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Optimal Control of Evolution Macro-Hybrid Mixed Variational Inclusions

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Abstract

Optimal control problems governed by primal and dual evolution macro-hybrid mixed variational state inclusions, in reflexive Banach spaces, are studied. This is a spatially localized macro-hybrid variational version of our mixed optimal control theory published in Appl. Math. Optim. 68, 2013, 445-473, where the solvability analysis of the state systems is given in terms of duality principles, and the mixed optimality analysis is performed via a perturbation conjugate duality method. Applications to nonlinear constrained problems from mechanics exemplify the theory.

Keywords: constrained optimal control, evolution macro-hybrid mixed variational inclusion, duality principle, composition duality method, perturbation conjugate duality method, set-valued variational analysis

Mathematics Subject Classification: 2010. 35K90, 49J20, 49J40, 58E30.

1. Introduction

The aim of this paper is to study optimal control problems governed by primal and dual evolution macro-hybrid mixed variational state inclusions, in functional frameworks of reflexive Banach spaces. In a mechanical sense, primal and dual evolution mixed variational inclusions are composed, generally, of primal-conservation and dual-constitutive interior field equations, with boundary conditions and constraints incorporated variationally as primal and dual components, respectively, via compositional dualizations.

Our interest is to determine optimal controls of macro-hybridized evolution mixed maximal monotone variational inclusions, applying our mixed optimal control theory established and proposed in (Alduncin, G., 2013), where the solvability analysis of the governing state systems is given in terms of duality principles, and the mixed optimality analysis is performed via a perturbation conjugate duality method. Macro-hibridization of variational inclusions corresponds to variational spatial localizations of global systems into families of subsystems, via nonoverlapping multidomain decompositions, synchronized in terms of dual transmission subdifferential constraint equations (Alduncin, G., 2007; Alduncin, G., 2011). In this manner we shall accomplish a macro-hybrid variational version of the previous mixed optimal control theory (Alduncin, G., 2013).

It is important to stress that macro-hybrid local primal and dual evolution mixed variational reformulations are fundamental in the treatment of big spatial mechanical systems with multi-space-time-scale physical response, highly nonhomogeneous and anisotropic material distribution, multi-constitutivity, as well as for parallel computing and multi-algorithmic resolution schemes. Moreover such macro-hybrid variational structures, significantly, permit numerical implementations in terms of variational internal non-conforming mixed finite element discretizations, with natural internal boundary parallel dual synchonizations.

Applications to nonlinear constrained problems from mechanics, of diffusion and quasistatic elastoviscoplastic deformation phenomena, illustrate the theory. These mechanical state systems correspond, precisely, to macro-hybrid versions of those mixed state systems worked out in (Alduncin, G., 2013) as representative examples.

The paper is organized as follows. Section 2, following (Alduncin, G., 2007, Alduncin, G., 2011), presents the primal and dual evolution macro-hybrid mixed functional frameworks for the theory, of reflexive Banach spaces, defining the admissibility subspaces of internal boundary, interface continuity transmission constraints, and stating the fundamental macro-hybrid composition duality result. The optimal control variational problem, as well as the primal and dual governing macro-hybrid mixed inclusion state systems, are given and analyzed in Section 3, where the state solvability is determined in terms of variational duality principles. Next, in Section 4, the necessary and sufficient macro-hybrid mixed optimality conditions of the present theory are established, applying the perturbation conjugate duality approach of the mixed optimal control theory proposed in (Alduncin, G., 2013). The applications of the macro-hybrid theory, to nonlinear multi-valued mechanical systems, are finally worked out in Section 5.

2. Preliminaries

For the stationary state functional framework of the theory, let $V(\Omega)$ and $Y^*(\Omega)$ be primal and dual mixed Ω-field reflexive Banach spaces with topological duals $V^*(\Omega)$ and $Y(\Omega)$, related to a spatial bounded domain $\Omega \subset \Re^d$, $d \in \{1,2,3\}$. Also, let $H(\Omega)$ and $Z^*(\Omega)$ be the corresponding primal and dual Hilbert pivot spaces; i.e., $V(\Omega) \subset H(\Omega) \subset V^*(\Omega)$ and $Y^*(\Omega) \subset Z^*(\Omega) \subset Y(\Omega)$, continuously and densely embedded. On the other hand, related to the boundary of the domain, $\partial\Omega$, assumed to be Lipschitz continuous, let $B(\partial\Omega)$ be the corresponding reflexive Banach primal boundary space with topological dual $B^*(\partial\Omega)$.

