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On the Trapped Surface Characterization of Black Hole Region in Vaidya Spacetime

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Abstract

Characterizing black holes by means of classical event horizon is a global concept because it depends on future null infinity. This means, to find black hole region and event horizon requires the notion of the entire spacetime which is a teleological concept. With this as a motivation, we use local approach as a complementary means of characterizing black holes. In this paper we apply Gauss divergence and covariant divergence theorems to compute the fluxes and the divergences of the appropriate null vectors in Vaidya spacetime and thus explicitly determine the existence of trapped and marginally trapped surfaces in its black hole region.

Keywords: Trapped surfaces, black holes, event horizons

1. Introduction

The most important predictions of general relativity are Black holes. They are the spacetime regions from which no signal can be seen by an observer far from the matter sources (Frolov and Zelnikov, 2011). These Black holes are formed as gravitational collapse of sufficiently massive objects, such as massive stars demonstrated by the work of (Chandrasekhar, 1983). Black holes are remarkably simple objects characterized by just a few numbers. As stated by Chandrasekhar the black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time (Hartle, 2003). The existence of black holes was first discussed by Michell and Laplace within the framework of the Newtonian theory in the 18th century, (Frolov and Zelnikov, 2011). They considered it as a star with a very strong gravitational field such that the Newtonian escape velocity $\sqrt{2GM/R}$ (where M and R are the mass and radius of the star respectively) is larger than the velocity of light. The inequality $R \leq 2GM/C^2$ for escape velocity also holds in general relativity (Penrose, 2004, Krishnan, 2013). The study of black holes has for many years depended on the event horizons as its boundary from which one can send signals to infinity (Senovilla, 2011). However, the study of black holes based on the concept of classical event horizon has the following drawbacks: to locate black hole region and event horizon requires the knowledge of the entire spacetime, the definition does not have direct relation with the notion of strong gravitational field as shown by (Ashtekar and Krishnan, 2004) and (Krishnan, 2012) for example in the Vaidya spacetime, we can have event horizon forming in a flat region. Another global feature of event horizons is their teleological nature (Gourgoulhon and Jaramillo, 2008). The event horizon, responds in advance to what will happen in the future. This was shown by Booth (2005) regarding the explicit example of two successive matter shells that collapsed to form a black hole: the first shell collapses to form the event horizon and the latter remains stationary for some time and then begins to grow before the second collapsing shell reaches it (Gourgoulhon and Jaramillo, 2008). If we consider black holes as physical objects, for instance in quantum gravity or numerical relativity, the above mentioned global nature of the event horizon creates a big problem (Gourgoulhon and Jaramillo, 2008). This global nature of the event horizon and these physical issues with the event horizon have motivated the need to use local approach specifically the notion of. trapped surface as a complementary means of characterizing black holes. A trapped surface S is a compact, space-like 2-dimensional sub-manifold of space-time on which the divergence of the outgoing null vector orthogonal to the surface converges. The notion of trapped surfaces, due to (Penrose, 1965b), entails that, in a very strong gravitational field such occurs in the gravitational collapse, outgoing light rays tend to converge. In a stationary black hole such as the Schwarzschild black hole, the event horizon and the Killing horizon are are one and same and characterize the boundary of the region in which trapped surfaces are located (Booth, Kunduri and OGrady, 2017). In a dynamical black hole such as the Vaidya spacetime, trapped surfaces are generally located inside the apparent horizon which is found in the event horizon (Ben-Dov, 2007). Therefore the event horizon is not in general the boundary of the region in which trapped surfaces are located (Ben-Dov, 2007). This means it is always possible to locate trapped surfaces inside the black hole region but there is no guarantee that the existence of black hole implies a trapped surface (Jakobsson, 2017). These trapped surfaces are very important in the Penrose singularity theorem and their presence indicates the development of singularity (Senovilla and Garfinkle, 2015) and therefore the formation of black holes.

In fact, in spacetimes which satisfy proper energy conditions, trapped surfaces lie inside the region of a black hole and their location does not involve the development of the whole future spacetime. The purpose of this paper is to explicitly demonstrate the existence of trapped surfaces and marginally trapped surfaces by computing the covariant divergences and the fluxes of null vectors in the Vaidya spacetime.

