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Periodic Properties of Solutions of Certain Second Order Nonlinear Differential Equations

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Abstract

Periodic properties of solutions play an important role in characterizing the behavior of solutions of sufficiently complicated nonlinear differential equations. Sufficient conditions are established which ensure the existence of periodic (or almost periodic) solutions of certain second nonlinear differential equations. Using the basic tool Lyapunov function, new result on the subject which improve some well known results in the literature with the particular cases of (1) for the existence of almost periodic or periodic solutions when the forcing term p is almost periodic or periodic in t uniformly in x and $\dot{x}$ are obtained. Our result further extends and improves on those that exist in the literature to the more general case considered.

Keywords: almost periodic or periodic solutions, second order nonlinear differential equations, Lyapunov's method

1. Introduction

We consider the second-order nonlinear ordinary differential equation,

$$\ddot{x} + \phi(x, \dot{x})\dot{x} + \psi(x, \dot{x}) = p(t, x, \dot{x}) \tag{1}$$

where $\phi, \psi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $p([0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathbb{R}$ is the real line $-\infty < t < \infty$. The functions ϕ, ψ and p depend only on the argument displayed explicitly and the dots denote differentiation with respect to t. ϕ and ψ satisfy condition of existence and uniqueness of solutions.

For many years now, several authors have dealt with ordinary differential equations and obtained useful results, see (Aleksandrov, A.Y. & Platonov, A.V., 2008; Cartwright, M.L. & Littlewood, J.E., 1947; Ezeilo, J.O.C., 1965; Hale, J.K., 1964; Loud, B.S., 1957; Lyapunov, A.M., 1966; Olutimo, A.L. & Akinwole, F.O., 2016; Omeike, M.O. & et al., 2014; Reissig, R. & et al., 1974; Reuter, G.E.H., 1951) and the references cited therein. The results obtained by these authors were based on Lyapunov's theory, see (Lyapunov, A.M., 1966; Yoshizawa, T., 1966) which ensure some qualitative behavior of solutions of the problem. However, the construction of these Lyapunov functionals remain a general problem. For second order differential equations, some special cases of (1) do exist for which ϕ and ψ are replaced by some other non-linear functions (at most) only x and $\dot{x}$. For example, (Cartwright, M.L. & Littlewood, J.E., 1947) studied the second order equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t)$$

and showed that if g is twice differentiable and satisfies $g(0) = 0$ and if further both f and g' are strictly positive, then all ultimately bounded solutions of the equations converges provided that $|g''(x)|$ is sufficiently small. (Loud, B.S., 1957) on his part showed that for the special case

$$\ddot{x} + c\dot{x} + g(x) = p(t)$$

in which c is a constant, proved convergence result without restrictions whatever on g provided that $c > 0$ is sufficiently large.

However, the search for periodic solutions describing the behavior of nonlinear systems is of interest because of the mathematical description of real physical systems modeled into nonlinear differential equations. Among the qualitative properties of solutions, periodicity is least discussed because of the difficulty in constructing a complete Lyapunov function. To this end, (Reuter, G.E.H., 1951) considered the differential equation of second order with a positive damping factor $kp(t)$,

$$\ddot{x} + kf(x)\dot{x} + g(x) = kp(t).$$

He proved that the differential equation has a unique almost periodic solution to which every other solution converges if $p(t)$ is almost periodic and $g(x)$ does not depart too far from linearity. Also, in a sequence of results, (Hale, J.K., 1964; Yoshizawa, T., 1964), gave simpler stability conditions to ensure the existence of almost periodic solutions of system of functional differential equations with ordinary differential equation as special cases. Similar results were also obtained by (Ezeilo, J.O.C., 1965; Tejumola, H.O., 1971). The method of those papers involves the use of Lyapunov's functions except in (Seifert, G., 1966) where Amerio's Theorem was used to obtain the existence of almost periodic solutions with certain stability conditions. Theoretically, the result obtained in (Tejumola, H.O., 1971) is a very interesting result. For example, many second-order differential equations which have been mentioned above are special cases of equation (1) and some known results were improved and extended by the result in (Tejumola, H.O., 1971). However, it is not easy to apply the Theorems in (Tejumola, H.O., 1971) to obtain new or better results since some restrictions on x and y are not necessary for the stability of many nonlinear systems.

