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Modules Whose Nonzero Endomorphisms Have E-small Kernels

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Abstract

Let R be a commutative ring and M an unital R-module. A submodule L of M is called essential submodule of M, if $L \cap K \neq \{0\}$ for any nonzero submodule K of M. A submodule N of M is called e-small submodule of M if, for any essential submodule L of M, $N + L = M$ implies $L = M$. An R-module M is called e-small quasi-Dedekind module if, for each $f \in End_R(M)$, $f \neq 0$ implies $Kerf$ is e-small in M. In this paper we introduce the concept of e-small quasi-Dedekind modules as a generalisation of quasi-Dedekind modules, and give some of their properties and characterizations.

Keywords: Essential submodules, e-small submodules, e-small quasi-Dedekind modules

1. Introduction

Throughout all rings are associative, commutative with identity and all modules are unitary R-module. A submodule L of M is called essential submodule of M, if $L \cap K \neq \{0\}$ for any nonzero submodule K of M. (Zhou, D. X. & all (2011)) introduce and study the concept of e-small submodules, where a submodule N of M is called e-small submodule of M if, for any essential submodule L of M, $N + L = M$ implies $L = M$. An R-module M is called e-small quasi-Dedekind module if, for each $f \in End_R(M), f \neq 0$ implies $Kerf$ is e-small in M. (Mijbass, A. S. (1997)) introduced and studied the concept of quasi-Dedekind module. In this paper we introduce and study the concept of e-small quasi-Dedekind as a generalization of quasi-Dedekind module. Also, we investigate the basic properties and characterizations of e-small quasi-Dedekind module. Finally we study the relations between e-small quasi-Dedekind modules and some classes of modules.

The notation $N \leq M$ means that N is a submodule of M and $N \leq^{\oplus} M$ denotes that N is a direct summand of M.

2. Preliminaries

Definition 1 *Let M be an R-module and $N \leq M$.*

1. *N is called essential submodule of M ($N \leq_e M$, for short) if, $N \cap K \neq \{0\}$ for any nonzero submodule K of M.*
2. *N is called small submodule of M ($N \ll M$, for short) if, for any submodule L of M, $N + L = M$ implies $L = M$*
3. *N is called e-small $N \ll_e M$, ($N \ll_e M$, for short) if, for any essential submodule L of M, $N + L = M$ implies $L = M$.*

Remark 1 *Each small submodule is e-small submodule. But the converse is not true in general for example: $N = \{\overline{0}, \overline{3}\}$ is a submodule of $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module. N is e-small but N is not small.*

Lemma 1 *(Zhou, D. X. & all, Proposition 2.5)*

1. *Let N, K and L are submodules of an R-module M such that $N \subseteq K$, if $K \ll_e M$, then $N \ll_e M$ and $K/N \ll_e M/N$.*
2. *If $K \ll_e M$ and $f : M \longrightarrow M'$ is a homomorphism, then $f(K) \ll_e M'$. In particular, if $K \ll_e M \subseteq M'$, then $K \ll_e M'$.*
3. *Assume that $K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_e M_1 \oplus M_2$ if and only $K_1 \ll_e M_1$ and $K_2 \ll_e M_2$.*

Lemma 2 *(Aidi, S. H. & all, (2015), Lemma 2.8)*
Let M be an R-module, let $K \leq N \leq M$ be submodules of M. If $K \ll_e M$ and $N \leq^{\oplus} M$, then $K \ll_e N$.

3. Some Properties Related to E-small Quasi-Dedekind Modules

Definition 2 *An R-module M is called e-small quasi-Dedekind if for all* $f \in End_R(M)$, $f \neq 0$ *implies* $Kerf \ll_e M$.

