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Existence and Uniqueness of Solution for Caginalp Hyperbolic Phase- Field System with a Polynomial Potential

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Abstract

We prove the existence and the uniqueness of solutions for Caginalp hyperbolic phase-field system with initial conditions, Dirichlet boundary homogeneous conditions and a regular potential of order $2p-1$, in bounded domain.

Keywords: Caginalp hyperbolic phase-field system, Dirichlet boundary homogeneous conditions, Polynomial potential

1. Introduction

We are interested on the study of the following Caginalp hyperbolic phase-field system in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ $(1 \leq n \leq 3)$

$$\epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \triangle u + f(u) = \alpha, \quad in \quad \mathbb{R}_+ \times \Omega, \tag{1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \triangle\frac{\partial \alpha}{\partial t} - \triangle\alpha = -u - \frac{\partial u}{\partial t}, \quad in \quad \mathbb{R}_+ \times \Omega, \tag{2}$$

with homogenous Dirichlet conditions

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0 \tag{3}$$

and initial conditions

$$u|_{t=0} = u_0, \frac{\partial u}{\partial t}|_{t=0} = u_1, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1, \tag{4}$$

where $\epsilon > 0$ is a relaxation parameter, $u = u(x,t)$ the order parameter and $\alpha = \alpha(t,x)$ are the unknown functions, f is a regular potential.

The hyperbolic system has been extensively studied with Dirichlet boundary condition and regular or singular potential (see, eg., (Agmon, S. (1965)), (Caginalp, G. (1986)), (1988)), (Goyaud, M. E. I., Moukamba, F., Moukoko, D., & Langa, F. D. R. (2015)), (Miranville, A., & Zelik, S. (2008)), (Moukoko D. (2014)), (Temam, R. (1997))).

Consider the following polynomial potential of order $2p-1$

$$f(s) = \sum_{k=1}^{2p-1} b_k s^k, \quad b_{2p-1} > 0, \quad p \geq 2.$$

The function f satisfies the following properties

$$\frac{1}{2}b_{2p-1}s^{2p} - c_1 \leq f(s).s \leq \frac{3}{2}b_{2p-1}s^{2p} + c_1, \quad c_1 > 0, \tag{5}$$

$$-\kappa \leq \frac{b_{2p-1}}{2p}s^{2p-2} - c_2 \leq f'(s) \leq 3pb_{2p-1}s^{2p-2} + c_2, \quad \forall s \in \mathbb{R}, \quad \kappa > 0, \quad c_2 > 0, \tag{6}$$

$$\frac{1}{4p}b_{2p-1}s^{2p} - c_3 \leq F(s) \leq \frac{3}{4p}b_{2p-1}s^{2p} + c_3 \quad \text{where} \quad F(s) = \int_0^s f(\tau)d\tau, \quad c_3 > 0. \tag{7}$$

We denote by $\|.\|$ the usual L^2-norm (with associated product scalar $((.,.))$, $\triangle$ denotes the Laplace operator with Dirichlet boundary conditions. Throughout this paper, C_i $(i = 1, ..., n)$ denote positive constants which may change from line to line, or even the same line.

2. Method

To prove our main results, we have to use classical methods of functional analysis applied in the theory of Partial Differential Equations.

3. Results

In this paper, we first prove the existence and the uniqueness of solutions theorems. The two first results being proven in a larger space, we will seek the solution with more regularity.

3.1 A priori Estimates

We multiply (1) by $\frac{\partial u}{\partial t}$ and (2) by $\gamma\frac{\partial \alpha}{\partial t}$ where $\gamma > 0$, integrate over Ω. This gives

$$\frac{d}{dt}\Big(\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\nabla u\|^2+2(F(u),1)+C\Big)+\|\frac{\partial u}{\partial t}\|^2\leq\|\alpha\|^2 \tag{8}$$

and

$$\frac{d}{dt}\Big(\gamma\|\frac{\partial \alpha}{\partial t}\|^2+\gamma\|\nabla\alpha\|^2\Big)+\gamma\|\frac{\partial \alpha}{\partial t}\|^2+2\gamma\|\nabla\frac{\partial \alpha}{\partial t}\|^2\leq 2\gamma\|u\|^2+2\gamma\|\frac{\partial u}{\partial t}\|^2. \tag{9}$$