The plan of the paper is as follows: section 2 discusses the covariant divergence which is the main tool for the computation of the divergences of both ingoing and outgoing null vectors. In this same section, Gauss's divergence theorem for the computation of the fluxes of a vector field is discussed. This section also contains discussions of local characterizations of black holes. A general discussion of the Vaidya spacetime is presented in the same section. In section 3, the existence of trapped and marginally trapped surfaces in the Vaidya spacetime is discussed after computing the covariant divergences and the fluxes of vector fields. This is the main result of this paper. Section 4 then gives the conclusion of the result.

2. Black Holes

2.1 Causal Structure

The idea that future events can be understood as consequences of initial conditions plus the laws of physics is causality (Carroll, 2004). The causal structure of spacetime is illustrated by figure 1 below. Associated with each event p in spacetime is a light cone. Half of the cone is labelled future and the other half past. Those events that can be reached by a material particle from p lie in the interior of the future light cone; these comprise the chronological future of p. The chronological future of p together with events lying on the cone itself comprises the causal future of p; physically, it represents events which in principle can be influenced by a signal emitted from p. In general relativity, the causal structure is locally of the same qualitative nature as in special relativity. But the main differences can occur globally because of spacetime singularity or the twisting of the directions of light cones as move from point to point (Wald, 1984). Before we can precisely define a black hole, we introduce some concepts regarding the causal structure of a given spacetime $(M; g)$. For a subset S of the manifold M, we have the following definitions.

Definition 2.1. *A causal curve is any smooth curve that is nowhere spacelike i.e. it is timelike or null everywhere (Carroll, 2004).*

Definition 2.2. *When any subset S of a manifold M is given, the causal future of S which is denoted $J^+(S)$, is the set of points that can be reached from S when future-directed causal curve is followed (Carroll, 2004).*

Definition 2.3. *The chronological future $I^+(S)$ is the set of points that can be reached when future-directed timelike curve is followed (Wald, 1984, Hawking and Ellis, 1973, Carroll, 2004) A curve of zero length is achronal but not causal; so a point p will always be in its own causal future $J^+(p)$, but not necessarily its own chronological future $I^+(p)$. The causal past $J^+(p)$ and chronological past I^- are defined the same way. A subset $S \subset M$ is called achronal if no two points in S are connected by a timelike curve (Carroll, 2004, Galloway and Vega, (2016))*

2.2 The Definition of a Black Hole

When a strongly asymptotically predictable spacetime M is given, the region B of a *black hole region* is defined as $B = M - J^-(p^+)$ where $J^-(p^+)$ is the causal past of future null infinity p^+. This means, B is the region of spacetime where there is no communication with p^+ (Jaramillo, 2011, Wald, 2001, Hawking and Ellis, 1973). In terms of spacetime, the boundary of a black hole is its event horizon (Bengtsson et al., 2013, Hayward, 2000, Booth, 2005) .In this context, the event horizon E, is defined by $E = \partial J - (p^+) \cap M$ (Bengtsson et al., 2013), or the boundary of the region B from where we can send signals to infinity (Senovilla, 2011).

2.3 Local Characterizations of Black Holes

2.3.1 Trapped Surface

A two-dimensional surface S in a four dimensional spacetime has two null directions normal to the surface at each point. Trapped surfaces are characterized by the covariant divergences of vectors orthogonal to the surface. Thus we can distinguish two future directed null vectors emerging from the surface S. If we denote the out-going and in-going null normals to the surface S by l^α and n^α respectively, then their respective covariant divergences are negative: θ_l and θ_n. Penrose defines a trapped surface S as a compact, space-like 2-dimensional sub-manifold of space-time on which $\theta_l\theta_n > 0$, where l^α and n^α are the two null normals to S (Ashtekar and Krishnan, 2004). In flat space, the out-going light rays diverge and the ingoing ones converge, i.e. $\theta_l > 0$ and $\theta_n < 0$ so there is no trapped surface.

