Journal of Mathematics Research; Vol. 10, No. 1; February 2018
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Edge-Maximal Graphs Containing No r Vertex-Disjoint Triangles

Mohammad Hailat[1]

[1] Department of Mathematical Scieces, University of South Carolina Aiken, USA

Correspondence: Mohammad Hailat, Department of Mathematical Scieces, University of South Carolina Aiken, SC 28801, USA. E-mail: mohammadh@usca.edu

Received: November 11, 2017 Accepted: December 14, 2017 Online Published: January 15, 2018

doi:10.5539/jmr.v10n1p110 URL: https://doi.org/10.5539/jmr.v10n1p110

Abstract

An important problem in graph theory is that of determining the maximum number of edges in a given graph G that contains no specific subgraphs. This problem has attracted the attention of many researchers. An example of such a problem is the determination of an upper bound on the number of edges of a graph that has no triangles. In this paper, we let $\mathcal{G}(n, V_{r,3})$ denote the class of graphs on n vertices containing no r-vertex-disjoint cycles of length 3. We show that for large n, $\mathcal{E}(G) \leqslant \lfloor \frac{(n-r+1)^2}{4} \rfloor + (r-1)(n-r+1)$ for every $G \in \mathcal{G}(n, V_{r,3})$. Furthermore, equality holds if and only if $G = \Omega(n, r) = K_{r-1, \lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil}$ where $\Omega(n, r)$ is a tripartite graph on n vertices.

Keywords: vertex-disjoint cycles, tripartite graphs

2010 Mathematics Subject Classification: Primary 05C38; Secondary 05C35

1. Introduction

In this paper, we only consider simple graphs with vertex set $V(G)$ and edge set $E(G)$. If an edge $e \in E(G)$ is incident with the two vertices u and v in $V(G)$, we write $e = uv = vu$. For a vertex $u \in V(G)$ the *neighborhood* of u, denoted by $N_G(u)$, is the set of vertices $v \in V(G)$ such that $uv \in E(G)$. The *degree* $d_G(u)$ is the cardinality of $N_G(u)$.

For vertex-disjoint subgraphs H_1 and H_2 of G, we let $E(H_1, H_2)$ to be the set of all edges that are incident to a vertex in H_1 and a vertex in H_2. That is $E(H_1, H_2) = \{uv \in E(G) \mid u \in V(H_1), v \in H_2\}$. We also define $\mathcal{E}(G)$ to be the cardinality of $E(G)$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$. The cycle on n vertices is denoted by C_n and the complete tripartite graph with partitioning sets of order m, n and k is denoted by $K_{m,n,k}$. For given graphs G_1 and G_2 we denote the union of G_1 and G_2 by $G_1 + G_2$ such that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. We also denote the joint of G_1 and G_2 by $G_1 \vee G_2$ such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E(G_1, G_2)$.

An important problem in extremal graph theory is the determination of maximum number of edges a graph has under a condition that the given graph has no specific subgraphs. Such an example is finding an upper bound for $\mathcal{E}(G)$ whenever G has no triangles (cycles of length 3) or, in general, G has no odd disjoint cycles. We have two types of disjoint cycles, the first type is *edge-disjoint cycles*, and the second type is *vertex-disjoint cycles*. Note that vertex-disjoint cycles are edge-disjoint cycles, but not vice-versa.

The determination of maximum number of edges in a graph that forbids certain subgraphs has attracted the attention of many graph researchers. For example, Höggkvist et al in (Höggkvist, R., Faudree, R. J., & Schelp, R. H., 1981) proved that $\mathcal{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ for a non bipartite graph G with n vertices that contains no odd cycle C_{2k+1} for all positive integers k. In (Bataineh, M., & Jaradat, M. M. M. , 2012), M. Bataineh and M. Jaradat proved that $\mathcal{E}(G) \leqslant \lfloor \frac{n^2}{4} \rfloor + r - 1$ for any graph $G \in \mathcal{G}(n; r, 2k+1)$ for large n and $r \geqslant 2$, $k \geqslant 1$, where $\mathcal{G}(n; r, 2k+1)$ is the set of all graphs on n vertices containing no r edge-disjoint cycles of length $2k+1$. In (Bataineh, M.), Bataineh proved that $\mathcal{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$ for every graph $G \in \mathcal{G}(n; V_{2k+1})$ where $\mathcal{G}(n; V_{2k+1})$ is the class of graphs on n vertices containing no vertex-disjoint cycles of length $2k+1$.

In this paper, we will generalize a result that is parallel to the result of (Bataineh, M., & Jaradat, M. M. M., 2012) in which we considered here no r vertex-disjoint cycles of length 3 instead of edge-disjoint cycles discussed in (Bataineh, M., & Jaradat, M. M. M., 2012).

