

An Instability Phenomenon in Hele-Shaw Displacements

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Abstract

We study the linear instability of the displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell, motivated by possible applications in Secondary Oil Recovery (SOR) process. Most numerical methods have usually failed when the Weissenberg numbers W_i (appearing in the constitutive relations) are near 1. We get an approximate formula of the growth rate with a blow-up for some particular $W_i = O(1)$. Therefore the instability at large Weissenberg numbers is due to the model, and not to the computational methods. Our growth rate is quite similar to Saffman-Taylor’s formula (obtained when a Newtonian liquid is displaced by air) if $W_1 = W_2$.

Keywords: Oldroyd-B fluids, Hele-Shaw displacement, linear stability

1. Introduction

A Hele-Shaw cell is a technical device consisting of two parallel plates, located at a small distance from each other. With some usual hypothesis - see (Hele-Shaw, 1898) - the averaged (across the plates) velocities of a Stokes flow are verifying an equation quite similar with the Darcy law for the flow in a porous medium, whose permeability is depending on the distance between the plates - see (Bear, 1972; Bird & Stewart, 1960; Lamb, 1933).

We can consider two immiscible fluids with different viscosities in a Hele-Shaw cell. The “contact” zone is a small region, where the viscosity display a (continuous) large variation. In the Hele-Shaw model, this zone is replaced by a sharp interface, where a viscosity *jump* exists. This model can be used for study the displacement process similar with the Secondary Oil Recovery (SOR) procedure: the oil (at low pressure) from a porous medium is pushed by a second fluid. The linear stability of the interface between two immiscible fluids of Darcy type in a Hele-Shaw cell or porous medium was studied in (Saffman & Taylor, 1958). The obtained growth constant is giving us the well-known Saffman-Taylor instability criterion: the interface is unstable if the displacing fluid is less viscous. The usual displacing fluids (some polymer-solute) used in (SOR) and the oil in a porous reservoir are often non-Newtonian fluids - the constitutive relations between the stress and the strain-rate tensors are non linear. Then seems be useful to study the “Saffman-Taylor” instability in the case of such fluids.

A large number of results in the field of non-Newtonian fluids are given in (Guillope & Saut, 1992; Nittman et al, 1985; Renardy, 2000; Schowlater, 1978; Truesdell & Noll, 1965; Zhao & Maher, 1993). The displacement of Maxwell upper-convected fluids by air in a Hele-Shaw cell are studied numerically in (Mora & Manna, 2009; Mora & Manna, 2010).

Some numerical results concerning the instability of the interface between air (as a displacing fluid) and an Oldroyd-B fluid in a Hele-Shaw were obtained in (Wilson, 1990). Here a blow-up of the growth constant for $W_2 = 0$, $W_1 > 2.5$ was reported, similar with the fractures observed in the flows of some complex fluids in Hele-Shaw cells - see (Nase et al, 2008), (Nittman et al, 1985; Zhao & Maher, 1993).

In this paper we study the modal linear stability of the interface appearing when an Oldroyd-B fluid is displaced by air in a Hele-Shaw cell. The Weissenberg numbers W_i appear in the constitutive relations (3) and we consider $W_i = O(1)$. The basic flow (the same as in (Wilson, 1990)) is described in section 2. The perturbations system is derived in section 3. We use the particular perturbations (26), depending on the arbitrary parameter α . If α checks the condition (43), then we can neglect some terms in the constitutive relations and we get approximate expressions of some components of the extra-stress tensor. An approximate formula of the growth rate of perturbations is given in section 4. We get dispersion curves which are similar with numerical results given in (Wilson, 1990). Our growth constant is similar with the Saffman-Taylor’s formula if $W_1 = W_2$. The main point is following: the particular perturbations (26) lead us to the blow-up of the growth constant (48) for $W_1 - W_2 \approx 0.4$. Then the large Weissenberg number instability is due to the model, and not to the computational methods. We conclude in section 5, where we further demonstrate that our basic flow becomes Newtonian if $W_1 = W_2$ (that means the relaxation and retardation time constants are equal in the constitutive relations (3)).

2. The Oldroyd-B Fluid and the Basic Flow

We consider a horizontal Hele-Shaw cell, parallel with the xOy plane and neglect the gravity. An Oldroyd-B fluid is displaced by air in the positive direction of Ox . The distance between plates is denoted by b and the cell length cell is l such that $\epsilon = b/l \ll 1$. We use the following notations:

the extra-stress and strain-rate tensors: $\underline{\tau}, \underline{\mathbf{S}}$;

the velocity, pressure and the fluid viscosity: $\underline{\mathbf{u}} = (u, v, w), p, \mu$;

the relaxation and retardation (time) constants: λ_1, λ_2 ;

the matrix containing the velocity derivatives: $\underline{\mathbf{V}}$;

the strain-rate tensor: $\underline{\mathbf{S}} := (\underline{\mathbf{V}} + \underline{\mathbf{V}}^T)/2; (\underline{\mathbf{V}}_{ij})^T := \underline{\mathbf{V}}_{ji}$.