Sum the two resulting differential inequalities with $1-2\gamma>0$. Due to the equation (7), we get

$$\frac{dE_1}{dt}+C_1\|\frac{\partial u}{\partial t}\|^2+\gamma\Big(\|\frac{\partial \alpha}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2\Big)\leq C_2\|\nabla\alpha\|+C_3\|\nabla u\|^2+2\int_\Omega F(u)dx+2c_3|\Omega|, \tag{10}$$

where

$$E_1=\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\nabla u\|^2+2\int_\Omega F(u)dx+\gamma\Big(\|\frac{\partial \alpha}{\partial t}\|^2+\|\nabla\alpha\|^2\Big).$$

Using (7) we have

$$2\int_\Omega F(u)dx+2c_3|\Omega|\geq\frac{1}{2p}b_{2p-1}\|u\|^{2p}_{L^{2p}},\quad b_{2p-1}>0.$$

Inserting the above estimate in (10), we obtain a differential inequality of the form

$$\frac{dE_1}{dt}+C_1\|\frac{\partial u}{\partial t}\|^2+\gamma\Big(\|\frac{\partial \alpha}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2\Big)\leq kE_1+C, C>0, \tag{11}$$

where the strictly positive constant k is independent of ϵ.

Applying Gronwall's lemma, we obtain for all $t\in(0,T)$

$$E_1(t)+\int_0^t\Big(C_1\|\frac{\partial u(s)}{\partial t}\|^2+\gamma\|\frac{\partial \alpha(s)}{\partial t}\|^2+2\gamma\|\nabla\frac{\partial \alpha(s)}{\partial t}\|^2\Big)ds\leq E_1(0)e^{kT}+C \tag{12}$$

which implies, in view of (7)

$$\begin{aligned}&\epsilon\|\frac{\partial u(t)}{\partial t}\|^2+\|\nabla u(t)\|^2+C\|u(t)\|^{2p}_{L^{2p}}+\gamma\Big(\|\frac{\partial \alpha(t)}{\partial t}\|^2+\|\nabla\alpha(t)\|^2\Big)+\int_0^t\Big(\|\frac{\partial u(s)}{\partial t}\|^2\\&+\gamma\|\frac{\partial \alpha(s)}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha(s)}{\partial t}\|^2\Big)ds\leq Ke^{kT}+C,\quad K>0.\end{aligned} \tag{13}$$

Then we deduce that

$$u\in L^\infty(0,T;L^{2p}(\Omega)\cap H^1_0(\Omega)),$$

$$\alpha\in L^\infty(0,T;H^1_0(\Omega)),$$

$$\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Omega))$$

and

$$\frac{\partial \alpha}{\partial t}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)).$$

The main result of this paper is the proof of the existence and the uniqueness theorems.

3.2 Existence and Uniqueness of Solutions

Theorem 1. (**Existence**) *Assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in (L^{2p}(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$*, then the system (1)-(2) has at least one solution* (u, α) *such that*

$$u \in L^\infty(0, T; L^{2p}(\Omega) \cap H_0^1(\Omega)),$$

$$\alpha \in L^\infty(0, T; H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \quad for \quad all \quad T > 0.$$

The proof is based on a priori estimates obtained in the previous section and on a standard Galerkin scheme.

Theorem 2. (**Uniqueness**) *If the assumptions of the theorem (1) hold. Then, the system (1)-(2) has a unique solution with the above regularity.*

Proof. Let $(u^{(1)}, \alpha^{(1)})$ and $(u^{(2)}, \alpha^{(2)})$ be two solutions of the system (1)-(7) with initial data $(u_0^{(1)}, u_1^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, u_1^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}) \in (L^{2p}(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, respectively. We set

$$(u, \alpha) = (u^{(1)}, \alpha^{(1)}) - (u^{(2)}, \alpha^{(2)})$$

and

$$(u_0, u_1, \alpha_0, \alpha_1) = (u_0^{(1)} - u_0^{(2)}, \quad u_1^{(1)} - u_1^{(2)}, \quad \alpha_0^{(1)} - \alpha_0^{(2)}, \quad \alpha_1^{(1)} - \alpha_1^{(2)}).$$