2. Important Lemmas and Theorems

In this section, we introduced necessary background that are needed in proving the main results of this paper.

2.1 Lemma. *(Bondy and Murty, 1976) Let G be a graph on n vertices. If* $\mathcal{E}(G) > \frac{n^2}{4}$*, then G contains a cycle of length* $2k+1$ *for each* $1 \leqslant k \leqslant \lfloor \frac{n+3}{4} \rfloor - \frac{1}{2}$.

2.2 Theorem. *(Batineh and Jaradat, 2012) Let* $k \geqslant 1$*,* $r \geqslant 2$ *be two integers and* $g \in \mathcal{G}(n; r, 2k+1)$*. For large n,*

$\mathcal{E}(G) \leqslant \lfloor \frac{n^2}{4} \rfloor + r - 1$. *Furthermore. equality holds if and only if* $G \in \Omega(n, r) = K_{r-1, \lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil}$.

2.3 Theorem. *(Bataineh, 2012) Let* $k \geqslant 1$ *be an integer and* $G \in \mathcal{G}(n, V_{2k+1})$. *Then for* $n > \max\{\frac{4k^3+15k^2+11k-5}{2}, 4(4k^2 + 8k - 3) + 1\}$, $\mathcal{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$. *Furthermore, equality holds if and only if* $G = \Omega(n, 2)$.

3. Main Result

In this section, we generalize a special case of Theorem 2.3 to the case where $G \in \mathcal{G}(n, V_r, 3)$. That is to the case where G is a graph on n vertices containing no r vertex-disjoint cycles of length 3. We start with $r = 2$.

3.1 Theorem. *Let k be a positive integer and* $G \in \mathcal{G}(n, 2, 3)$. *Then for large n,* $\mathcal{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$. *Furthermore, equality holds if and only if* $G = \Omega(n, 2)$.

Proof. Since $G \in \mathcal{G}(n, 2, 2k + 1)$, then G has no two vertex-disjoint cycles of length 3. Suppose first that G has no cycle of length 3. Then for $n \geqslant 11$, we have $3 \leqslant \lfloor \frac{n+3}{3} \rfloor$, so that, using Lemma 2.1 we have:

$$\begin{aligned}
\mathcal{E}(G) &\leqslant \left\lfloor \frac{n^2}{4} \right\rfloor \\
&= \left\lfloor \frac{((n-1)+1)^2}{4} \right\rfloor \\
&\leqslant \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \frac{2(n-1)}{4} + \frac{1}{4} + 1 \\
&\leqslant \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1) \qquad \text{for } n \geqslant 11
\end{aligned}$$

Now if G has a cycle of length 3, then for large n, $\mathcal{E}(G) \leqslant \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1$ by Theorem 2.3. Note that if $G = \Omega(n, 2) = K_{1, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ then

$$\mathcal{E}(G) = \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1)$$

Therefore, equality holds if and only if $G = \Omega(n, 2)$. □

To prove the main theorem we have to introduce Turán graphs since these graphs play a major role in the proof.

3.2 Definition. The complete s-partite graph on n vertices with part sizes being $\left\lceil \frac{n}{s} \right\rceil$ or $\left\lfloor \frac{n}{s} \right\rfloor$ is called *Turán graph*. We denote this graph by $T_{n,s}$.

Note that Turán graph is K_{s+1} free, where K_{s+1} is the complete graph on $(s + 1)$-vertices. In (Conlon, D.), David Conlon introduced the following statement of Turán's theorem.

3.3 Theorem. *(Turán) If G is an n-vertex* K_{s+1}*-free graph, then it contains at most* $\mathcal{E}(T_{n,s})$ *edges.*

In addition, Conlon introduced three different proofs of Turáns Theorem. In proof 2 (Zykovs Symmetrization), he concluded that the set of vertices of a K_{s+1}-free graph G on n vertices with maximum number of edges can be partitioned into two equivalence classes. In these classes, vertices in the same class are non-adjacent and vertices in different classes are adjacent. Since the graph G is K_{s+1}-free, it must be a complete s-partite graph. Note that $T_{n,s}$ is the unique graph that maximizes the number of edges among such graphs.

3.4 Theorem. *Let G be a graph that has* $(r - 1)$ *vertex-disjoint cycles* $C_1, C_2, \ldots, C_{r-1}$, *but no r vertex disjoint cycles of length 3 and let* $H = G - \bigcup_{i=1}^{r-1} G(C_i)$. *Then* $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leqslant 2(r-1)(n-r+1) - 4(r-1)^2$ *and* $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leqslant 3(r-1)^2$.