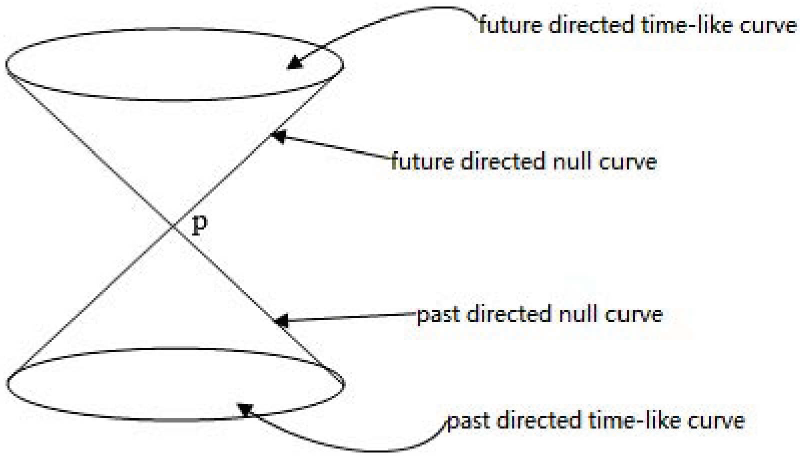

Figure 1. Light cone at p

2.3.2 Marginally Trapped Surface

The surface S is said to be marginally trapped (MTS) if $\theta_l = 0$ and $\theta_n < 0$ (Gourgoulhon, 2017). The singularity theorems of (Penrose, 1965a, Hawking and Penrose, 1970, Senovilla and Garfinkle, 2015) have shown that the presence of such surfaces is the signature of a spacetime containing a black hole. Hayward, (1994) defines marginally trapped surface as a spatial 2-surface S on which one null expansion vanishes.

2.3.3 The Trapped Region and the Trapping Boundary

An outer trapped surface defined by Hawking as a compact spacelike 2-dimensional submanifold in (M, g_{ab}) such that the expansion of the outgoing null geodesics orthogonal to the surface is negative (Ashtekar and Krishnan, 2004). This definition does not matter whether the ingoing null geodesics are converging or not but it includes for convenience the case $\theta = 0$.

Trapped Region: Hawking defines the trapped region $T(M)$ in a surface M as the set of all points in M, through which there passes an outer-trapped surface, lying entirely in M. The spacetime region T containing trapped surface is called the trapped region (Ashtekar and Krishnan, 2004). Schnetter et al., (2006), define a trapped region as the region in which trapped surfaces can be located, it can be in the full spacetime or on a Cauchy surface. Hayward, (1994), defines a trapped region as a subset of space-time through each point of which there passes a trapped surface. Trapping boundary is defined as a connected component of the boundary of an inextendible trapped region (Hayward, 1994). Under certain assumptions, Hayward was able to show that the trapping boundary is foliated by marginally trapped surfaces (MTSs), i.e., a compact, space-like 2-dimensional sub-manifolds on which the expansion of one of the null normals, say l^μ vanishes and that of the other, n^μ is everywhere negative (Ashtekar and Krishnan, 2004). A trapping horizon is defined as a hypersurface of M foliated by spacelike 2-surfaces S such that the expansion scalar θ_l of one of the null geodesics orthogonal to S vanishes. A trapping horizon can be either spacelike or null (Hayward, 1994).

2.3.4 Apparent Horizon

When a spacelike 3-surface is given, the outer boundary of a region containing outer trapped surfaces that lie in the 3-surface is called the apparent horizon (Ben-Dov, 2007). On a given spatial hypersurface, all (marginally) outer trapped surfaces can be found. Here, the outermost marginally outer trapped surface on the spatial slice is called the apparent horizon. In the practice of numerical relativity, the apparent horizon serves as the definition of the boundary of a black hole (Jakobsson, 2017).

Both of the above definitions of apparent horizon are dependent on the given spatial slicing of the spacetime. There may exist trapped surfaces lying not in one of the given spatial slices that extend beyond the apparent horizon. For instance, there are slicings of the Schwarzschild spacetime, reaching the singularity, which do not have trapped surface in any spatial slice (Wald and Iyer, 1991), even though the whole interior of the Schwarzschild black hole is filled with trapped surfaces. In general, the apparent 3-horizon is neither unique nor continuous because a different foliation of the spacetime into spacelike surfaces can result in a different location of the apparent horizon through the spacetime (Ben-Dov, 2007).