Then, (u, α) satisfies the following system

$$\epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \triangle u + f(u^{(1)}) - f(u^{(2)}) = -\alpha \tag{14}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -u - \frac{\partial u}{\partial t} \tag{15}$$

$$u|_{t=0} = u_0, \frac{\partial u}{\partial t}|_{t=0} = u_1, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1.$$

Multiplying (14) by $\frac{\partial u}{\partial t}$ and (15) by $\beta\frac{\partial \alpha}{\partial t}$ where $\beta > 0$, integrating over Ω, we obtain

$$\frac{d}{dt}\Big(\epsilon \parallel \frac{\partial u}{\partial t} \parallel^2 + \parallel \nabla u \parallel^2 \Big) + \parallel \frac{\partial u}{\partial t} \parallel^2 + 2\int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}dt \leq \parallel \alpha \parallel^2 \tag{16}$$

and

$$\frac{d}{dt}\Big(\beta \parallel \frac{\partial \alpha}{\partial t} \parallel^2 + \beta \parallel \nabla\alpha \parallel^2 \Big) + \beta \parallel \frac{\partial \alpha}{\partial t} \parallel^2 + 2\beta \parallel \frac{\partial \nabla\alpha}{\partial t} \parallel^2 \leq 2\beta\Big(\parallel u \parallel_{H^1}^2 + \parallel \frac{\partial u}{\partial t} \parallel^2 \Big). \tag{17}$$

Summing (16) and (17), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\epsilon \parallel \frac{\partial u}{\partial t} \parallel^2 + \parallel \nabla u \parallel^2 + \beta \parallel \frac{\partial \alpha}{\partial t} \parallel^2 + \beta \parallel \nabla\alpha \parallel^2 \Big) + (1 - 2\beta) \parallel \frac{\partial u}{\partial t} \parallel^2 + \beta \parallel \frac{\partial \alpha}{\partial t} \parallel^2 \\ &+ 2\beta \parallel \frac{\partial \nabla\alpha}{\partial t} \parallel^2 \leq -2\int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}dt + \parallel \alpha \parallel^2 + 2\beta \parallel u \parallel_{H^1}^2, \quad 1 - 2\beta > 0. \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} f(u^{(1)}) - f(u^{(2)}) &= \sum_{k=1}^{2p-1} b_k((u^{(1)})^k - (u^{(2)})^k) \\ &= (u^{(1)} - u^{(2)})\Big(b_1 + b_2(u^{(1)} + u^{(2)}) + \sum_{k=3}^{2p-1} b_k \sum_{j=0}^{k-1}(u^{(1)})^{k-j-1}.(u^{(2)})^j\Big), \end{aligned}$$

which implies

$$| f(u^{(1)}) - f(u^{(2)}) | \leq | u | \Big(| b_1 | + | b_2 | (| u^{(1)} | + | u^{(2)} |) + \sum_{k=3}^{2p-1} | b_k | \sum_{j=0}^{k-1} | u^{(1)} |^{k-j-1} . | u^{(2)} |^j \Big).$$

There exits $C > 0$ such that $| b_k |\leq C, \quad \forall k \in \{1, 2, \cdots, 2p-1\}$.

Applying Young's inequality, we obtain

$$\begin{aligned} | u^{(1)} |^{k-j-1} . | u^{(2)} |^j &\leq \frac{k-1-j}{k-1}(| u^{(1)} |^{k-j-1})^{\frac{k-1}{k-1-j}} + \frac{j}{k-1}(| u^{(2)} |^j)^{\frac{k-1}{j}} \\ &\leq \frac{k-j-1}{k-1} | u^{(1)} |^{k-1} + \frac{j}{k-1} | u^{(2)} |^{k-1}, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{j=0}^{k-1} | u^{(1)} |^{k-j-1} . | u^{(2)} |^j &\leq \sum_{j=0}^{k-1} \frac{k-j-1}{k-1} | u^{(1)} |^{k-1} + \sum_{j=0}^{k-1} \frac{j}{k-1} | u^{(2)} |^{k-1} \\ &\leq \frac{k}{2}\Big(| u^{(1)} |^{k-1} + | u^{(2)} |^{k-1} \Big). \end{aligned}$$