Proof. Note that H is K_3 free graph since, otherwise, G would have r vertex-disjoint cycles of length 3, a contradiction to the assumption. Let H' be a graph on the vertices of H with a maximum number of edges. Note that $|V(H)| = |V(H')| = n - 3(r - 1) = (n - r + 1) + 2(r - 1)$, $\mathcal{E}(H) \leqslant \mathcal{E}(H')$, and $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leqslant \mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H')$.

Let $n' = n - 3(r - 1) = |V(H')|$. Since H' is K_{2+1}-free graph then, using proof 2 of Turáns theorem, H' is $T_{n',2}$ and the vertices of H' can be partitioned into two equivalent classes H'_1 and H'_2 where $|V(H'_1)| = \lceil \frac{n'}{2} \rceil$ and $|V(H'_2)| = \lfloor \frac{n'}{2} \rfloor$. Note that vertices in H_1' are non-adjacent and, also, vertices in H_2' are non-adjacent, but vertices in H_1' are adjacent to vertices in H_2'. In Figure 1, let

$$C_1 = v_1u_1w_1v_1$$
$$\vdots$$
$$C_{r-1} = v_{r-1}u_{r-1}w_{r-1}v_{r-1}$$

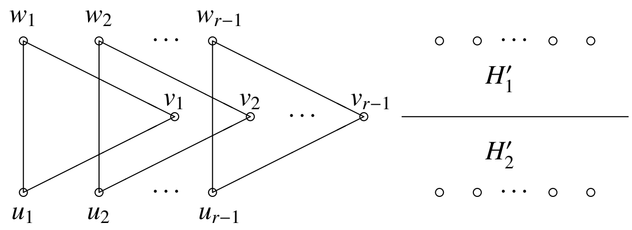

Figure 1

Note that $|H_1'| = \left\lceil \frac{n-3(r-1)}{2} \right\rceil$ and $|H_2'| = \left\lfloor \frac{n-3(r-1)}{2} \right\rfloor$, so that

$$\begin{aligned}\mathcal{E}(v_1, H') &\leqslant \left\lfloor \frac{n-3(r-1)}{2} \right\rfloor + \left\lceil \frac{n-3(r-1)}{2} \right\rceil = n - 3(r-1)\\ &= (n-r+1) - 2(r-1)\end{aligned}$$

In Figure 1, if v_1 is adjacent to a vertex $x \in H_1'$ and a vertex $y \in H_2'$ then $C_1' = v_1xyv_1$ is a is a cycle of length 3. If u_1 or w_1 is adjacent to vertices in H_1' and H_2' then we can form another cycle $C_1'' = u_1x'y'u_1$. Now if we replace C_1 with C_1' and C_1'' then we will have r vertex-disjoint cycles of length 3, a contradiction. It follows that w_1 and u_1 are adjacent to H_1' or H_2' but not to both. Therefore,

$$\begin{aligned}\mathcal{E}(u_1, H') &\leqslant \frac{1}{2}((n-r+1) - 2(r-1)) = \frac{1}{2}(n-r+1) - (r-1)\\ \mathcal{E}(w_1, H') &\leqslant \frac{1}{2}((n-r+1) - 2(r-1)) = \frac{1}{2}(n-r+1) - (r-1)\end{aligned}$$

so that,

$$\begin{aligned}\mathcal{E}(G(C_1), H) &\leqslant \mathcal{E}(G(C_1), H')\\ &\leqslant \mathcal{E}(v_1, H') + \mathcal{E}(u_1, H') + \mathcal{E}(w_1, H')\\ &\leqslant (n-r+1) - 2(r-1) + \frac{1}{2}(n-r+1) - (r-1) + \frac{1}{2}(n-r+1) - (r-1)\\ &\leqslant 2(n-r+1) - 4(r-1)^2.\end{aligned}$$

It follows that

$$\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leqslant 2(r-1)(n-r+1) - 4(r-1)^2.$$

Note that, without loss of generality, $w_1, w_2, \ldots, w_{r-1}$ are adjacent to vertices in H_2' and $v_1, v_2, \ldots, v_{r-1}$ are adjacent to vertices in H_1'. Note, also, that $K = G - G(\{v_1, \ldots, v_{r-1}\})$ is K_3 free and, therefore, K can be partitioned into two sets K_1' and K_2' where $w_1, \ldots, w_{r-1} \in K_1'$, $H_1' \subset K_1'$, $v_1, \ldots, v_{r-1} \in K_2'$, and $H_2' \subset K_2'$. Therefore, $w_iw_j \notin E(G)$ and $v_iv_j \notin E(G)$. Also, $v_iv_j \notin G$ since, otherwise, we will have r-vertex disjoint cycle of length 3. It follows that

$$\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leqslant |K_{r-1,r-1,r-1}| = 3(r-1)^2.$$

□

Now we have our main result.