The flow equations, the free-divergence condition and the constitutive relations are

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z}, \quad p_y = \tau_{21,x} + \tau_{22,y} + \tau_{23,z}, \quad p_z = \tau_{31,x} + \tau_{32,y} + \tau_{33,z}, \tag{1}$$

$$u_x + v_y + w_z = 0, \tag{2}$$

$$\underline{\tau} + \lambda_1 \underline{\tau}^\nabla = 2\mu[\underline{\mathbf{S}} + \lambda_2 \underline{\mathbf{S}}^\nabla], \quad \lambda_1 > \lambda_2 \geq 0, \tag{3}$$

$$\underline{\tau}^\nabla = \underline{\mathbf{u}} \cdot \nabla \underline{\tau} - (\underline{\mathbf{V}} \underline{\tau} + \underline{\tau} \underline{\mathbf{V}}^T), \quad \underline{\mathbf{S}}^\nabla = \underline{\mathbf{u}} \cdot \nabla \underline{\mathbf{S}} - (\underline{\mathbf{V}} \underline{\mathbf{S}} + \underline{\mathbf{S}} \underline{\mathbf{V}}^T). \tag{4}$$

The last two relations are not containing the partial time derivatives $\underline{\tau}_t$ and $\underline{\mathbf{S}}_t$, because we consider a steady flow. $p_x, p_y, p_z, \tau_{ij,x}, \tau_{ij,y}, \tau_{ij,z}$ are denoting the x, y, z partial derivatives of τ_{ij} and p . The following boundary conditions are used:

a) No-slip condition for the velocity components on the plates.

b) Laplace's law in the neighborhood of the basic air-fluid interface.

We study the linear stability of the following basic flow, denoted by the super index 0 :

$$\nabla p^0 = (p_x^0(x), 0, 0), \quad \mathbf{v}^0 = (u^0(z), 0, 0), \tag{5}$$

$$\mathbf{V}^0 = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}^0 = \frac{1}{2}(\mathbf{V}^0 + \mathbf{V}^{0T}). \tag{6}$$

The basic extra-stress tensor is given by the following equation

$$\underline{\tau}^0 - \lambda_1(\mathbf{V}^0 \underline{\tau}^0 + \underline{\tau}^0 \mathbf{V}^{0T}) = \mu\{2\mathbf{S}^0 - \lambda_2(2\mathbf{V}^0 \mathbf{S}^0 + 2\mathbf{S}^0 \mathbf{V}^{0T})\}, \tag{7}$$

where

$$\mathbf{V}^0 \underline{\tau}^0 + \underline{\tau}^0 \mathbf{V}^{0T} = \begin{pmatrix} 2u_z^0 \tau_{31}^0 & u_z^0 \tau_{32}^0 & u_z^0 \tau_{33}^0 \\ u_z^0 \tau_{32}^0 & 0 & 0 \\ u_z^0 \tau_{33}^0 & 0 & 0 \end{pmatrix}, \quad 2(\mathbf{V}^0 \mathbf{S}^0 + \mathbf{S}^0 \mathbf{V}^{0T}) = \begin{pmatrix} 2(u_z^0)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the components of the basic extra-stress are

$$\tau_{33}^0 = 0, \quad \tau_{22}^0 = 0, \quad \tau_{23}^0 = 0, \quad \tau_{12}^0 = 0, \tag{8}$$

$$\tau_{13}^0 = \mu u_z^0, \quad \tau_{11}^0 = 2(\lambda_1 - \lambda_2)\mu(u_z^0)^2 \tag{9}$$

and we get the basic flow equations:

$$p_x^0 = \tau_{11,x}^0 + \tau_{12,y}^0 + \tau_{13,z}^0, \quad p_y^0 = \tau_{21,x}^0 + \tau_{22,y}^0 + \tau_{23,z}^0, \quad p_z^0 = \tau_{31,x}^0 + \tau_{32,y}^0 + \tau_{33,z}^0. \tag{10}$$

The equations (5) - (10) (used also in (Wilson, 1990)) give us

$$p_z^0 = 0, \quad p_y^0 = 0, \quad p_x^0(x) = \tau_{13,z}^0(z). \tag{11}$$

We conclude that a negative constant G exists such that $p_x^0(x) = \mu u_{zz}^0 = G$, therefore the basic velocity u^0 can be obtained in terms of G :

$$u^0 = \frac{1}{2\mu} p_x^0(z^2 - bz) = \frac{1}{2\mu} G(z^2 - bz). \tag{12}$$

The following characteristic velocity U is introduced

$$U = \langle u^0 \rangle := \frac{1}{b} \int_0^b u^0 dz = -\frac{b^2}{12\mu} p_x^0, \tag{13}$$

then we have the relations

$$u^0 = (G/2\mu)z(z - b) = -(6U/b^2)z(z - b). \tag{14}$$

The basic air-fluid interface is

$$x = \langle u^0 \rangle t = Ut. \tag{15}$$

As in (Wilson, 1990), the basic pressure can depend on the time t :

$$p^0 = G(x - \langle u^0 \rangle t) = G(x - Ut), \quad x > \langle u^0 \rangle t = Ut. \tag{16}$$

In the following we consider the moving coordinate system $\bar{x} = x - Ut$, then the basic interface is $\bar{x} = 0$. But (with no confusion) we still use the notation $x = 0$ for the basic interface.

3. The Perturbations System

The small perturbations of the basic solution are denoted by u, v, w, p, τ ,

$$\mathbf{V} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = (\mathbf{V} + \mathbf{V}^T)/2. \tag{17}$$

The perturbation of the basic interface is denoted by η . We consider that a fluid element that was originally on the interface remains here, then it follows

$$\eta_t = u \tag{18}$$

(in other words, the interface is material).

In the frame of the linear stability, the free-divergence relation is also verified by the components of the velocity perturbation, then $u_x + v_y + w_z = 0$. We integrate across the plates, we use the condition $w = 0$ on $z = 0, z = b$, then we get

$$\int_0^b (u_x + v_y) = 0.$$

In this paper we consider the particular perturbations such that

$$u_x + v_y = 0,$$

(which verify the above condition) and we get $w_z = 0$. Then the boundary conditions are giving us $w = 0$, which is obtained in (Wilson, 1990) by using a numerical method .