Example 1 *Every semi-simple module is e-small quasi-Dedekind.*

Remark 2 *It is clear that every quasi-Dedekind R-module is an e-small quasi-Dedekind R-module. But the converse is not true in general, for example* $\mathbb{Z}$ *as a* $\mathbb{Z}$*-module is e-small quasi-Dedekind but it is not quasi-Dedekind.*

Remark 3

1. *The epimorphic image of e-small quasi-Dedekind module is not necessary e-small quasi-Dedekind; for example* $\mathbb{Z}$ *as a* $\mathbb{Z}$*-module is e-small quasi-Dedekind. Let* $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z}$*, where* π *is the natural projection.* $\mathbb{Z}/12\mathbb{Z}$ *as a* $\mathbb{Z}$*-module is not e-small quasi-Dedekind.*

2. *The direct sum of e-small quasi-Dedekind modules is not necessarily an e-small quasi-Dedekind module; for example each of* $\mathbb{Z}/4\mathbb{Z}$ *and* $\mathbb{Z}/3\mathbb{Z}$ *as* $\mathbb{Z}$*-module is e-small quasi-Dedekind. But* $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$ *is not an e-small quasi-Dedekind* $\mathbb{Z}$*-module, since* $\mathbb{Z}/12\mathbb{Z}$ *is not e-small quasi-Dedekind.*

Proposition 1 *Let* M_1, M_2 *be R-modules such that* $M_1 \cong M_2$.
Then M_1 *is an e-small quasi-Dedekind R-module if and only if* M_2 *is an e-small quasi-Dedekind R-module.*

Proof. $\Rightarrow$) Let $f \in End_R(M_2), f \neq 0$. Since $M_1 \cong M_2$, there exists an isomorphism $g : M_1 \longrightarrow M_2$ and $g^{-1} : M_2 \longrightarrow M_1$. Let $h = g^{-1} \circ f \circ g \in End_R(M_1)$. It is clear that $h \neq 0$ and so $g(kerh) \ll_e M_2$ by lemma 1. On the other hand we have $g(Kerh) = Kerf$. Hence $Kerf \ll_e M_2$. So M is e-small quasi-Dedekind.
$\Leftarrow$) The proof of the converse is similarly.

Proposition 2 *Every direct summand of an e-small quasi-Dedekind module is an e-small quasi-Dedekind module.*

Proof. Let $M = N \oplus K$ such that M is an δ-small quasi-Dedekind R-module.
Let $f : K \longrightarrow K, f \neq 0$. We have $h = i \circ f \circ p \in End_R(M), h \neq 0$, where p is the natural projection and i is the inclusion mapping. Hence $Kerh \ll_e M$. But $Kerf \subseteq Kerh$, so by lemma 1, $Kerf \ll_e M$. On the other hand $Kerf \leq K$ implies $Kerf \ll_e K$ by lemma 2. Thus K is an e-small quasi-Dedekind R-module.

Proposition 3 *Let M be an R-module and let* $N, L \leq M$ *with* $M = N + L$ *and* $M/N \cap N$ *e-small quasi-Dedekind. Then* M/N *and* M/L *are e-small quasi-Dedekind R-modules.*

Proof. Since $M/N \cap L = (N/N \cap L) \oplus (L/N \cap L)$, by proposition 2, $N/N \cap L$ and $L/N \cap L$ are e-small quasi-Dedekind. But $N/N \cap L \cong M/L$ and $L/N \cap L \cong M/N$, so M/L and M/N are e-small quasi-Dedekind.

Definition 3 *An R-module is M called N-e-small quasi-Dedekind if, for every* $0 \neq \phi \in Hom_R(M, N)$, $Ker\phi \ll_e M$.

In view of the above definition, an R-module M is e-small quasi-Dedekind if and only if M is M-e-small quasi-Dedekind.

Theorem 1 *Let* M_1 *and* M_2 *be two R-modules and let* $M = M_1 \oplus M_2$*. If M is e-small quasi-Dedekind, then* M_i *is* M_j*-e-small quasi-Dedekind for all* $i, j = 1, 2$.