Our motivation comes from the papers mentioned above. With respect to our observations in the literature, periodic properties of solutions for (1) have not been discussed. Thus, it is worthwhile to study the subject for (1). The aim of this paper is to establish sufficient conditions that ensure the existence of periodic properties of solutions of (1) under the conditions that the forcing term p is almost periodic or periodic in t uniformly in x and $\dot{x}$ using a complete Lyapunov function in a simple approach. The result obtained here is not only new but will include the notion of the properties studied in (Ezeilo, J.O.C., 1965; Tejumola, H.O., 1971), extend and improve those results obtained by (Hale, J.K., 1964; Reuter, G.E.H., 1951; Yoshizawa, T., 1964) to the more general equation (1). It may be useful to researchers as it plays an important role in characterizing the behavior of solutions of sufficiently complicated nonlinear differential equations.

1.1 Definition A continuous function $f : \mathbb{R} \to x$ is called almost periodic if for each $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ such that every interval of length $\ell(\varepsilon)$ contains a number τ with property that

$$|f(t+\tau) - f(t)| < \varepsilon \;\; for\ each\ t \in \mathbb{R}.$$

1.2 Definition A continuous function $f : \mathbb{R} \to x$ is said to be periodic with period ω for all $t \in \mathbb{R}$ such that

$$f(t+\omega) = f(t) \;\; for\ all\ t \in \mathbb{R}.$$

2. Main Result

Our main result is the following result.

Theorem 1 In addition to the basic assumptions imposed on the functions ψ, ϕ and p appearing in (1), we assume that there exist positive constants a, b, υ and δ_o such that the followings hold:

1) $\phi(x,y) \geq a$, $\frac{\psi(x,y)}{y} \geq b$, $y \neq 0$ and $y\psi(x,0) \leq 0$;

2) $\frac{\psi(x,y)}{x} \geq \delta_o$, $\frac{\psi(x,0)}{x} \geq \upsilon$, $x \neq 0$;

3) $p(t,x,y) \equiv r(t,x,y+q)$ satisfies

$$r(t,x_2,y_2+q) - r(t,x_1,y_1+q) \leq \sigma(t)\{|x_2(t) - x_1(t)| + |y_2(t) - y_1(t)|\},$$

for arbitrary t and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ holds and $\sigma(t)$ satisfy

$$\int_{-\infty}^{\infty} \sigma^{\alpha}(t)dt < \infty,$$

for some constant α in the range $1 \leq \alpha \leq 2$. Suppose further that there exists a solution $x(t)$ of equation (1) satisfying

$$|x(t)|^2 + |\dot{x}(t)|^2 \leq D_o, \quad for \;\; t \in \mathbb{R}. \tag{2}$$

Then,

i) If $q(t)$ and $r(t,x,\dot{x})$ are almost periodic in t, for $|x(t)|^2 + |\dot{x}(t)|^2 \leq D_o$, then $x(t)$ is almost periodic in t.

ii) If $q(t)$ and $r(t,x,\dot{x})$ are periodic in t, with period ω, for $|x(t)|^2 + |\dot{x}(t)|^2 \leq D_o$, then $x(t)$ is periodic with period ω.