We introduce the small perturbations in the constitutive equations and in the expressions of the upper convected derivatives and get

$$\tau^0 + \tau + \lambda_1(\tau^0 + \tau)^\nabla = 2\mu[\mathbf{S}^0 + \mathbf{S} + \lambda_2(\mathbf{S}^0 + \mathbf{S})^\nabla], \tag{19}$$

$$(\tau^0 + \tau)^\nabla = u^0 \tau_x - [\mathbf{V}^0 \tau^0 + \tau^0 \mathbf{V}^{0T}] - [\mathbf{V}^0 \tau + \mathbf{V} \tau^0 + \tau^0 \mathbf{V}^T + \tau \mathbf{V}^{0T}], \tag{20}$$

$$(\mathbf{S}^0 + \mathbf{S})^\nabla = u^0 \mathbf{S}_x - [\mathbf{V}^0 \mathbf{S}^0 + \mathbf{S}^0 \mathbf{V}^{0T}] - [\mathbf{V}^0 \mathbf{S} + \mathbf{V} \mathbf{S}^0 + \mathbf{S}^0 \mathbf{V}^T + \mathbf{S} \mathbf{V}^{0T}]. \tag{21}$$

In the frame of the linear stability (that means by neglecting the second order terms in perturbations) it follows

$$\tau + \lambda_1(u^0 \tau_x - \mathbf{E}) = \mu\{2\mathbf{S} + \lambda_2(u^0 2\mathbf{S}_x - \mathbf{F})\}, \tag{22}$$

$$\mathbf{E} := \mathbf{V}^0 \tau + \mathbf{V} \tau^0 + \tau^0 \mathbf{V}^T + \tau \mathbf{V}^{0T},$$

$$\mathbf{F} := 2[\mathbf{V}^0 \mathbf{S} + (\mathbf{V}^0 \mathbf{S})^T + \mathbf{V} \mathbf{S}^0 + (\mathbf{V} \mathbf{S}^0)^T]. \tag{23}$$

We have

$$\tau^0 \mathbf{V}^T = (\mathbf{V} \tau^0)^T, \quad \tau \mathbf{V}^{0T} = (\mathbf{V}^0 \tau)^T, \quad \mathbf{S}^0 \mathbf{V}^T = (\mathbf{V} \mathbf{S}^0)^T, \quad \mathbf{S} \mathbf{V}^{0T} = (\mathbf{V}^0 \mathbf{S})^T,$$

and by using (6), (9) and (17) we get

$$\mathbf{V}^0 \boldsymbol{\tau} = \begin{pmatrix} u_z^0 \tau_{31} & u_z^0 \tau_{32} & u_z^0 \tau_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{V} \boldsymbol{\tau}^0 = \begin{pmatrix} u_x \tau_{11}^0 + u_z \tau_{31}^0 & 0 & u_x \tau_{13}^0 \\ v_x \tau_{11}^0 + v_z \tau_{31}^0 & 0 & v_x \tau_{13}^0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$2\mathbf{V}^0 \mathbf{S} = \begin{pmatrix} u_z^0 u_z & u_z^0 v_z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 2\mathbf{V} \mathbf{S}^0 = \begin{pmatrix} u_z u_z^0 & 0 & u_x u_z^0 \\ v_z u_z^0 & 0 & v_x u_z^0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathbf{E} = \begin{pmatrix} 2(u_z^0 \tau_{31} + u_x \tau_{11}^0 + \tau_{13}^0 u_z) & (u_z^0 \tau_{32} + v_x \tau_{11}^0 + v_z \tau_{13}^0) & (u_z^0 \tau_{33} + u_x \tau_{13}^0) \\ (u_z^0 \tau_{32} + v_x \tau_{11}^0 + v_z \tau_{13}^0) & 0 & \tau_{13}^0 v_x \\ (u_z^0 \tau_{33} + u_x \tau_{13}^0) & \tau_{13}^0 v_x & 0 \end{pmatrix}; \tag{24}$$

$$\mathbf{F} = \begin{pmatrix} 4u_z^0 u_z & 2u_z^0 v_z & u_x u_z^0 \\ 2u_z^0 v_z & 0 & v_x u_z^0 \\ u_x u_z^0 & v_x u_z^0 & 0 \end{pmatrix}. \tag{25}$$

4. The Sability Analysis

We consider the following perturbations of the basic solution:

$$u = f(z)EXP \cos(ny), \quad v = f(z)EXP \sin(ny), \quad EXP = \exp(-n[\alpha + x] + \sigma t),$$

$$\sigma = \text{growth constant}, \quad f(z) = \beta u^0(z), \quad \beta = O(\epsilon^2), \quad \text{dimension of } \alpha = \text{length}. \tag{26}$$

We obtain a formula for the growth constant σ corresponding to the perturbations (26), which is displaying a blow-up (in terms of the wavenumbers n) for some particular values of the Weissenberg numbers.

Near the basic interface $x = 0$ we use the following *Laplace's law* as in (Renardy, 2000; Wilson, 1990) (see (13) and (18) for definitions of G, η):

$$(G \langle \eta \rangle + \langle p \rangle) - \langle \tau_{11} \rangle = \gamma (\langle \eta_{yy} + \eta_{zz} \rangle), \quad \eta = u/\sigma. \tag{27}$$

Here $(\eta_{yy} + \eta_{zz})$ is the approximate expression of the total curvature of the perturbed interface and γ is the surface tension on the interface. The above relation gives the growth constant expression:

$$\sigma = \frac{\gamma \langle u_{yy} + u_{zz} \rangle - G \langle u \rangle}{\langle p - \tau_{11} \rangle}.$$

The problem is to compute $\langle p - \tau_{11} \rangle$ in terms of the basic and perturbed velocities. For this, in the following we search a particular solution $\tau_{33}, \tau_{13}, \tau_{23}, \tau_{12}, p_x, p_z, \tau_{11}$ in terms of u, v, u^0, v^0 .