3.5 Theorem. *Let $G \in \mathcal{G}(n, V_r, 3)$. Then for large n,*

$$\mathcal{E} \leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).$$

Furthermore, equality holds if and if $G = \Omega(n, r)$.

Proof. Suppose G is a graph on n vertices containing no r vertex-disjoint triangles. That is G has no r vertex-disjoint cycles each of length 3. We prove the theorem by induction on r. For $r = 2$, the result holds by Theorem 3.1.

Now suppose the result is true for $r - 1$. We need to show that it is true for r. For this, let $G \in \mathcal{G}(n, r, 3)$. If G contains no $(r - 1)$ vertex-disjoint cycles of length 3, then by induction:

$$\begin{aligned}
\mathcal{E}(G) &\leqslant \left\lfloor \frac{(n-(r-1)+1)^2}{4} \right\rfloor + ((r-1)-1)(n-(r-1)+1) \\
&= \left\lfloor \frac{(n-r+2)^2}{4} \right\rfloor + (r-2)(n-r+2) \\
&\leqslant \frac{(n-r+1)^2 + 2(n-r+1) + 1 + 4((r-1)-1)(n-(r-1)+1))}{4} + 1 \\
&= \frac{(n-r+1)^2}{4} + \frac{2(n-r+1) + 4(r-1)(n-r+1) + 4(r-1) - 4(n-r+1) - 4}{4} + 1 \\
&= \frac{(n-r+1)^2}{4} + (r-1)(n-r+1) - \frac{1}{2}(n-r+1)(r-1) - 1 + 1 \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1), \quad \text{for } n \geqslant 3r-3.
\end{aligned}$$

Therefore, we now assume that G has $(r-1)$ vertex-disjoint cycles of length 3 and has no r vertex-disjoint cycles of length 3. Let C_1, C_2,...,C_{r-1} be such cycles. Let $H = G - \bigcup_{i=1}^{r-1} G(C_i)$, so that H has no cycle of length 3 since, otherwise, G will have r vertex-disjoint cycles of length 3. Note that $n' = |V(H)| = n - 3(r-1) = (n-r+1) - 2(r-1)$. Since H has no cycle of length 3 then, using Lemma 2.1, we have;

$$\begin{aligned}
\mathcal{E}(H) \leqslant \left\lfloor \frac{n'^2}{4} \right\rfloor &= \left\lfloor \frac{((n-r+1) - 2(r-1))^2}{4} \right\rfloor \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor - (r-1)(n-r+1) + (r-1)^2
\end{aligned}$$

Also, using Theorem 3.4, we have

$$\begin{aligned}
\mathcal{E}(H, \bigcup_{i=1}^{r-1} G(C_i)) &\leqslant 2(r-1)(n-r+1) - 4(r-1)^2 \\
\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) &\leqslant 3(r-1)^2
\end{aligned}$$

so that,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(H) + \mathcal{E}(H, \bigcup_{i=1}^{r-1} G(C_i)) + \mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor - (r-1)(n-r+1) + (r-1)^2 \\
&\qquad + 2(r-1)(n-r+1) - 3(r-1)^2 + 3(r-1)^2 \\
&\leqslant \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).
\end{aligned}$$

Furthermore, we conclude that equality holds for $\Omega(n,r) = K_{r-1,\lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil}$ since

$$\begin{aligned}
\mathcal{E}(\Omega(n,r)) &= (r-1)\left\lfloor \frac{n-r+1}{2} \right\rfloor + (r-1)\left\lceil \frac{n-r+1}{2} \right\rceil + \left\lceil \frac{n-r+1}{2} \right\rceil \left\lfloor \frac{n-r+1}{2} \right\rfloor \\
&= (r-1)\lfloor n-r+1 \rfloor + \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).
\end{aligned}$$

□

References

Bataineh, M. *Edge-maximal graphs containing no vertex-disjoint cycles of length* $2k+1$. To appear in Jordan Journal of Mathematics and Statistics.

Bataineh, M., & Jaradat, M. M. M. (2012). Edge Maximal C2k+1-edge disjoint free graphs. *Discussion Mathematicae Graph Theory, 32*, 271-278.

Bondy, J., & Murty, U. (1976). *Graph Theory with Applications*, London, UK: The MacMillan Press.

Conlon, D. *Extremal Graph Theory*, Lecture 1.

Höggkvist, R., Faudree, R. J., & Schelp, R. H. (1981). Pancyclic graphs-connected Ramsey number. *Ars Comb, 11*, 37-49.

Copyrights