Then we obtain

$$\begin{aligned} | f(u^{(1)}) - f(u^{(2)}) | &\leq | u | \Big(| b_1 | + | b_2 | (\frac{1}{2p-2} | u^{(1)} |^{2p-2} + \frac{1}{2p-2} | u^{(2)} |^{2p-2} + C) \\ &\quad + \sum_{k=3}^{2p-1} \frac{k}{2} | b_k | (| u^{(1)} |^{k-1} + | u^{(2)} |^{k-1})\Big) \\ &\leq C | u | \Big(| b_1 | + | b_2 | \frac{1}{2p-2}(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2}) \\ &\quad + \sum_{k=3}^{2p-1} | b_k | \frac{(k-1)k}{4(p-1)}(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2}) + 1\Big) \\ &\leq C | u | \Big(\frac{1}{2p-2}(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2}) \\ &\quad + \frac{| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2}}{4(p-1)} \sum_{k=3}^{2p-1} k(k-1) + 1\Big), \end{aligned}$$

which on yields to

$$| f(u^{(1)}) - f(u^{(2)}) |\leq C | u | \Big(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2} + 1\Big).$$

Therefore, one gets

$$\int_\Omega |f(u^{(1)}) - f(u^{(2)})||\frac{\partial u}{\partial t}|dx \leq C \int_\Omega | u | \Big(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2} + 1\Big)|\frac{\partial u}{\partial t}|dx. \tag{19}$$

In this proof, we consider the two cases $n = 1$ and $n = 2$ or 3.

- **For n=1**.

$u^{(1)}, u^{(2)} \in H_0^1(\Omega)$, then $u^{(1)}, u^{(2)} \in C(\bar{\Omega})$ and are both bounded.

There exits $C > 0$ such that $\sup_{x\in\bar{\Omega}} | u^{(1)}(x) |\leq C$, it is the same for $u^{(2)}$. Then we have

$$\begin{aligned} \int_\Omega |f(u^{(1)}) - f(u^{(2)})||\frac{\partial u}{\partial t}|dx &\leq C \int_\Omega | u | \Big(| u^{(1)} |^{2p-2} + | u^{(2)} |^{2p-2} + 1\Big)|\frac{\partial u}{\partial t}|dx \\ &\leq C\Big(\| u^{(1)} \|_{L^\infty}^{2p-2} + \| u^{(2)} \|_{L^\infty}^{2p-2} + 1\Big) \int_\Omega | u | |\frac{\partial u}{\partial t}|dx \\ &\leq C | u | |\frac{\partial u}{\partial t}| \\ &\leq C \| u \|_{H^1} \| \frac{\partial u}{\partial t} \| . \end{aligned}$$

- **For n=2 or 3**.

We have $H^1(\Omega) \subset L^q(\Omega)$ for $q \in [1, 6]$.

Applying the Hölder inequality to (19), we obtain the following inequality

$$\int_\Omega |f(u^{(1)}) - f(u^{(2)})||\frac{\partial u}{\partial t}|dx \leq C \parallel u \parallel_{L^6} \Big(\parallel| u^{(1)} |^{2p-2}\parallel_{L^3} + \parallel| u^{(2)} |^{2p-2}\parallel_{L^3} +1\Big) \parallel \frac{\partial u}{\partial t} \parallel .$$