Assume now that r is the perturbation such that p the continuous function $p(t,x,\dot{x})$ is separable in the form

$$p(t,x,\dot{x}) = q(t) + r(t,x,\dot{x}),$$

with $q(t) + r(t, 0, 0)$ continuous in their respective arguments, where

$$|q(t)| = \int_0^t |q(s)ds| \leq d_o, \quad d_o > 0.$$

Thus, the equation (1) can be written in the equivalent system form as

$$\begin{aligned} \dot{x} &= y + q, \\ \dot{y} &= -\phi(x, y)y - \psi(x, y) + p(t, x, y + q) - \phi(x, y)q. \end{aligned} \tag{3}$$

Our main tool is the proof of the results is the continuous function $V = V(x, y)$ defined for any $x, y \in \mathbb{R}$ by

$$2V(x, y) = (ax + y)^2 + 2\int_0^x \psi(\xi, 0)d\xi + y^2. \tag{4}$$

The following results are immediate from (4).

lemma Assume that all the conditions on functions ϕ, ψ in Theorem 1 are satisfied. Then, there exist positive constants d_1 and d_2 such that

$$d_1(x^2 + y^2) \leq V(x, y) \leq d_2(x^2 + y^2). \tag{5}$$

Proof: Using (i) of Theorem 1, we have that

$$\begin{aligned} 2V(x, y) &\geq (ax + y)^2 + x^2\frac{\psi(x, 0)}{x} + y^2, \\ 2V(x, y) &\geq \upsilon x^2 + y^2. \end{aligned}$$

Also, from (4), we have

$$2V(x, y) \leq a^2x^2 + 2axy + 2y^2 + 2\upsilon x^2$$

by (i) of Theorem 1 and Schwartz's inequality, we have

$$2axy \leq 2a(x^2 + y^2),$$

so that

$$2V(x, y) \leq (a^2 + 2a + 2\upsilon)x^2 + (2a + 2)y^2.$$

If we choose, $\delta_1 = \min\{\upsilon, 1\}$ and $\delta_2 = \max\{a^2 + 2a + 2\upsilon; 2a + 2\}$, then, inequality (5) holds.

lemma Let the hypothesis (i) and (ii) of Theorem 1 hold. Then, there exist positive constants d_3, d_4 and d_5 such that

$$\dot{V}(t) \leq -(d_3 - d_4\sigma(t))V(t) + d_5V^{\frac{1}{2}}(t).$$

Proof: On using (3), a direct computation of $\frac{dV}{dt}$ gives after simplification

$$\begin{aligned} \dot{V}(t) = &- a[\phi(x, y) - a]xy - ax\psi(x, y) - \Big[2\phi(x, y) - a\Big]y^2 \\ &- 2y\psi(x, y) + y\psi(x, 0) \\ &+ [x\frac{\psi(x, 0)}{x} - ay]q(t) \\ &+ [ax + 2y]r(t, x, y + q), \end{aligned}$$

where by (i) of Theorem 1,

$$[\phi(x, y) - a] \geq 0.$$

$$\begin{aligned} ax\psi(x, y) &\geq ax^2\frac{\psi(x, y)}{x} \\ &\geq a\delta_o x^2. \end{aligned}$$

$$\frac{2\psi(x,y)}{y}y^2 \geq 2by^2.$$

and

$$y\psi(x,y) \leq 0.$$

It follows that

$$\begin{aligned}\dot{V} \leq & - \quad a\delta_o x^2 - 2by^2 + (\upsilon|x| - a|y|)q(t) \\ & + \quad (a|x| + 2|y|)r(t,x,y+q).\end{aligned}$$

That is,

$$\begin{aligned}\dot{V} \leq & - \quad \delta_1(x^2+y^2) + \delta_2(x^2+y^2)^{\frac{1}{2}} \\ & + \quad \delta_3(x^2+y^2)^{\frac{1}{2}}|r(t,x,y+q)|.\end{aligned}$$

Thus,

$$\dot{V} \leq \quad - \quad \delta_1(x^2+y^2) + \delta_2(x^2+y^2)^{\frac{1}{2}} + \delta_3(x^2+y^2)^{\frac{1}{2}}|r(t,x,y+q)|,$$

where $\delta_1 = \min\{a\delta_o, 2b\}$, $\delta_2 = \max\ \sqrt{2}d_o\{\upsilon, a\}$ and $\delta_3 = \max\ \sqrt{2}\{a, 2\}$.