2.4 Covariant Divergence of a Vector Field

The covariant divergence of a vector field is given by the scalar

$$\mathrm{div}\bar{V} = \partial_i V^i + \Gamma^i_i V^k \tag{1}$$

This is a scalar in all frames and reduces to the familiar form in a Cartesian system. Applying the Christoffel symbol

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} \right] \tag{2}$$

Equation (1) can be written compactly as follows

$$\mathrm{div}\bar{V} = \nabla_i V^i = \frac{\partial V^i}{\partial x^i} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} V^i = \frac{1}{\sqrt{g}} \left[\sqrt{g} \frac{\partial V^i}{\partial x^i} + \frac{\partial \sqrt{g}}{\partial x^j} V^i \right] = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} V^i)}{\partial x^i} \tag{3}$$

In this compact form, we only need to calculate g and its derivative, but not the Christofel symbols themselves to evaluate the covariant divergence of a vector filed.

2.5 Gauss's Divergence Theorem

The divergence theorem applies to a closed surface S. By a closed surface S, we mean a surface consisting of one connected piece which does not intersect itself and which completely encloses a single finite region D of space called its interior. The closed surface S is then said to be the boundary of D. A sphere, a cube and a torus are examples of closed surfaces.

Theorem 2.4 (Gauss's divergence theorem)The integral of the divergence of a vector field over a region V equals the flux of the field through the surface S bounding V provided the field is suitably smooth inside V and S (Borisenko and Tarapov, 1979). Consider a region V, in which a vector field $\bar{g}$ is continuous and differentiable, the divergence of this vector field is given by

$$\int_V \nabla \cdot \bar{g} dV = \oint_S \bar{g}\bar{n} dS \tag{4}$$

Here, S is a closed surface and surrounds a very small region V completely at point r.

Definition 2.4. *The measure of how much vector field crosses a given surface is called the flux of a vector field. The divergence theorem relates the total flux of a vector field out of a closed surface S to the integral of the divergence of the vector over the enclosed volume V. But if there is a massive source inside the surface, its gravitational field has an attractive or converging effect. Close enough to a massive source; the outgoing null vectors converge and the flux becomes negative i.e. $\nabla \cdot \bar{l} < 0$. A surface S is said to be trapped if and only if both fluxes are negative: $\nabla \cdot \bar{l} = \theta_l < 0$ and $\nabla \cdot \bar{n} = \theta_n < 0$. The surface where the flux of outgoing null vector becomes zero and the flux of ingoing null vector is less than zero is said to be marginally trapped (MTS) i.e. $\nabla \cdot \bar{l} = \theta_l = 0$ and $\nabla \cdot \bar{n} = \theta_n = 0$.*

2.6 Gravitational Singularity

Gravitational field of a point mass is

$$\bar{g} = -\frac{GM}{r^2} \hat{r} \tag{5}$$

This field is defined everywhere except at $r = 0$ where it blows up to infinity; there is a singularity there. How do we treat this singularity? According to Gauss's law; if you draw a spherical surface or any closed shape around a mass M and measure the flux through that surface, the flux will be the same no matter how the mass inside is distributed, as it only depends on the quantity of mass enclosed. As far as real world measurement is concern, it does not matter that a point mass has an infinite gravitational field at the center or at the distance point of measurement, the gravitational field is finite and the same as if the mass were diffused into a cloud without a singularity. The singularity matters in showing how the usual mathematical approach fails, thereby indicating a more sophisticated approach is needed to handle that case. This may be done by applying Gauss's theorem to the gravitational field to derive the gravitational flux. Equation (4) presents us with an interesting paradox when we consider the vector field equation (5). On one hand, the divergence of this vector field is zero and the other hand $-4\pi GM$. We immediately notice that, we have a big problem according to the divergence theorem, i.e. $-4\pi GM$

This paradox is resolved by noting that $\nabla\cdot\left(-GM\frac{\hat{r}}{r^2}\right) = 0$ is valid only at $r \neq 0$. To reconcile the two sides of the divergence theorem, we therefore introduce a singular function known as the delta distrbution $\delta^3(r)$ defined by the identity

$$\delta^3(r) = \nabla\cdot\left(-GM\frac{\hat{r}}{r^2}\right) \tag{6}$$

with the property that

$$\delta^3(r) = \begin{cases} 0, & r \neq 0 \\ \infty, & r = 0 \end{cases} \tag{7}$$