Proof. Suppose that $M = M_1 \oplus M_2$ is e-small quasi-Dedekind. Then, by proposition 2, M_1 and M_2 are e-small quasi-Dedekind. Thus M_1 is M_1-e-small quasi-Dedekind and M_2 is M_2-e-small quasi-Dedekind. Let $0 \neq f \in Hom_R(M_1, M_2)$. Then $h = i \circ f \circ p \in End_R(M)$, where p is the natural projection and i is the inclusion mapping. It is clear that $h \neq 0$. So $Kerh \ll_e M_1 \oplus M_2$, because $M = M_1 \oplus M_2$ is e-small quasi-Dedekind. On the other hand, we may assume that $Kerf \oplus \{0\} \subseteq Kerh$. Thus $Kerf \oplus \{0\} \ll_e M_1 \oplus M_2$. Then by lemma 1, $Kerf \ll_e M_1$ and so M_1 is M_2-e-small quasi-Dedekind.

Theorem 2 *Let M be an R-module. Then M is e-small quasi-Dedekind if and only if* $Hom(M/N, M) = \{0\}$*, for all* $N \not\ll_e M$.

Proof. $\Rightarrow$) Suppose on the contrary that there exists $N \not\ll_e M$ such that $Hom(M/N, M) \neq \{0\}$. Then there exists $\varphi : M/N \longrightarrow M, \varphi \neq 0$. Hence $\varphi \circ \pi \in End_R(M)$, where π is the canonical projection. It is clear that $\varphi \circ \pi \neq 0$ and so $Kerf(\varphi \circ \pi) \ll_e M$. Since $N \subseteq Ker(\varphi \circ \pi)$, $N \ll_e M$ by lemma 1. This is a contradiction.
$\Leftarrow$) Suppose that there exits $f \in End_R(M), f \neq 0$ such that $Kerf \not\ll_e M$. Define $g : M/Kerf \longrightarrow M$ by $g(m + Kerf) = f(m)$, for all $m \in M$. It is clear that $g \neq 0$. So $Hom(M/Kerf, M) \neq \{0\}$ which is a contradiction.

Remark 4 *If M is an e-small quasi-Dedekind R-module, and* $N \leq M$*. Then it is not necessary that* M/N *is an e-small quasi-Dedekind R-module; for example the* $\mathbb{Z}$*-module* $M = \mathbb{Z}$ *is e-small quasi-Dedekind. Let* $N = 12\mathbb{Z} \leq \mathbb{Z}$*, then*

$M/N = \mathbb{Z}/12\mathbb{Z}$ is not an e-small quasi-Dedekind R-module.

Definition 4 *Let M be an R-module, put $Z(M) = \{m \in M : ann_R(m) \leq_e R\}$. M is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. M is called Goldie torsion if $M = Z_2(M)$.*

Proposition 4 *Let M be an e-small quasi-Dedekind R-module such that M/U is semi-simple nonsingular for all $U \not\ll_e M$. Then M/N is an e-small quasi-Dedekind R-module, for all $N \leq M$.*

Proof. Let $K/N \not\ll_e M/N$. So by lemma 1, $K \not\ll_e M$. Suppose that $Hom((M/N)/(K/N), M/N) \neq \{0\}$. Since $Hom((M/N)/(K/N), M/N) \cong Hom(M/K, M/N)$, there exists $f : M/K \longrightarrow M/N$ such that $f \neq 0$. Since M/K is semi-simple nonsingular, so by (Lam T.Y(1999), Exer. 12A.), there exists $g : M/K \longrightarrow M$ such that $\pi \circ g = f$, where π is the canonical projection. Hence $\pi \circ g(M/K) = f(M/K) \neq 0$, so $g \neq 0$. But $g \in Hom(M/K, M)$ and $K \not\ll_e M$. Thus $Hom(M/K, M) \neq \{0\}$ and $K \not\ll_e M$, which is a contradiction. Thus M/N is an e-small quasi-Dedekind R-module.