It follows that

$$\dot{V} \leq \quad - \quad \delta_1(x^2+y^2) + \delta_4(x^2+y^2)^{\frac{1}{2}}[r(t,x,y+q)+1],$$

where $\delta_4 = max\{\delta_2, \delta_3\}$.

So that since

$$|r(t,x,y,z+q)| \leq \delta_4\sigma(t)[(x^2+y^2)^{\frac{1}{2}} + 1],$$

by (iii) of Theorem 1 and (5), we have

$$\begin{aligned}\dot{V} \leq & \quad - \quad \delta_1(x^2+y^2) + \delta_4\sigma(t)(x^2+y^2) + \delta_4(x^2+y^2)^{\frac{1}{2}} \\ \dot{V} \leq & \quad - \quad (\delta_5 - \delta_6\sigma(t))V + \delta_7 V^{\frac{1}{2}},\end{aligned}$$

where $\delta_5 = \frac{\delta_1}{d_2}$, $\delta_6 = \frac{\delta_3}{d_1}$ and $\delta_7 = \frac{\delta_2}{d_1}$.

It can be further verified that

$$\dot{V} \leq \quad - \quad \delta_8(x^2+y^2) + \delta_9(x^2+y^2)^{\frac{1}{2}}|\varphi|, \tag{6}$$

where $\varphi = r(t, x_2, y_2+q) - r(t, x_1, y_1+q)$.

Proof of Theorem 1

Consider the function

$$W(t) = V\Big((x(t-\tau) - x(t)), (y(t-\tau) - y(t))\Big)$$

where V is the function defined in (4) with x, y replaced by $(x(t+\tau) - x(t))$ and $(y(t+\tau) - y(t))$ respectively. Then, by (5) we have positive constants d_6 and d_7 such that

$$d_6 S(t) \leq W(t) \leq d_7 S(t), \tag{7}$$

where

$$S(t) = \{|x(t+\tau) - x(t)|^2 + |y(t+\tau) - y(t)|^2\}.$$

Differentiating $W(t)$ along the system (3), we get as in (6),

$$\begin{aligned}\dot{W}(t) \quad \leq & \quad -\delta_8\{|x(t+\tau) - x(t)|^2 + |y(t+\tau) - y(t)|^2\} \\ & +\delta_9\{|x(t+\tau) - x(t)|^2 + |y(t+\tau) - y(t)|^2\}^{\frac{1}{2}}|\varphi|,\end{aligned} \tag{8}$$

where $\varphi = r((t+\tau), x(t), y(t) + q(t+\tau) - r(t, x, y+q)$ with δ_8 and δ_9 being finite constants.

Inequality (8) can be arranged as

$$\begin{aligned}\dot{W}(t) &\leq -\delta_8\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2\} \\ &\quad +\delta_9'\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2\}^{\frac{1}{2}}|\varphi| \\ &\quad +\delta_{10}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2\}^{\frac{1}{2}} \\ \times &\quad |r(t+\tau), x(t), y(t) + q(t+\tau) - r(t, x(t), y+q)|. \end{aligned} \tag{9}$$

Since the perturbation r is uniformly almost periodic in t. Then, given arbitrary $\varepsilon > 0$, we can find $\tau > 0$ such that $|q(t+\tau) - q(t)| \leq \ell\varepsilon^2$,

$$|r(t+\tau), x(t), y(t) + q(t+\tau) - r(t, x(t), y+q)| \leq \ell\varepsilon^2, \tag{10}$$

where ℓ is a constant whose value will be determined later. Thus, (9) becomes

$$\dot{W}(t) \quad \leq \quad -\delta_8 S(t) + \delta_9' S^{\frac{1}{2}}|\varphi| + \delta_{10} S^{\frac{1}{2}}(t)\ell\varepsilon^2. \tag{11}$$

By (2) of Theorem 1,

$$\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2\}^{\frac{1}{2}} \leq \mathcal{D}_o.$$

Inequality (11) becomes,

$$\dot{W}(t) + \delta_8 S(t) \leq \delta_9' S^{\frac{1}{2}}|\varphi| + \delta_{10}\mathcal{D}_o\ell\varepsilon^2. \tag{12}$$

Let α be any constant such that $1 \leq \alpha \leq 2$ and set $\rho = 1 - \frac{1}{2}\alpha$, so that $0 \leq \rho \leq \frac{1}{2}$.