We define the dimensionless quantities

$$x' = x/l, \quad y' = y/l, \quad z' = z/b, \quad \epsilon = b/l \ll 1, \quad u' = u/U, \quad v' = v/U,$$

$$p' = p(l/\mu U), \quad \gamma' = \gamma(l/\mu U), \quad n' = nl, \quad \sigma' = \sigma(l/U), \quad \alpha' = \alpha/l, \quad t' = t(U/l),$$

$$\{\tau'_{11}, \tau'_{12}, \tau'_{22}\} = \{\tau_{11}, \tau_{12}, \tau_{22}\}(l/\mu U), \quad \{\tau'_{13}, \tau'_{23}, \tau'_{33}\} = \{\tau_{13}, \tau_{23}, \tau_{33}\}(b/\mu U),$$

$$W_i = \lambda_i(U/b), \tag{28}$$

where W_i are the Weissenberg numbers. From (12), (13) and (26) we get

$$u' = \beta u^{0'} EXP' \cos(n'y') = \beta 6z'(1 - z')EXP' \cos(n'y'),$$

$$v' = \beta u^{0'} EXP' \sin(n'y') = \beta 6z'(1 - z')EXP' \sin(n'y'),$$

$$EXP' = \exp(-n'[\alpha' + x'] + \sigma' t').$$

In the following we use *only dimensionless quantities, then we omit the '.*

The flow equations and the dimensionless quantities (28) give us

$$p_x \frac{\mu U}{\rho^2} - \tau_{11,x} \frac{\mu U}{\rho^2} = \tau_{12,y} \frac{\mu U}{\rho^2} + \tau_{13,z} \frac{\mu U}{b^2}, \quad p_x - \tau_{11,x} = \tau_{12,y} + \tau_{13,z} \frac{1}{\epsilon^2}, \tag{29}$$

$$p - \tau_{11} = (-1/n)\{\tau_{12,y} + \tau_{13,z} \frac{1}{\epsilon^2}\}. \tag{30}$$

Proposition 1. We prove that

$$\tau_{33} = 0, \quad \tau_{31} = u_z, \quad \tau_{32} = v_z \tag{31}$$

are possible solutions of the perturbations system (22).

Proof. From (22) and (28) we get

$$\tau_{33} + W_1 \epsilon u^0 \tau_{33,x} = 0, \tag{32}$$

$$\tau_{31} + W_1 \epsilon u^0 \tau_{31,x} - W_1 \epsilon (u_z^0 \tau_{33} + u_x \tau_{13}^0) = u_z + W_2 \epsilon (u^0 u_{zx} - u_x u_z^0), \tag{33}$$

$$\tau_{32} + W_1 \epsilon (u^0 \tau_{32,x} - \tau_{13}^0 v_x) = v_z + W_2 \epsilon (u^0 v_{zx} - v_x u_z^0). \tag{34}$$

The equation (32) gives us the possible solution $\tau_{33} = 0$. We use (26), then in the right part of (33) - (34) we get $W_2 \epsilon (u^0 u_{zx} - u_z^0 u_x) = 0, W_2 \epsilon (u^0 v_{zx} - v_x u_z^0) = 0$. If $\tau_{31} = u_z, \tau_{32} = v_z$, in the left part of (33), (34) we get

$$W_1 \epsilon (u^0 \tau_{31,x} + u_z^0 \tau_{33} - u_x \tau_{13}^0) = W_1 \epsilon (u^0 u_{zx} - u_x u_z^0) = 0,$$

$$W_1 \epsilon (u^0 \tau_{32,x} - \tau_{13}^0 v_x) = W_1 \epsilon (u^0 v_{zx} - v_x u_z^0) = 0.$$

Then the system (32) - (34) is verified by the expressions (31). As a consequence, we obtain

$$p_z = \tau_{31,x} + \tau_{32,y} = \mu(u_{zx} + v_{zy}) = 0. \tag{35}$$

□

From (31)₂ we get $\tau_{31,z} = u_{zz}$. We use (13), (27), (30) and we obtain the dimensionless form of the Laplace's law as follows:

$$\begin{aligned} \frac{-12\mu U}{b^2} \cdot \frac{\langle Uu \rangle}{\sigma(U/l)} + \frac{\mu U}{l} \langle p - \tau_{11} \rangle &= \frac{\gamma\mu U}{\sigma(U/l)} [\langle Uu_{yy} \rangle \frac{1}{l^2} + \langle Uu_{zz} \rangle \frac{1}{b^2}], \\ \frac{-12 \langle u \rangle}{\epsilon^2} \frac{\sigma}{\sigma} + (-1/n)\{\tau_{12,y} + u_{zz} \frac{1}{\epsilon^2}\} &= \frac{\gamma}{\sigma} \langle u_{yy} + u_{zz} \frac{1}{\epsilon^2} \rangle. \end{aligned} \tag{36}$$

The term $\langle u_{yy} \rangle$ can not be neglected, being related with the curvature of the perturbed interface.