Additional property of the delta distribution is

$$\int_V f(r)\delta(r-a)dV = \begin{cases} f(a), & \text{if } a \text{ is located inside } V \\ 0, & \text{if } a \text{ is located outside } V \end{cases} \tag{8}$$

Using the delta distribution $\delta^3(r)$, we can write

$$\nabla\cdot\bar{g} = -4\pi GM\delta^3(r) \tag{9}$$

And thus,

$$\int_V \nabla\cdot\bar{g}dV = -4\pi GM\int_V \delta^3(r)dV = -4\pi GM \tag{10}$$

Hence, according to the field (10) the flux of the field (5) through a surface S surrounding the origin is not zero.

2.7 Vaidya Spacetime

The Vaidya metric is obtained by making the Schwarzschild metric time dependent. This is done by replacing M in the Schwarzschild metric with $M(v)$ where v is the Eddington-Finkelstein time coordinates. In the ingoing Eddington-Finkelstein coordinate $(v, r\theta, \phi)$ the Vaidya spacetime is as follows

$$ds^2 = -fdv^2 + 2dvdr + r^2d\Omega, \qquad f = 1 - \frac{2M(v)}{r} \tag{11}$$

The stress-energy tensor for this metric

$$8\pi T_{\alpha\beta} = \frac{2}{r^2}\frac{dM}{dv}\partial_\alpha v\partial_\beta v \tag{12}$$

Now in this metric, the vector

$$k_\alpha = -\partial_\alpha v,\ k^\alpha = -\partial_r,\ k\cdot k = 0,\ k^\alpha\nabla_\alpha k^\beta = 0 \tag{13}$$

is tangent to ingoing null geodesics. Hence the Vaidya stress-energy tensor is

$$T_{\alpha\beta} = \frac{2}{4\pi r^2}\frac{dM}{dv}k_\alpha k_\beta \tag{14}$$

For k^α to be timelike, this would have been interpreted as the stress-energy tensor of dust with density $\rho = \frac{1}{4\pi r^2}\frac{dM}{dv}$

But k^α is actually null so the ingoing Vaidya metric is sourced by radially infalling null dust. It is require that the null dust has positive density: $\rho > 0 \implies \frac{dM}{dr} > 0$. We have $v = t - r_*$ implies the black-hole mass increases in time (Poisson, 2004).

3. Trapped Surface and Marginally Trapped Surface in Vaidya Spacetime

3.1 Trapped and Marginally Trapped Surfaces in Vaidya Spacetime Applying Covariant Divergence of a Vector Field

To apply covariant divergence, a set of null normal orthogonal to the constant (r, v) surfaces are

$$l = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\frac{\partial}{\partial r}, \qquad n = -\frac{\partial}{\partial r} \tag{15}$$

and their covariant divergences are respectively given by

$$\theta_l(v, r) = \nabla\cdot\bar{l} = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{q} = \frac{r - 2M(v)}{r^2} \tag{16}$$

$$\theta_n(v, r) = \nabla\cdot\bar{n} = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{q} = \frac{-2}{r} \tag{17}$$

Thus the surface with $r = 2M(v)$ and fixed v are marginally trapped surfaces. However, in the region $r < 2M(v)$, which is the region of the black hole, we get both divergences to be negative and the surface is said to be trapped.

3.2 Trapped and Marginally Trapped Surfaces in Vaidya Spacetime Applying the Flux of a Vector Field

The divergence theorem tells us that the flux of g across S can be found by integrating the divergence of g over the region enclosed by S. The field $\bar{g} = -\frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface. The outgoing and ingoing null vectors are respectively given by

$$l = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\frac{\partial}{\partial r}, \qquad n = -\frac{\partial}{\partial r} \tag{18}$$