Then, (12) becomes

$$\dot{W} + \delta_8 S(t) \leq \delta_9' S^{\rho} W^* + \delta_{10}\mathcal{D}_o\ell\varepsilon^2 \tag{13}$$

and $W^* = S^{(\frac{1}{2}-\rho)}(|\varphi| - \frac{\delta_8}{\delta_9'}S^{\frac{1}{2}}(t))$.

We consider two cases

1) $|\varphi| \leq \frac{\delta_8}{\delta_9'}S^{\frac{1}{2}}$ and

2) $|\varphi| > \frac{\delta_8}{\delta_9'}S^{\frac{1}{2}}$

separately, we find that in either case, there exists some constants $\delta_{11} > 0$ such that $W^* \leq \delta_{11}|\varphi|^{2(1-\rho)}$. Thus, the inequality (13) becomes

$$\frac{dW}{dt} + \delta_8 S \leq \delta_{12} S^{\rho}\sigma^{2(1-\rho)} S^{(1-\rho)} W(t) + \delta_{10}\mathcal{D}_o\ell\varepsilon^2$$

where $\delta_{12} \geq 2\delta_9'\delta_{11}$. This follows that

$$\frac{dW}{dt} + ((\delta_{13} - \delta_{14})\sigma^{\alpha}(t))W(t) \leq \delta_{10}\mathcal{D}_o\ell\varepsilon^2 \tag{14}$$

after using (7) on W, with δ_{13}, δ_{14} as positive constants.

On integrating (14) from t_o to t $(t \geq t_o)$, we obtain

$$\begin{aligned} W(t) &\leq \delta_{15} W(t_o) \exp\{-\delta_{13}(t-t_o)\} + \delta_{14}\int_{t_o}^{t}\sigma^{\alpha}(s)d(s)\} \\ &+ \delta_{16}\ell\varepsilon^2, \end{aligned} \tag{15}$$

where $\delta_{15} = \frac{\delta_9'}{\delta_8}$ and $\delta_{16} = \frac{\delta_{15}\delta_{10}\mathcal{D}_o}{\delta_{13}}$.

If

$$\int_{t_o}^{t}\sigma^{\alpha}(s)d(s) < \delta_{13}\delta_{14}^{-1}(t-t_o),$$

then, the exponential index remains negative for all $(t - t_o) \geq 0$. As $t = (t - t_o) \rightarrow \infty$ and that $W(t_o)$ is finite in (15), we have that

$$W(t) \leq \delta_{16}\ell\varepsilon^2 \ \ for \ \ any \ \ t.$$

Since W(t) satisfies (7),

$$W(t) \leq d_6^{-1}\delta_{16}\ell\varepsilon^2.$$

By definition of $W(t)$ in (7), we have that

$$|x(t+\tau) - x(t)| + |y(t+\tau) - y(t)| \leq \left(\frac{2\ell\delta_{16}}{d_6}\right)^{\frac{1}{2}}\varepsilon. \tag{16}$$

choose $\ell = \frac{d_6}{2\delta_{16}}$, inequality (16) implies

$$|x(t+\tau) - x(t)| + |y(t+\tau) - y(t)| \leq \varepsilon, \tag{17}$$

where τ is chosen to satisfy (10) is relatively dense and hence (17) implies that the solutions $(x(t), y(t))$ or equivalently $x(t), \dot{x}(t)$ of (1) are uniformly almost periodic in t.