Proposition 2. If $\exp(\sigma t) < 1$, then $\forall x \geq 0$ we have

$$\begin{aligned} \max_n \{v_x\} &= \frac{3\beta}{2\alpha e}, & \max_n \{v_{xx}\} &= \frac{6\beta}{\alpha^2 e^2}, & \max_n \{v_{xxx}\} &= \frac{81\beta}{2\alpha^3 e^3}, \\ \max_n \{u_x\} &= \frac{3\beta}{2\alpha e}, & \max_n \{u_{xx}\} &= \frac{6\beta}{\alpha^2 e^2}, & \max_n \{u_{xxx}\} &= \frac{81\beta}{2\alpha^3 e^3}. \end{aligned} \tag{37}$$

Proof. From (14) we have $u^0 = 6z(1 - z) \leq 6 \cdot (1/4) = 3/2$ and get

$$u_x \leq (3\beta/2)n \exp(-n\alpha), \quad u_{xx} \leq (3\beta/2)n^2 \exp(-n\alpha), \quad u_{xxx} \leq (3\beta/2)n^3 \exp(-n\alpha).$$

Consider the function $F1(n) = ne^{-n\alpha}$, then

$$dF1/dn = (1 - n\alpha)e^{-n\alpha}, \quad dF1/dn(1/\alpha) = 0 \Rightarrow \max_n F1 = F1(1/\alpha) = 1/(\alpha e).$$

The maximal values of $n^2 e^{-n\alpha}, n^3 e^{-n\alpha}$ give us the relations (37). As a consequence, when $\alpha > 81/(12e)$ we obtain $\max_n \{u_{xxx}\} < \max_n \{u_{xx}\}$. □

Remark 1. In the Figure 1 we see that $\epsilon n^3 \exp(-an)$ can be neglected in front $n^2 \exp(-an)$, when $\alpha = 2, \epsilon = 0.02$. In practice, we have $\epsilon \approx 10^{-3}$ and numerical estimates give $\epsilon n^3 \exp(-an) < 0.0003$. This is the point in our procedure for obtaining an approximate growth constant. □

Proposition 3. If $\exp(\sigma t) < 1, \beta = O(\epsilon^2), (W_1 - W_2) = O(1), \alpha > 11(W_2 - W_1)/e$, then $\tau_{12,y}$ can be approximated by the formula

$$\tau_{12,y} = (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon. \tag{38}$$

Proof. The constitutive relations (22) and the dimensionless quantities (28) give us

$$\tau_{12,y} \frac{\mu U}{l^2} + \lambda_1 u^0 \tau_{12,xy} \frac{\mu U^2}{\beta} - \lambda_1 [u_z^0 \mu v_{zy} \frac{U^2}{b^2 l} + v_{xy} 2\mu(\lambda_1 - \lambda_2)(u_z^0)^2 \frac{U^3}{l^2 b^2} + \mu u_z^0 v_{zy} \frac{U^2}{b^2 l}] =$$

$$\mu\{(u_y + v_x)_y \frac{U}{l^2} + \lambda_2 u^0 (u_y + v_x)_{xy} \frac{U^2}{l^3} - 2\lambda_2 u_z^0 v_{zy} \frac{U^2}{b^2 l}\}.$$

Recall $W_i = \lambda_i U/b$, $\epsilon = b/l$, then we get

$$\begin{aligned} \tau_{12,y} + W_1 u^0 \tau_{12,xy} \epsilon - W_1 [2u_z^0 v_{zy}/\epsilon + 2(W_1 - W_2)v_{zy}(u_z^0)^2] = \\ (u_y + v_x)_y + W_2 u^0 (u_y + v_x)_{xy} \epsilon - 2W_2 u_z^0 v_{zy}/\epsilon. \end{aligned} \tag{39}$$

We introduce the expression (38) in (39) and obtain

$$\begin{aligned} \tau_{12,y} = -W_1 u^0 \epsilon \{(u_y + v_x)_{yx} + 2(W_1 - W_2)u_z^0 v_{zyx}/\epsilon\} + \\ (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon + 2W_1(W_1 - W_2)v_{xy}(u_z^0)^2 + W_2 u^0 (u_y + v_x)_{xy} \epsilon. \end{aligned} \tag{40}$$

From the Fourier expansion (26) we have $2W_1(W_1 - W_2)[v_{xy}(u_z^0)^2 - u^0 u_z^0 v_{zyx}] = 0$, then the last relation (40) becomes

$$\tau_{12,y} = (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon + (W_2 - W_1)u^0 (u_y + v_x)_{xy} \epsilon. \tag{41}$$

The equation (14) and the dimensionless quantities (28) give us $u^0 = 6(z - z^2) \leq 3/2$, $u_z^0 = 6(1 - 2z) \leq 6$. We need estimates of the maximal values of the terms in the right part of the expression (41). We see that:

$$u_{yy}, v_{xy} \text{ contain the factor } n^2 e^{-n\alpha}, \quad u_{yxy}, v_{xxy} \text{ contain the factor } n^3 e^{-n\alpha}.$$

Then we can use **Proposition 2** and get

$$\begin{aligned} \max_n (u_y + v_x)_y &= 2 \cdot \frac{3\beta}{2} \cdot \frac{4}{\alpha^2 e^2} = \frac{12\beta}{\alpha^2 e^2} = \frac{12}{\alpha^2 e^2} \epsilon^2; \\ \max_n (W_1 - W_2)u_z^0 v_{zy} \frac{1}{\epsilon} &= (W_1 - W_2) \frac{36\beta}{\alpha e \epsilon} = O(1) \frac{36}{\alpha e} \epsilon; \\ \max_n (W_2 - W_1)u^0 (u_y + v_x)_{xy} \epsilon &= |W_2 - W_1| \frac{9\beta}{2} \cdot \frac{27}{\alpha^3 e^3} \epsilon = O(1) \frac{243}{2\alpha^3 e^3} \epsilon^3. \end{aligned} \tag{42}$$