Since

$$\begin{aligned} \parallel| u^{(1)} |^{2p-2}\parallel_{L^3} &= \Big(\int_\Omega (| u^{(1)} |^{2p-2})^3 dx\Big)^{\frac{1}{3}} \\ &= \Big(\int_\Omega (| u^{(1)} |^{3(2p-2)}) dx)^{\frac{1}{3(2p-2)}}\Big)^{2p-2} \\ &= \parallel u^{(1)} \parallel^{2p-2}_{L^{3(2p-2)}}, \end{aligned}$$

similarly for $\parallel| u^{(2)} |^{2p-2}\parallel_{L^3}$.
Note that $L^{6p}(\Omega) \subset L^{3(2p-2)}(\Omega)$, then there exists $C > 0$ such that $\parallel u^{(1)} \parallel_{L^{3(2p-2)}} \leq C \parallel u^{(1)} \parallel_{L^{6p}}$, which implies

$$\begin{aligned} \parallel| u^{(1)} |^{2p-2}\parallel_{L^3} &\leq \parallel u^{(1)} \parallel^{2p-2}_{L^{3(2p-2)}} \\ &\leq C \parallel u^{(1)} \parallel^{2p-2}_{L^{6p}}, \end{aligned}$$

thus

$$\begin{aligned} \parallel u^{(1)} \parallel_{L^{6p}} &= \Big(\int_\Omega (| (u^{(1)}) |^{6p}) dx\Big)^{\frac{1}{6p}} \\ &= \Big((\int_\Omega (| u^{(1)} |^{p})^6 dx)^{\frac{1}{6}}\Big)^{\frac{1}{p}} \\ &= \parallel| u^{(1)} |^p\parallel^{\frac{1}{p}}_{L^6} \\ &\leq C \parallel| u^{(1)} |^p\parallel^{\frac{1}{p}}_{H^1} \\ &\leq C \parallel u^{(1)} \parallel_{H^1} . \end{aligned}$$

Then, we have

$$\parallel| u^{(1)} |^{2p-2}\parallel_{L^3} \leq C \parallel u^{(1)} \parallel^{2p-2}_{H^1} \tag{20}$$

and

$$\parallel| u^{(2)} |^{2p-2}\parallel_{L^3} \leq C \parallel u^{(2)} \parallel^{2p-2}_{H^1} . \tag{21}$$

Hence

$$\begin{aligned} \int_\Omega |f(u^{(1)}) - f(u^{(2)})||\frac{\partial u}{\partial t}|dx &\leq C \parallel u \parallel_{L^6} \Big(\parallel u^{(1)} \parallel^{2p-2}_{H^1} + \parallel u^{(2)} \parallel^{2p-2}_{H^1} +1\Big) \parallel \frac{\partial u}{\partial t} \parallel \\ &\leq C \parallel u \parallel_{L^6} \parallel \frac{\partial u}{\partial t} \parallel \\ &\leq C \parallel u \parallel_{H^1} \parallel \frac{\partial u}{\partial t} \parallel . \end{aligned}$$

Finally, for $n = 1, \quad 2 \quad \text{or} \quad 3$, we obtain

$$\frac{d}{dt}\Big(\epsilon \parallel \frac{\partial u}{\partial t} \parallel^2 + \parallel \nabla u \parallel^2 +\beta \parallel \frac{\partial \alpha}{\partial t} \parallel^2 +\beta \parallel \nabla \alpha \parallel^2 \Big) \leq C\Big(\parallel u \parallel^2_{H^1} + \parallel \alpha \parallel^2 \Big), C > 0.$$

We deduce the continuous dependence of the solution with respect to the initial conditions, hence the uniqueness of the solution of problem (1)-(7), is proved.

The existence and the uniqueness of the solution to problem (1)-(7) being proven in a larger space, we now establish the solution with more regularity.

Remark 1 In the proof of the uniqueness solution, we have obtained that when the assumptions of theorem (1) hold, the solution (u,α) of the system (1)-(2) is such that

$$\| | u |^{2p-2}\|_{L^3}^{\frac{1}{p}} \leq C.$$

Theorem 3 *Assume* $(u_0,u_1,\alpha_0,\alpha_1) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, *then the system (1)-(7) possesses a unique solution* (u,α) *such that*

$$u,\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H_0^1(\Omega)),$$

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega))$$

and

$$\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0,T;L^2(\Omega)) \quad for \quad all \quad T>0.$$

Proof In view of theorems (1) and (2), the system (1)-(7) possesses an uniqueness solution (u,α) such that

$$u \in L^\infty(0,T;L^{2p}(\Omega)\cap H_0^1(\Omega)),$$

$$\alpha \in L^\infty(0,T;H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Omega))$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)) \quad \text{for} \quad \text{all} \quad T>0.$$