Proposition 5 *Let M be an R-module. The following statements are equivalent:*

1. *M is e-small quasi-Dedekind.*
2. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N = Img$ for some $g^2 = g \in End_R(M)$.*
3. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $Kerf + N = M$, then there exist a unique complete set (g, g_1) of orthogonal idempotents in $End_R(M)$ and $N_1 \leq M$ with $N = Mg$ and $N_1 = Mg_1$.*

Proof. 1) $\Rightarrow$ 2) Suppose that M is δ-small quasi-Dedekind. Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. For any complement L to N in M, we have $N \oplus L \leq_e M$. It is clear that $N + L + Kerf = M$. So $N \oplus L = M$, because M is e-small quasi-Dedekind. So there exists $g^2 = g \in End_R(M)$ with $N = Img$.
2) $\Rightarrow$ 3) Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ such that $Kerf + N = M$. Then $f(N) = f(M)$. So, by 2) and (Anderson, F.W.,& all (1973), Corollary 5.8 and Corollary 6.20), the result is obtained.
3) $\Rightarrow$ 1) Let $0 \neq f \in End_R(M)$. Let $Kerf + N = M$ where $N \leq_e M$. By 3) and (Anderson, F.W.,& all (1973), Corollary 6.20), $N \leq^{\oplus} M$. Thus $M = N$ and so M is e-small quasi-Dedekind.

Recall that a submodule N of an R-module M is called fully invariant if $f(N) \subseteq N$ for any $f \in End_R(M)$.
An idempotent e in a ring R is called left semicentral if $xe = exe$ for each $x \in R$.
N is a closed submodule of M if N has no proper essential extension inside M.

Proposition 6 *Let M be an R-module. Then the following conditions are equivalent:*

1. *M is e-small quasi-Dedekind.*
2. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then N is closed in M.*

Proof. 1) $\Rightarrow$ 2) By proposition 5, $N \leq^{\oplus} M$. So N is closed.
2) $\Rightarrow$ 1) Let $0 \neq f \in End_R(M)$. Let $Kerf + N = M$ where $N \leq_e M$. Then $f(N) = f(M)$. By 2) N is closed in M. Thus $M = N$ and so M is e-small quasi-Dedekind.

Proposition 7 *Let M an R-module such that for any $N \leq M$, $Z_2(M) \subseteq N$. Then the following statements are equivalent:*

1. *M is e-small quasi-Dedekind.*
2. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $Kerf + N = M$, then M/N is nonsingular.*
3. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $Kerf + N = M$, then N is closed in M.*

Proof. 1) $\Rightarrow$ 2) Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. By proposition 6, N is closed in M. By our assumption,

$Z_2(M) \subseteq N$. So by (Asgari, S. & all(2012), Proposition 1.2), M/N is nonsingular.
2) $\Rightarrow$ 3) . Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ with $Kerf + N = M$. Then $f(N) = f(M)$. By 2), M/N is nonsingular. Hence by (Asgari, S. & all(2012), Proposition 1.2), N is closed in M.
3) $\Rightarrow$ 1) Follows from proposition 6.

Proposition 8 *Let M be an e-small quasi-Dedekind R-module such that for any nonzero $f \in End_R(M)$, there exists $N \leq M$ with $f(N) = f(M)$. Then the following assertions are verified:*

1. *For any fully invariant submodule $L \leq M$, $N + L$ is fully invariant in M if and only if $g_1 End(M)N \subseteq L$ for some $g_1^2 = g_1 \in End_R(M)$.*
2. *N is fully invariant in M if only if there exists $g^2 = g \in End_R(M)$ with g semicentral.*
3. *There exists $N' \leq M$ such that N is a fully invariant submodule of M if and only if $Hom_R(N, N') = \{0\}$.*
4. *There exists an epimorphism $g \in Hom_R(M, N)$ and a monomorphism $h \in Hom_R(N, M)$ such that $M = Kerg \oplus Imh$.*
5. *$N = eE(M) \cap M$ for some $e^2 = e \in End_R(E(M))$.*

Proof. Since M is e-small quasi-Dedekind, by proposition 5, $N \leq^{\oplus} M$. Thus $N = gM$ for some $g^2 = g \in End_R(M)$. It is clear that $g_1 = (1 - g)$ is an idempotent in $End_R(M)$.