The condition

$$\alpha > 11|W_2 - W_1|/e \tag{43}$$

gives us

$$\alpha > \frac{243|W_2 - W_1|}{24e} \quad \text{and} \quad \frac{243\beta|W_2 - W_1|}{2\alpha^3 e^3} \epsilon < \frac{12\beta}{\alpha^2 e^2} \epsilon. \tag{44}$$

As $\epsilon \ll 1$, from (42) - (44) we get

$$\max_n (W_2 - W_1)u^0 (u_y + v_x)_{xy} \epsilon < \max_n (u_y + v_x)_y \epsilon \ll \max_n (u_y + v_x)_y.$$

We conclude that $(W_2 - W_1)u^0 (u_y + v_x)_{xy} \epsilon$ can be neglected in front of $(u_y + v_x)_y$ and (41) gives the relation (38). □

Remark 2. The above result is an important improvement compared with (Wilson, 1990), where instead of (26) was used the expansion

$$u \propto \exp(-nx + \sigma t \cos(ny)), \quad v \propto \exp(-nx + \sigma t) \sin(ny). \tag{45}$$

By using (45), the second and third partial derivatives of (u, v) with respect to x, y contain the factors $n^2 \exp(-nx + \sigma t)$ and $n^3 \exp(-nx + \sigma t)$, which are not bounded in terms of n when $x \rightarrow 0$, even if $\exp(\sigma t) < 1$. Moreover, the expansion (45) is not giving the explicit expression of the perturbations amplitude, which in our paper is $6\beta|z(1 - z)|$. Therefore the expansion (26) and the condition (43) allow us to avoid the singularity near $x = 0$ and to neglect, in a rigorous way, some terms in the constitutive relations. □

The expression (38) of $\tau_{12,y}$ allow us to get the growth constant formula, as follows. The relations (29), (38) give us

$$\begin{aligned} p - \tau &= (-1/n)\{(u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy} \frac{1}{\epsilon} + u_{zz} \frac{1}{\epsilon^2}\} = \\ &(-1/n)[O(\epsilon^2) + O(\epsilon) + O(1)]. \end{aligned} \tag{46}$$

Consider $O(\gamma) = 1$. Then (36) and (46) give the magnitudes of the terms appearing in the dimensionless Laplace's law :

$$\frac{O(1)}{\sigma} - \frac{1}{n}\{O(\epsilon^2) + O(\epsilon) + O(1)\} = \frac{O(1)}{\sigma}\{O(\epsilon^2) + O(1)\}. \tag{47}$$

We insert (46) in (36). As $\langle u^0 \rangle = 1, \langle (u_z^0)^2 \rangle = 12$, we get

$$\frac{-12}{\epsilon^2 \sigma} + (-1/n)\{-2n^2 + 24n(W_1 - W_2)\frac{1}{\epsilon} - 12\frac{1}{\epsilon^2}\} = \frac{\gamma}{\sigma}(-n^2 - 12\frac{1}{\epsilon^2}).$$

From the last relation we obtain the following

Proposition 4. The growth rate σ corresponding to the approximate solution (38) is given by the expression

$$\sigma = \frac{n(1 - \gamma) - \gamma(\epsilon^2/12)n^3}{n^2(\epsilon^2/6) - 2n(W_1 - W_2)\epsilon + 1}. \tag{48}$$

□

Remark 3. In the case $\exp(\sigma t) < 1$, from (48) we obtain the following important results:

a) The denominator of (48) is $\neq 0$ for $W_1 - W_2 < 1/\sqrt{6} \approx 0.408$. Indeed, we have

$$W_1 - W_2 < 1/\sqrt{6} \Rightarrow \Delta = (W_1 - W_2)\epsilon^2 - (\epsilon^2/6) < 0.$$

b) $\sigma \rightarrow \infty$ when $W_2 = 0, W_1 < 1/\sqrt{6}, W_1 \rightarrow 1/\sqrt{6}$.

c) If $0.3 < W_1 < 0.408$ and $W_2 = 0$ we have a destabilization effect compared with the Saffman-Taylor growth constant σ_{S-T} below:

$$\sigma_{S-T} = n - \gamma(\epsilon^2/12)n^3. \tag{49}$$

In Figure 2 we plot σ given by (48) (on the vertical axis) in terms of n (on the horizontal axis), for $\gamma = 0.1, \epsilon = 0.006, W_2 = 0$ and W_1 increasing from 0.1 until 0.38. The plots are similar with the numerical results given in (Wilson, 1990). We also also plotted the Saffman-Taylor growth rate (49). □

Proposition 5. If the hypothesis of **Proposition 3** hold, then $\tau_{11,x}$ can be approximated by the formula

$$\tau_{11,x} = 2u_{xx} + 4(W_1 - W_2)u_z^0 u_{zx}/\epsilon. \tag{50}$$

Proof. The dimensionless form of the constitutive relations (22) is

$$\begin{aligned} \tau_{11,x} \frac{\mu U}{l^2} + \lambda_1 u^0 \tau_{11,xx} \frac{\mu U^2}{l^3} - 2\lambda_1 [u_z^0 \mu u_{zx} \frac{U^2}{b^2 l} + u_{xx} 2\mu(\lambda_1 - \lambda_2)(u_z^0)^2 \frac{U^3}{l^2 b^2} + \mu u_z^0 u_{zx} \frac{U^2}{b^2 l}] = \\ \mu \{ 2u_{xx} \frac{U}{l^2} + \lambda_2 u^0 2u_{xxx} \frac{U^2}{l^3} - 4\lambda_2 u_z^0 u_{zx} \frac{U^2}{b^2 l} \}, \end{aligned}$$

then we obtain

$$\begin{aligned} \tau_{11,x} + W_1 u^0 \tau_{11,xx} \epsilon - 2W_1 [u_z^0 u_{zx}/\epsilon + u_{xx} 2(W_1 - W_2)(u_z^0)^2 + u_z^0 u_{zx}/\epsilon] = \\ 2u_{xx} + 2W_2 u^0 u_{xxx} \epsilon - 4W_2 u_z^0 u_{zx}/\epsilon. \end{aligned}$$