Multiply (1) by $-\triangle\frac{\partial u}{\partial t}$ and integrate over Ω, we have

$$\begin{aligned}\frac{d}{dt}\Big(\epsilon \| \nabla\frac{\partial u}{\partial t}\|^2 + \|\triangle u\|^2\Big) + 2\|\nabla\frac{\partial u}{\partial t}\|^2 &= -2(\nabla f(u),\nabla\frac{\partial u}{\partial t}) + 2(\nabla\alpha,\nabla\frac{\partial u}{\partial t})\\ &= -2(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t}) + 2(\nabla\alpha,\nabla\frac{\partial u}{\partial t})\\ &= -2\int_\Omega f'(u)\nabla u\nabla\frac{\partial u}{\partial t}dx + 2(\nabla\alpha,\nabla\frac{\partial u}{\partial t}).\end{aligned}$$

We deduce the following inequality

$$\frac{d}{dt}\Big(\epsilon \| \nabla\frac{\partial u}{\partial t}\|^2 + \|\triangle u\|^2\Big) + 2\|\nabla\frac{\partial u}{\partial t}\|^2 \leq 2\int_\Omega |f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx + 2\|\nabla\alpha\|\|\nabla\frac{\partial u}{\partial t}\|. \tag{22}$$

Therefore, in view of (6) and remark (1), we have

$$\begin{aligned}\int_\Omega |f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx &\leq \int_\Omega\Big(3pb_{2p-1}|u^{2p-2}|+c_2\Big)|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\\ &\leq \Big(3pb_{2p-1}\||u|^{2p-2}\|_{L^3}^{\frac{1}{p}}+c_2\Big)\|\nabla u\|_{L^6}\|\nabla\frac{\partial u}{\partial t}\|\\ &\leq C\Big(\|u\|_{H^1}^{2p-2}+1\Big)\|\triangle u\|\|\nabla\frac{\partial u}{\partial t}\|\\ &\leq C\|\triangle u\|\|\nabla\frac{\partial u}{\partial t}\|.\end{aligned}$$

Inserting estimate according to the using (22), we obtain

$$\begin{aligned}
\frac{d}{dt}\Big(\epsilon \parallel \nabla\frac{\partial u}{\partial t} \parallel^2 + \parallel \triangle u \parallel^2 \Big) + 2 \parallel \nabla\frac{\partial u}{\partial t} \parallel^2 &\leq C \parallel \triangle u \parallel\parallel \nabla\frac{\partial u}{\partial t} \parallel +2 \parallel \nabla\alpha \parallel\parallel \nabla\frac{\partial u}{\partial t} \parallel \\
&\leq C \parallel \triangle u \parallel^2 +2 \parallel \nabla\alpha \parallel^2 + \parallel \nabla\frac{\partial u}{\partial t} \parallel^2 \\
\frac{d}{dt}\Big(\epsilon \parallel \nabla\frac{\partial u}{\partial t} \parallel^2 + \parallel \triangle u \parallel^2 \Big) + \parallel \nabla\frac{\partial u}{\partial t} \parallel^2 &\leq C \parallel \triangle u \parallel^2 +2 \parallel \nabla\alpha \parallel^2 .
\end{aligned}$$

By using Gronwall's lemma, we deduce that

$$u \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H_0^1(\Omega)).$$

Multiply (2) by $-\triangle\frac{\partial\alpha}{\partial t}$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \triangle\alpha \parallel^2 \Big) + 2 \parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 +2 \parallel \triangle\frac{\partial \alpha}{\partial t} \parallel^2 = 2(u,\triangle\frac{\partial \alpha}{\partial t}) + 2(\frac{\partial u}{\partial t},\triangle\frac{\partial \alpha}{\partial t}),$$

we deduce the following inequality

$$\frac{d}{dt}\Big(\parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \triangle\alpha \parallel^2 \Big) + 2 \parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 +2 \parallel \triangle\frac{\partial \alpha}{\partial t} \parallel^2 \leq 2 \parallel u \parallel^2 + \parallel \triangle\frac{\partial \alpha}{\partial t} \parallel^2 +2 \parallel \frac{\partial u}{\partial t} \parallel^2,$$

which implies

$$\frac{d}{dt}\Big(\parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \triangle\alpha \parallel^2 \Big) + 2 \parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \triangle\frac{\partial \alpha}{\partial t} \parallel^2 \leq 2 \parallel u \parallel^2 +2 \parallel \frac{\partial u}{\partial t} \parallel^2 .$$