1. $\Rightarrow$) Let $x \in g_1 End(M)g$ and $m \in M$. Then $xm = xgm \in gM + L = N + L$. On the other hand we have $xm = g_1 xm \in g_1 L \leq L$.
(⇐ Let $s \in End_R(M)$. Then $s(gM) = (gs + g_1 s)gM \subseteq gM + g_1 End_R(M)gM \subseteq gM + L$.
2. $\Rightarrow$) Let $s \in End_R(M)$ and $m \in M$. By 1), $N = gM$ for some $g^2 = g \in End_R(M)$. Then $sgm = gm'$ for some $m' \in M$. It follows that $gsgm = g^2 m' = gm' = smg$. Hence g is semicentral.
(⇐ Let $s \in End_R(M)$ and $m \in M$. Thus $sgm = gsgm \in gM$ and so N is fully invariant in M.
3. Since $N \leq^{\oplus} M$, there exists $N' \leq M$ with $M = N \oplus N'$. In this case, it is is well know that N is a fully invariant submodule of M if and only if $Hom_R(N, N') = \{0\}$.
4. Since $N \leq^{\oplus} M$, there exists an epimorphism $g \in Hom_R(M, N)$ and a homomorphism $h \in Hom_R(N, M)$ such that $g \circ h = 1_N$. It follows that h is a monomorphism. It is clear that hg an indempotent and so $M = Kerg \oplus Imh$.
5. Since $N \leq^{\oplus} M$, N is a closed submodule of M. Since $N \subseteq M \subseteq E(M)$, $E(M)$ contains a copy of $E(N)$. Thus $N \leq_e E(N) \cap M$ implies that $N = E(N) \cap M$. Since $E(N)$ is injective, $E(N) = eE(M)$ for some $e^2 = e \in End_R(E(M))$. So the result is obtained.

Proposition 9 *Let M be an e-small quasi-Dedekind R-module such that for any nonzero $f \in End_R(M)$, there exists $N \leq M$ with $f(N) = f(M)$. Let L be a fully invariant submodule of M such that $L \leq_e N$. Then the following assertions are verified:*

1. *$g_1 End_R(M)N \subseteq Z(M)$ for some $g_1^2 = g_1 \in End_R(M)$.*
2. *$N + Z(M)$ is fully invariant in M.*
3. *If $Z(M) \subseteq N$, then N is fully invariant. Moreover, if $L \leq_e K$, then $K \subseteq N$. In particular, $Z_2(M) \subseteq N$.*

Proof.

1. We have $N = gM$ for some $g^2 = g \in End_R(M)$. Then $g_1 = (1 - g)$ is an idempotent in $End_R(M)$. Let $m \in M$. Then $gmI \subseteq L$ for some $I \leq_e R$. Thus $g_1 End_R(M)gmI \subseteq N \cap g_1 M = \{0\}$. It follows that $g_1 End_R(M)N \subseteq Z(M)$.
2. Result from (1) and proposition 8 (1).
3. By (2), N is a fully invariant submodule of M. Let $k \in K$. Thus $kI \subseteq L$ for some $I \leq_e R$. So $g_1 k \in Z(M) \subseteq N$. On the other hand $k = gk + g_1 k \in N$. Thus $K \subseteq N$.

Proposition 10 *Let M be a nonsingular R-module. Then the following conditions are equivalent:*

1. *M is e-small quasi-Dedekind.*

2. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N \leq^{\oplus} M$.*

3. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $Kerf + N = M$ and $((N + Z_2(M))/Z_2(M)) \leq_e M/Z_2(M)$, then $M = N$.*

Proof.
1) $\Rightarrow$ 2) Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. Then the result follows directly from proposition 5.
2) $\Rightarrow$ 3) Let $0 \neq f \in End_R(M)$. Suppose that there exists $N \leq M$ with $Kerf + N = M$. Then $f(N) = f(M)$ and by 2), there exists $L \leq M$ such that $M = N \oplus L$. By hypothesis, $((N + Z_2(M))/Z_2(M)) \leq_e M/Z_2(M)$. Thus by (Asgari, S. & all(2012), Proposition 1.1), M/N is Goldie torsion. On the other hand $M = N \oplus L$ implies that $M/N \cong L$ is Goldie torsion. It follows that $L = \{0\}$. This implies that $M = N$.
3) $\Rightarrow$ 1) Let M an R-module and a nonzero $f \in End_R(M)$ such that $Kerf + N = M$, where $N \leq_e M$. We have M/N is singular and so Goldie torsion. Thus by 3) and (Asgari, S. & all(2012), Proposition 1.1), $M = N$. So M is e-small quasi-Dedekind.