To show that the solutions are also periodic, we assume that

$$\begin{aligned} q(t+\omega) &= q(t) \\ r(t+\omega, x(t), y(t) + q(t)) &= r(t, x(t), y(t)), \end{aligned}$$

for $(x^2 + y^2) \leq D_1$, for some constants $D_1 > 0$.

Since the perturbation $r(t, x, y + q)$ has period ω in t, we replace τ in the definition of $W(t)$ with ω. The terms in the left hand side of (10) is identically zero, thus we may have inequality (17) as

$$|x(t+\omega) - x(t)| + |y(t+\omega) - y(t)| \leq 0.$$

Thus,

$$|x(t+\omega) - x(t)| + |y(t+\omega) - y(t)| = 0.$$

which implies that

$$x(t+\omega) = x(t) \ \ and \ \ y(t+\omega) = y(t).$$

That is, $x(t), y(t)$ or equivalently $x(t), \dot{x}(t)$ of (1) are periodic in t with period ω.

Conclusion

Analysis of nonlinear systems literary shows that Lyapunov's theory in periodic properties of solutions is rarely scarce. The second Lyapunov's method allows to predict and characterize the periodic behavior of solutions of sufficiently complicated nonlinear physical system. The solutions of second order nonlinear differential equation (1) are periodic and almost periodic uniformly in x and $\dot{x}$ according to Lyapunov's theory if the conditions of Theorem 1 hold as $t \rightarrow \infty$.

References

Aleksandrov, A. Y., & Platonov, A. V. (2008). Conditions of ultimate boundedness of solutions for a class of nonlinear systems. *Nonlinear Dynamic and Systems Theory, 8*(2), 109 - 122.

Cartwright, M. L., & Littlewood, J. E. (1947). On nonlinear differential equations of the second order. *Ann. of Math., 48*, 472 - 494.

Ezeilo, J. O. C. (1965). A note on the convergence of solutions of certain second order differential equations. *Portugaliae Mathematical, 24*, 49 - 58.

Hale, J. K. (1964). Periodic and almost periodic solutions of funtional-differential equations. *Arch. Rat. Mech. Anal., 15*, 289 - 304.

Loud, B. S. (1957). Boundedness and convergence of solutions of $x'' + cx' + g(x) = e(t)$. *Duke Math. J., 24*, 63 - 72. https://doi.org/10.1215/S0012-7094-57-02412-2

Lyapunov, A. M. (1966). *Stability of motion, Academic Press*. London .

Olutimo A. L., & Akinwole F. O. (2016). Stability and boundedness of solutions of certain non-autonomous third-order nonlinear differential equations. *Journal of Applied Mathematics and Physics, 4*, 149-155. https://doi.org/10.4236/jamp.2016.41018

Omeike, M. O., Olutimo A. L., & Oyetunde, O. O. (2014). Boundedeness of solutions of certain system of second-order ordinary differential equations. *Acta Univ., Palaki Olomuc., Fac. rer. nat., Mathematica, 53*(1), 107-115.

Reissig, R., Sansone, G., & Conti, R. (1974). *Nonlinear differential equations of higher order.* Translated from the German, Noordhoff International Publishing, Leyden.

Reuter, G. E. H. (1951). On certain non-linear differential equations with almost periodic solutions, *J. London Math. Soc., 26*, 215 - 221. https://doi.org/10.1112/jlms/s1-26.3.215

Seifert, G. (1966). Almost periodic solutions for almost periodic system of ordinary differential equations. *Journal of Differential Equations, 2*, 305 - 319.

Tejumola, H. O. (1971). Boundedness criteria for solutions of some second order differential equations. *Academic Nazionale Dei Lincei, serie VII, 50*(4), 204 - 209.

Yoshizawa, T. (1964). Extreme stability and existence of almost periodic solutions of functional-differential equations. *Arch. Rat. Mech. Anal., 17*, 148 - 170.

Yoshizawa, T. (1966). Stability theory by Lyapunov's second method. *Math. Soc. Japan*.

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