We insert the expression (50) in the last relation and get

$$\begin{aligned} \tau_{11,x} + W_1 \epsilon u^0 [2u_{xxx} + 4(W_1 - W_2)u_z^0 u_{zxx}/\epsilon] - 2W_1 [u_z^0 u_{zx}/\epsilon + u_{xx} 2(W_1 - W_2)(u_z^0)^2 + u_z^0 u_{zx}/\epsilon] = \\ 2u_{xx} + 2W_2 u^0 u_{xxx} \epsilon - 4W_2 u_z^0 u_{zx}/\epsilon. \tag{51} \end{aligned}$$

As $f(z) = \beta u^0$, from (26) we have $u^0 u_z^0 u_{zxx} - u_{xx}(u_z^0)^2 = 0$, then it follows

$$\tau_{11,x} = 2u_{xx} + 4(W_1 - W_2)u_z^0 u_{zx}/\epsilon + 2(W_2 - W_1)\epsilon u^0 u_{xxx}.$$

As in **Proposition 3**, the condition (43) allows us to neglect $\{2(W_2 - W_1)\epsilon u^0 u_{xxx}\}$ in front of $\{2u_{xx}\}$. Then the last above expression of $\tau_{11,x}$ gives us the equation (50). □

We compute now the expression of p_x , by using (38) and (50) :

$$\begin{aligned} p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z}/\epsilon^2 = \\ 2u_{xx} + 4(W_1 - W_2)u_z^0 u_{zx}/\epsilon + (u_y + v_x)_y + 2(W_1 - W_2)u_z^0 v_{zy}/\epsilon + u_{zz}/\epsilon^2. \end{aligned}$$

As $u_{xx} + u_{yy} = 0, \quad u_{xx} + v_{xy} = 0, \quad u_{zx} + v_{zy} = 0$, it follows

$$p_x = 2(W_1 - W_2)u_z^0 u_{zx} / \epsilon + u_{zz} / \epsilon^2.$$

We have obtained $p_z = 0$ - see (35) - then the last above relation gives us

$$p_{xz} = 2(W_1 - W_2)(u_z^0 u_{zx})_z / \epsilon + u_{zzz} / \epsilon^2 = 0. \tag{52}$$

□

Now we can justify the expression of f used in (26), as follows.

Proposition 6. As before, we suppose $\exp(\sigma t) < 1, \quad (W_1 - W_2) = O(1)$. If the condition (43) holds and

$$\beta = \epsilon^2 / 27, \tag{53}$$

then $f(z) = \beta u^0(z)$ verifies (52) with the precision order $O(\epsilon)$.

Proof. We have $\max_n \{n \exp(-n\alpha)\} = 1/(\alpha e)$, then $|(u_z^0 u_{zx})_z| \leq 144\beta/(\alpha e)$. It follows

$$PXZ1 := 2(W_1 - W_2)|(u_z^0 u_{zx})_z| \frac{1}{\epsilon} \leq 2(W_1 - W_2)144 \frac{\epsilon}{27} \cdot \frac{1}{11(W_1 - W_2)} < \epsilon \ll 1.$$

As $u_{zzz}^0 = 0$, we conclude that $f(z) = \beta u^0(z)$ verifies (52) with the precision order $O(\epsilon)$. The condition (53) is in agreement with the hypothesis $\beta = O(\epsilon^2)$ used in (26). □

5. Conclusions

A large part of the numerical results concerning the flow of Oldroyd-B fluids failed when the Weissenberg is near 1. In this paper we prove that this kind of instability is due to the model, and not to the numerical methods - at least in the case of Hele-Shaw displacements.

The basic solution (5) - (10) is considered, which governs the steady displacement of an Oldroyd-B fluid by air in a Hele-Shaw cell. We obtain the perturbations system (22) - (25) and get the growth constant corresponding to the particular perturbations (26). The dimensionless quantities are introduced in the relations (28) and we get the dimensionless (significant) constitutive relations (32), (33), (34), (39), (51), where the Weissenberg numbers W_i are verifying $W_i = O(1), i = 1, 2$.

The perturbations (26) display bounded values for partial derivatives of (u, v) in terms of $x, y, \forall x \geq 0$. The condition (43) imposed on the parameter α (appearing in the perturbations (26)) allows us to obtain the approximate expressions (38), (50), (48) for some components of the extra-stress tensor τ and for the growth constant σ .

If $\alpha = 0$, then the perturbations (26) leads us to very high values of the derivatives of (u, v) with respect to x, y near the interface $x = 0$. In this case, the estimates and the expressions given in **Propositions 3** and **5** can not be obtained - see also **Remark 2**.

In the case $W_1 = W_2$, the growth rate (48) is quite similar with the Saffman-Taylor formula for a Newtonian fluid displaced by air in a Hele-Shaw cell. As we consider a 3D Hele-Shaw cell, the formula (48) contains the new term $(-\gamma n)$ in the numerator and the new term $n^2(\epsilon^2/6)$ in the denominator.