Definition 5

1. *An R-module M is called prime if $Ann_R(M) = Ann_R(N)$ for each $0 \neq N \leq M$.*

2. *An R-module M is called faithful if $Ann_R(M) = \{0\}$.*

Proposition 11 *Let M be a prime faithful R-module. Then the following conditions are equivalent:*

1. *M is e-small quasi-Dedekind.*

2. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N \leq^{\oplus} M$.*

3. *For any nonzero $f \in End_R(M)$, if there exists $N \leq M$ such that $Kerf + N = M$ and $(N + Z_2(M)) \leq_e M$, then $M = N$.*

Proof. Suppose that M is prime. Then $Ann_R(M)$ is a prime ideal of R. Also M is a torsion-free module $\overline{R}$-module, where $\overline{R} = R/Ann_R(M)$. We have $\overline{R} = R/Ann_R(M) \cong R$, because M is faithful. Hence M is a torsion-free module over a integral domain. Thus by (Lam, T. Y. (1999), P.247), M is nonsingular. Thus the result follows from proposition 10 and (Asgari, S. & all(2012), Proposition 1.1).

Remark 5 *Let $N \leq M$ and $f \in End_R(M), f \neq 0$. If $f(N) \ll_e f(M)$, then it is not necessarily that $N \ll_e M$ for example: let $\mathbb{Z}/12\mathbb{Z}$ as $\mathbb{Z}$-module, and let $N =< \overline{3} >\leq \mathbb{Z}/12\mathbb{Z}$. Let $f = 4\overline{x} \in End_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z})$. It is clear that $f \neq 0$ and $f(N) = f(< \overline{3} >) = \{\overline{0}\} \ll_e f(\mathbb{Z}/12\mathbb{Z}) = f(M)$, but $< \overline{3} > \not\ll_e \mathbb{Z}/12\mathbb{Z}$.*

Proposition 12 *Let M be an e-small quasi-Dedekind nonsingular R-module. Let a nonzeo $f \in End_R(M)$ such that for each $N \leq M$, $f(M)/f(N)$ is singular. If $f(N) \ll_e f(M)$, then $N \ll_e M$.*

Proof. Let $N + K = M$ where $K \leq_e M$. Then $f(N) + f(K) = f(M)$. Since M is nonsingular, $f(M)$ is nonsinguar. By hypothesis, $f(M)/f(K)$ is singular, so, $f(K) \leq_e f(M)$. Since $f(N) \ll_e f(M)$, $f(K) = f(M)$. It follows that $M = Kerf + K$. Thus $K = M$ and so $N \ll_e M$.

Corollary 1 *Let M be an e-small quasi-Dedekind nonsingular R-module and a nonzero surjective $f \in End_R(M)$. Suppose that for each $N \leq M$, $M/f(N)$ is singular. Then $N \ll_e M$ if and only if $f(N) \ll_e M$.*

Proof. Suppose that $N \ll_e M$. Then by lemma 1, $f(N) \ll_e M$. The converse follows directly from proposition 12.

Definition 6

1. *A submodule N of an R-module M is called δ-small ($N \ll_\delta M$, for short) if whenever $N + L = M$ and M/L is singular then $L = M$.*

2. *An R-module M is called δ-Hollow if every proper submodule of M is δ-small in M.*

3. *A pair (P, f) is a δ-projective cover of an R-module M, if P is a projective R-module and $f : P \longrightarrow M$ is an epimorphism and $Kerf \ll_\delta P$.*

Remark 6

1. *Every δ-small submodule is e-small but not conversely (see (Zhou, D. X. & all (2000), P. 1052).*
2. *Obviously, every δ-hollow is e-small quasi-Dedekind but not conversely; for example $\mathbb{Q}$ as $\mathbb{Z}$-module is e-small quasi-Dedekind, but it is not δ-hollow.*

