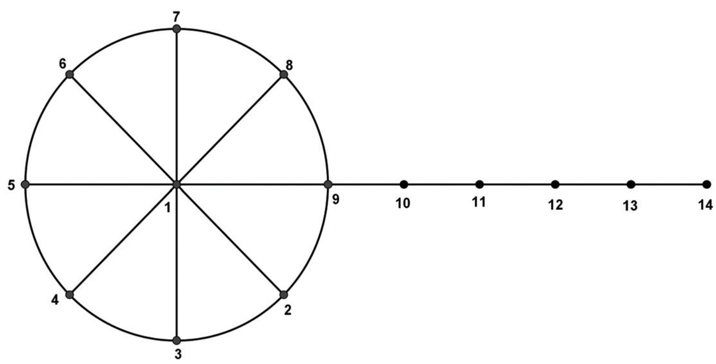

Figure 3. Graph obtained by identifying a rim vertex of W_8 with an end vertex of P_6 and its prime labeling.

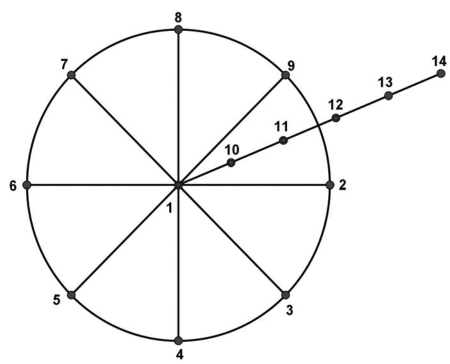

Figure 4. The graph obtained by identifying the apex vertex of W_8 with end vertex of P_6 and its prime labeling

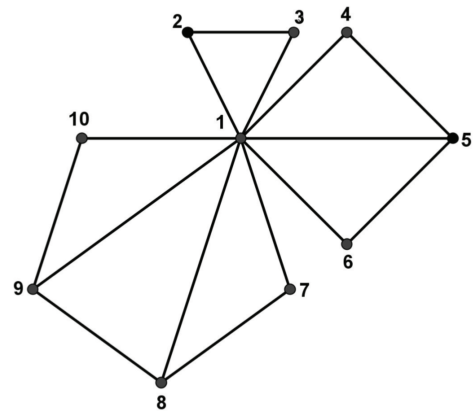

Figure 5. Graph obtained by identifying the apex vertices of f_3, f_4, f_5 and its prime labeling.

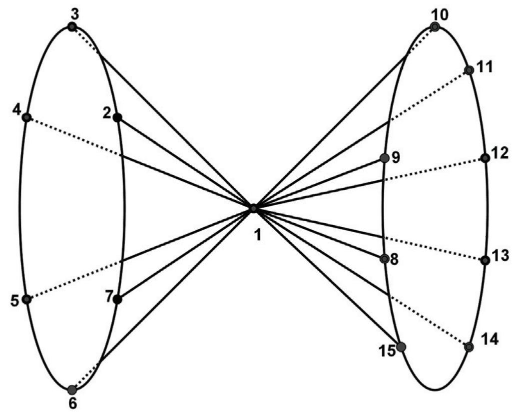

Figure 6. The graph obtained by identifying the apex vertices of W_6 and W_8 and its prime labeling.

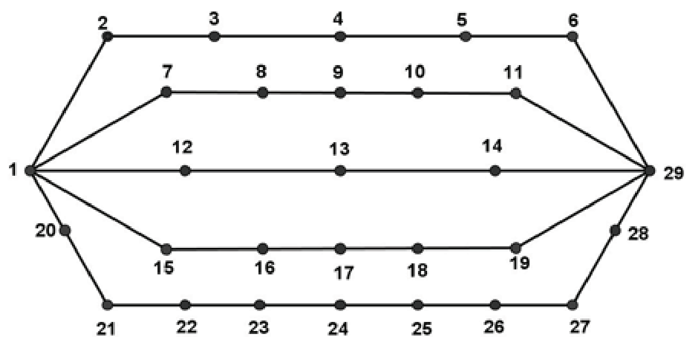

Figure 7. A 5-ply and its prime labeling