Hence

$$\alpha \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)).$$

Multiplying (1) by $\frac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we find

$$\begin{aligned}
\frac{d}{dt} \parallel \frac{\partial u}{\partial t} \parallel^2 +2\epsilon \parallel \frac{\partial^2 u}{\partial t^2} \parallel^2 &= 2(\triangle u,\frac{\partial^2 u}{\partial t^2}) - 2(f(u),\frac{\partial^2 u}{\partial t^2}) + 2(\alpha,\frac{\partial^2 u}{\partial t^2}) \\
&= (\frac{2\triangle u}{\sqrt{\epsilon}},\ \sqrt{\epsilon}\frac{\partial^2 u}{\partial t^2}) - (\frac{2}{\sqrt{\epsilon}}f(u),\ \sqrt{\epsilon}\frac{\partial^2 u}{\partial t^2}) + (\frac{2\alpha}{\sqrt{\epsilon}},\ \sqrt{\epsilon}\frac{\partial^2 u}{\partial t^2}) \\
&\leq \frac{2}{\epsilon} \parallel \triangle u \parallel^2 + \frac{3\epsilon}{2} \parallel \frac{\partial^2 u}{\partial t^2} \parallel^2 + \frac{2}{\epsilon} \parallel f(u) \parallel^2 + \frac{2}{\epsilon} \parallel \alpha \parallel^2 \\
&\leq \frac{2}{\epsilon}\Big(\parallel \triangle u \parallel^2 + \parallel f(u) \parallel^2 + \parallel \alpha \parallel^2 \Big) + \frac{3\epsilon}{2} \parallel \frac{\partial^2 u}{\partial t^2} \parallel^2 \\
&\leq \frac{2}{\epsilon}\Big(\parallel \triangle u \parallel^2 + \parallel f(u) \parallel^2 + \parallel \alpha \parallel^2 \Big),
\end{aligned}$$

then

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)).$$

Multiply (2) by $\frac{\partial^2 \alpha}{\partial t^2}$ and integrate over Ω, on gets

$$\begin{aligned}
\frac{d}{dt}\Big(\parallel \frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 \Big) + 2 \parallel \frac{\partial^2 \alpha}{\partial t^2} \parallel^2 &= 2(\triangle\alpha,\frac{\partial^2 \alpha}{\partial t^2}) - 2(u,\frac{\partial^2 \alpha}{\partial t^2}) - 2(\frac{\partial u}{\partial t},\frac{\partial^2 \alpha}{\partial t^2}) \\
&\leq C\Big(\parallel \triangle\alpha \parallel^2 + \parallel u \parallel^2 + \parallel \frac{\partial u}{\partial t} \parallel^2 \Big) + \parallel \frac{\partial^2 \alpha}{\partial t^2} \parallel^2 \\
\frac{d}{dt}\Big(\parallel \frac{\partial \alpha}{\partial t} \parallel^2 + \parallel \nabla\frac{\partial \alpha}{\partial t} \parallel^2 \Big) + \parallel \frac{\partial^2 \alpha}{\partial t^2} \parallel^2 &\leq C\Big(\parallel \triangle\alpha \parallel^2 + \parallel u \parallel^2 + \parallel \frac{\partial u}{\partial t} \parallel^2 \Big),
\end{aligned}$$

hence

$$\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; L^2(\Omega)).$$

Then the proof of the theorem (3) is complete.

4. Discussion

In this paper, we have confirmed the existence and the uniqueness of the solution for hyperbolic Caginalp phase-field system with all regular potential. We next study the existence of global and exponential attractors. We can also complete this work by studying this system with any other types of boundary conditions.

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