Proposition 13 *Let (P, f) is a δ-projective cover of an R-module M such that P is δ-Hollow. Then M is e-small quasi-Dedekind.*

Proof. We have $f : P \longrightarrow M$ is an epimorphism. Let $g \in End_R(M)$ such that $g \neq 0$. $f^{-1}(Kerg)$ is a proper submodule of P. Suppose that P is δ-Hollow, then $f^{-1}(Kerg) \ll_\delta P$ and by (Zhou, Y.Q.(2000), Lemma 1.3), $f(f^{-1}(Kerg)) \ll_\delta M$. But $f(f^{-1}(Kerg)) = Kerg$, then $Kerg \ll_e M$. Hence M is e-small quasi-Dedekind.

Definition 7 *A ring R is called δ-semiperfect if every simple R-module has projective δ-cover.*

Recall that a module is M called weakly co-Hopfian if for any endomorphism $f \in End_R(M)$, $f(M) \leq_e M$.

Proposition 14 *Let R be an artinian principal ideal ring such that every projective R-module is δ-hollow. Then every weakly co-Hopfian R-module is e-small quasi-Dedekind.*

Proof. Let M be a weakly co-Hopfian R-module. Since R is a artinian principal ideal ring, then by (Barry, M.,& all (2010), Therorem 3.8), M is finitely generated. Thus by (Zhou, Y.Q.(2000), Theorem 3.6), M has a projective δ-cover (P, f) because R is δ-semiperfect. By hypothesis, P is δ-Hollow. So by proposition 13, M is e-small quasi-Dedekind.

Definition 8 *An R-module M is called coretractable if for any proper submodule N of M, there exists a nonzero homomorphism $f : M \longrightarrow M$ with $f(N) = \{0\}$, that is $Hom_R(M/N, M) \neq \{0\}$.*

Proposition 15 *Let M be a coretractable R-module such that for any*
$0 \neq f \in E, ann_E(Kerf) \leq_e E$ where $E = End_R(M)$. Then M is e-small quasi-Dedekind.

Proof. Let $f \in E$ such that $f \neq 0$. Let K be a proper essential submodule K of M. There exists $0 \neq g \in E$ with $g(K) = \{0\}$. Since $ann_E(Kerf) \leq_e E$, there exists $h \in E$ such that $0 \neq hg \in ann_E(Kerf)$. Therefore, $hg(Kerf + L) = \{0\}$ and hence $Kerf + K \neq M$. Thus $Kerf \ll_e M$ and so M is e-small quasi-Dedekind.

Proposition 16 *Let M be a nonzero coretractable R-module. If E is uniform as E-module, then M is e-small quasi-Dedekind.*

Proof. Let $f \in E$ such that $f \neq 0$. Suppose that E is uniform. Since $ann_E(Kerf) \neq \{0\}$, it is an essential ideal of E. By proposition 15, $Kerf \ll_e M$ and so M is esmall-quasi-Dedekind.

Definition 9 *Let R be a ring.*

1. *An element $x \in R$ is left quasi-regular in case 1-x has a left inverse in R. Similarly $x \in R$ is right quasi-regular in case 1-x has a right inverse in R.*
2. *An ideal I of R is left quasi-regular in case each element of I is left quasi-regular.*

Proposition 17 *Let R be a ring such that every proper ideal of R is quasi-regular. Then R is an e-small quasi-Dedekind R-module.*

Proof. Let $f \in End(R)$ such that $f \neq 0$. Let $R = Kerf + J$, with $J \leq_e R$. Then there exists $x \in Kerf$ and $j \in J$ with 1=$x + j$. So j=1-x is inversible whence $1 \in J$ and $J = R$. Thus $Kerf \ll_e R$ and so R is e-small quasi-Dedekind.

Remark 7 *Let R be a nonzero ring such that any ideal in R is free as an R-module. Then R is an e-small quasi-Dedekind ring.*

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