Their fluxes are given by

$$\phi_l = 2\pi GM\left(\frac{r - 2M(v)}{r}\right) \tag{19}$$

$$\phi_n = -4\pi GM \tag{20}$$

For $r > 2M(v)$, i.e. in flat space, the flux of S along the outgoing null vector is positive: $\phi_l > 0$ whereas that along ingoing null vector is negative: $\phi_n < 0$. But if $r < 2M(v)$ i.e. in the region of the black hole, both fluxes are negative. Such surfaces are said to be trapped. For $r = 2M(v)$ hypersurface, $\phi_l = 0$, $\phi_n < 0$ is called marginally trapped surface

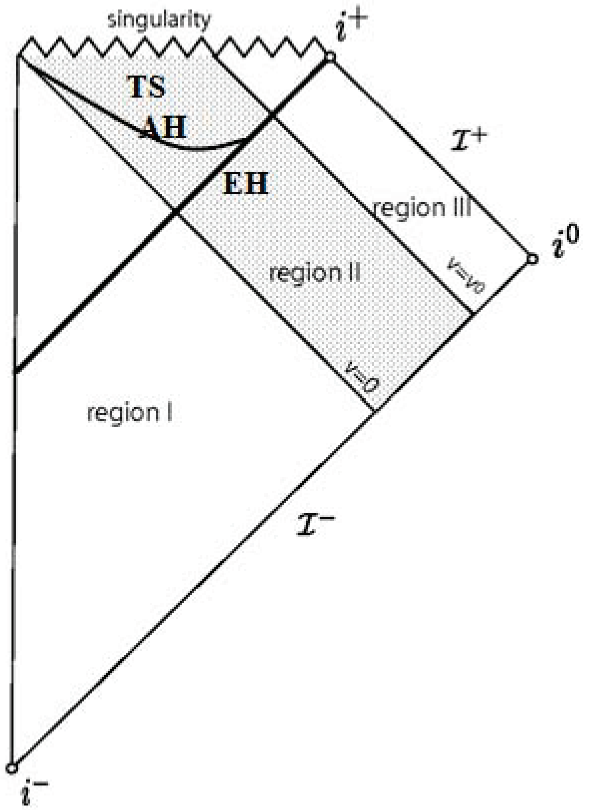

Figure 2. Penrose-Carter conformal diagram for Vaidya spacetime. Region I is flat and II is the Vaidya spacetime region. Region III is the Schwarzschild spacetime region. The event horizon EH is different from the $r = 2M(v)$ surface. The two meet in the Schwarzschild region. The Apparent Horizon (Marginally trapped tube) AH is described by $r = 2M(v)$. Trapped surface (TS) lies inside the apparent horizon.

4. Conclusion

The existence of trapped and marginally trapped surfaces in Vaidya spacetime (dynamic black hole) have been investigated applying Gauss divergence and covariant divergence theorems. The fluxes and the divergences of the appropriate null vectors in Vaidya spacetime have been computed. With these fluxes and divergences, we have been able to actually determine that, trapped and marginally trapped surfaces exist in Vaidya spacetime. The trapped surface, which is a closed two-surface S with the property that for both ingoing and outgoing null vectors orthogonal to S, the divergences are both negative everywhere on S. In terms of fluxes, both outward and inward fluxes are negative. The three dimensional boundary of the region of the spacetime in which trapped surfaces are located the - trapped region is the trapping horizon and its two dimensional intersection with a spacelike hypersurface is called the apparent horizon. The apparent horizon is therefore a marginally trapped surface - a closed two surface on which one of the divergences or fluxes vanishes. We also observed that for a general spacetime like the Vaidya spacetime, the event horizon and the apparent horizon are different hypersurfaces. From figure 2, the two meet only in the final Schwarzschild region. As stated in (Poisson, 2004) that unless the null energy condition is violated, the apparent horizon always lies in the event horizon in the dynamic situations like the Vaidya spacetime. This proves the statement by (Ben-Dov, 2008) that the boundary of the region in which trapped surfaces is located is not in general, the event horizon.

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References

Ashtekar, A., & Krishnan, B. (2004). Living rev. rel. 7 10 (www.livingreviews.org/lrr-2004-10).

Ben-Dov, I. (2007). *"outer trapped surfaces in vaidya spacetimes"* arxiv:grqc/ 0611057.

Bengtsson, I., Jacobsson, E., & Senovilla, J. M. M. (2013) *Trapped surfaces in oppenheimer-snyder black holes*; arxiv-1306.6486.