On the other hand, when $W_1 = W_2$ (that means $\lambda_1 = \lambda_2$) our basic flow is Newtonian. Indeed, consider the tensor $\mathbf{A} = \tau^0 - \mu 2\mathbf{S}^0 = a_{ij}, \quad \mathbf{A} = \mathbf{A}^T$, then the constitutive relations (7) gives us

$$\begin{aligned} \tau^0 - \lambda_1(\mathbf{V}^0 \tau^0 + \tau^0 \mathbf{V}^{0T}) &= \mu\{2\mathbf{S}^0 - \lambda_1(\mathbf{V}^0 2\mathbf{S}^0 + 2\mathbf{S}^0 \mathbf{V}^{0T})\}, \\ \mathbf{A} - \lambda_1(\mathbf{V}^0 \mathbf{A} + \mathbf{A} \mathbf{V}^{0T}) &= 0. \end{aligned} \tag{54}$$

We have:

$$\mathbf{V}^0 = \begin{pmatrix} 0 & 0 & u_z^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V}^0 \mathbf{A} + \mathbf{A} \mathbf{V}^{0T} = \begin{pmatrix} 2u_z^0 a_{31} & u_z^0 a_{32} & u_z^0 a_{33} \\ u_z^0 a_{32} & 0 & 0 \\ u_z^0 a_{33} & 0 & 0 \end{pmatrix} \tag{55}$$

and the relation (54) becomes

$$\begin{pmatrix} a_{11} - 2\lambda_1 u_z^0 a_{31} & a_{12} - \lambda_1 u_z^0 a_{32} & a_{13} - \lambda_1 u_z^0 a_{33} \\ a_{21} - \lambda_1 u_z^0 a_{32} & a_{22} & a_{23} \\ a_{31} - \lambda_1 u_z^0 a_{33} & a_{32} & a_{33} \end{pmatrix} = 0. \tag{56}$$

First we get $a_{22} = a_{23} = a_{33} = 0$. The third row gives $a_{31} = 0$, from the second row it follows $a_{21} = 0$ and the first row is giving $a_{11} = 0$. Thus $\mathbf{A} = 0$ and $\tau^0 - \mu 2\mathbf{S}^0 = 0$ - that means the basic flow (54) is Newtonian.

When $W_1 - W_2 \approx 0.408$, the growth constant formula (48) gives us a strong destabilization effect, compared with (Saffman & Taylor, 1958) - see **Remark 3** and Figure 2. Our dispersion curves are similar with the numerical results of (Wilson, 1990). We conclude that the instability for $W_i = O(1)$ is a property specific to the used flow pattern - in our case a Hele-Shaw displacement.

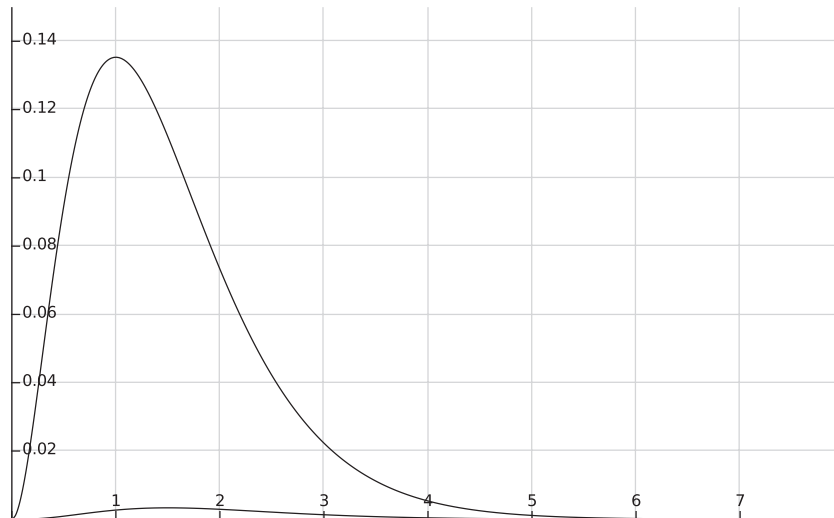


Figure 1. $\epsilon n^3 \exp(-n\alpha)$ (lower) and $n^2 \exp(-n\alpha)$ (upper) for $\alpha = 2, \epsilon = 0.02$.

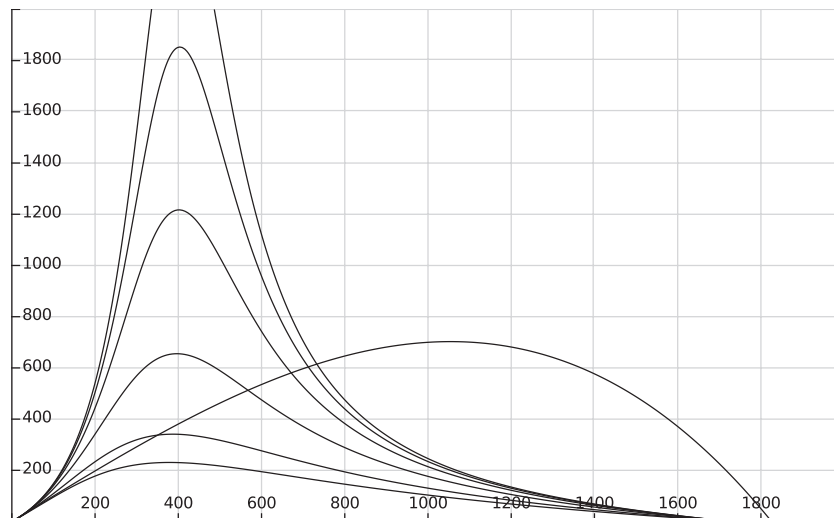


Figure 2. Plot of (48) compared with (49). $\epsilon = 0.006, \gamma = 0.1, W_2 = 0, W_1 = 0.1$ (lower), 0.2, 0.3, 0.35, 0.37, 0.38 (upper).

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