Booth, I. (2005). *Black hole boundaries.* Can. J. phys., 831073. https://doi.org/10.1139/p05-063

Booth, I. Kunduri, H. K., & O'Grady, A. (2017). *unstable marginally outer trapped surfaces in static spherically symmetric spacetimes.* Physics Reviewed D, 96.

Borisenko, A. I., & Tarapov, I. E. (1979). *Vector analysis with applications*. Dover Publications, Inc. New York.

Carroll, S. M. (2004). Spacetime and geometry. *an introduction to general relativity*. Addison Wesley, 1301 Sansome St. San Francisco, CA94111.

Chandrasekhar, S. (1983). *The mathematical theory of black holes*. Oxford University press. New York.

Frolov, V. P., & Zelnikov, A. (2011). *Introduction to black hole physics*. OUP, Oxford. https://doi.org/10.1093/acprof:oso/9780199692293.001.0001

Gourgoulhon, E. (2017). *Geometry and physics of black holes*, Lecture notes.

Gourgoulhon, E., & Jaramillo, J. L. (2008). *New theoretical approaches to black holes*, arxiv.org.

Hartle, J. B. (2003). *An introduction to Einstein's General Relativity*. Addison Wesley, San Francisco. https://doi.org/10.1119/1.1604390

Hawking, S. W., & Penrose, R. (1970). *The singularities of gravitational collapse and cosmology*. Proc. R. Soc. Lond, 314(1519):529548. https://doi.org/10.1098/rspa.1970.0021

Hawking, S. W., & Ellis, G. F. R. *The Large Scale Structure of Space- time.* Cambridge University Press, Cambridge, 1973.

Hayward, S. A. (1994). *"general laws of black hole dynamics"* phys. rev. d49 6467 [arxiv; gr-qc/9303006].

Hayward, S. A. (2000). *Black hole new horizons*, world scientific singapore.

Jakobsson, E. (2017). *Shapes of Spacetimes - Collected tales of black holes*. Department of Physics, Stocholm University.

Jaramillo, J. L. (2011). *An introduction to local black hole horizons in the 3+1 approach to general relativity,* max-planck-institute ur gravitations physik, albert einstein institut, am muhlenberg 1, golm d-14476, germany.

Krishnan, B. (2012). *The spacetime in the neighborhood of a general isolated black hole*. Class. Quant. Grav, 29:205006. https://doi.org/10.1088/0264-9381/29/20/205006

Krishnan, B. (2013). *Quasi-local black hole horizons*, max planck institute for gravitational physics (albert einstein institute), callinstr. 38, 30167 hannover, germany. page 1.

Penrose, R. (1965). *Gravitational collapse and space-time singularities*. Phys. Rev. Lett, 14:57:59. https://doi.org/10.1098/rspa.1965.0058

Penrose, R. (1965). *Zero rest mass fields including gravitation: asymptotic behaviour*. Proc. Roy. Soc. Lond, A284159-203.

Penrose, R. (2004). *The road to reality. A complete guide to the laws of the universe.* Published by Jonathan Cape.

Poisson, E. (2004). *A Relativist's Toolkit (The Mathematics of Black-Hole Mechanics)*. Cambridge University Press: Cambridge, UK.

Schnetter, E., Krishnan, B., & Beyer, F. (2006). *Introduction to dynamical horizons in numerical relativity.* Physical Review D (Particles, Fields, Gravitation, and Cosmology), *74*(2), 024028.

Senovilla., & Garnkle. (2015). *The 1965 penrose singularity theorem*. Classical and Quantum Gravity, 32. https://doi.org/10.1088/0264-9381/32/12/124008

Senovilla, J. M. M. (2011). *Trapped surfaces.* In International Journal of Modern Physics: Conference Series World Scientific Publishing Com- pany, arXiv-1107.1344.

Wald, R. M. (1984). *General Relativity*. University of Chicago Press Chicago. https://doi.org/10.7208/chicago/9780226870373.001.0001

Wald, R. M., & Iyer, V. (1991). Phys. rev. d44r3719.

Wald, R. M. (2001). *The thermodynamics of black holes.* living rev. relativ, 4, 6. https://doi.org/10.12942/lrr-2001-6