Following our studies (Alduncin, G., 2007; Alduncin, G., 2011) on evolution mixed variational inclusions, for the macro-hybrid functional frameworks of the optimal control problems to be treated, the spatial bounded domain Ω of the systems is decomposed in terms of disjoint and connected subdomains $\{\Omega_e\}$, $\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e$, with internal boundaries $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1, 2, ..., E$, and interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \le e < k \le E$, both assumed Lipschitz continuous. Accordingly, the primal and dual Ω-field stationary spaces $V(\Omega)$ and $Y^*(\Omega)$ are considered to be decomposable in the sense

$$\begin{aligned} V(\Omega) &= \{\{v_e\} \in \boldsymbol{V}_{\{\Omega_e\}} \equiv \textstyle\prod_{e=1}^{E} V(\Omega_e) : \{\pi_{\Gamma_e} v_e\} \in \boldsymbol{Q}\}, \\ Y^*(\Omega) &= \boldsymbol{Y}^*_{\{\Omega_e\}} \equiv \textstyle\prod_{e=1}^{E} Y^*(\Omega_e), \end{aligned} \tag{1}$$

as well as the Hilbert pivot spaces to be such that $H(\Omega) = \boldsymbol{H}_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} H(\Omega_e)$, and similarly $Z^*(\Omega) = \boldsymbol{Z}^*_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} Z^*(\Omega_e)$. Here $\boldsymbol{Q} \subset \boldsymbol{B}_{\{\Gamma_e\}} \equiv \prod_{e=1}^{E} B(\Gamma_e)$ is the primal admissibility subspace of internal boundary interface continuity transmission, characterized by

$$\begin{aligned} \boldsymbol{Q} = \{\{\pi_{s_{\Gamma_e}} v_e\} \in \boldsymbol{B}_{\{\Gamma_e\}} :& \\ &\langle \{\delta^*_{\Gamma_e} y^*_e\}, \{\pi_{\Gamma_e} v_e\} \rangle_{B_{\{\Gamma_e\}}} = 0,\ \forall \{y^*_e\} \in \boldsymbol{Y}^*_{\{\Omega_e\}}, \{v_e\} \in \boldsymbol{V}_{\{\Omega_e\}}\}, \end{aligned} \tag{2}$$

where $[\pi_{\Gamma_e}]$ is the continuous linear internal boundary primal trace operator of the product space $\boldsymbol{V}_{\{\Omega_e\}}$ with values in $\boldsymbol{B}_{\{\Gamma_e\}}$, and $[\delta^*_{\Gamma_e}]$ is the continuous linear internal boundary dual trace operator of the product space $\boldsymbol{Y}^*_{\{\Omega_e\}}$ into $\boldsymbol{B}^*_{\{\Gamma_e\}} \equiv \prod_{e=1}^{E} B^*(\Gamma_e)$. Such local internal boundary trace operators are assumed to satisfy the compatibility conditions (Girault, V. & Raviart, P.-A., 1986).

$$(\boldsymbol{C}_{[\pi_{\Gamma_e}]}) \quad [\pi_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{V}_{\{\Omega_e\}}, \boldsymbol{B}_{\{\Gamma_e\}}) \text{ is surjective,}$$

$$(\boldsymbol{C}_{[\delta^*_{\Gamma_e}]}) \quad [\delta^*_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{Y}^*_{\{\Omega_e\}}, \boldsymbol{B}^*_{\{\Gamma_e\}}) \text{ is surjective,}$$

which are fundamental for compositional dualization in a macro-hybrid variational sense.

Notice that the primal admissibility subspace $\boldsymbol{Q}$ is in fact the polar subspace (orthogonal under duality) to the dual admissibility subspace $\boldsymbol{Q}^* \subset \boldsymbol{B}^*_{\{\Gamma_e\}}$ of internal boundary interface continuity transmission, given by

$$\begin{aligned} \boldsymbol{Q}^* = \{\{\delta^*_{\Gamma_e} y^*_e\} \in \boldsymbol{B}^*_{\{\Gamma_e\}} :& \\ &\langle \{\delta^*_{\Gamma_e} y^*_e\}, \{\pi_{\Gamma_e} v_e\} \rangle_{B_{\{\Gamma_e\}}} = 0,\ \forall \{y^*_e\} \in \boldsymbol{Y}^*_{\{\Omega_e\}}, \{v_e\} \in \boldsymbol{V}_{\{\Omega_e\}}\}. \end{aligned} \tag{3}$$

Moreover the following $[\pi_{\Gamma_e}]$-compositional dualization result is valid (Alduncin, G., 2007).

Lemma 1 *Under primal compatibility condition* $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$*, macro-hybrid compositional dualization, for* $\{u_e\} \in \boldsymbol{V}_{\{\Omega_e\}}$ *and* $\{\lambda^*_e\} \in \boldsymbol{B}^*_{\{\Gamma_e\}}$*,*

$$\{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I_{Q^*}}(\{\lambda^*_e\}) \iff \{\pi^T_{\Gamma_e} \lambda^*_e\} \in \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) \tag{4}$$

holds true, where I_Q *and* I_{Q^*} *are indicator* $\boldsymbol{Q}$*- and* $\boldsymbol{Q}^*$*-subspace functionals, conjugate to each other.*

Proof. From the fact that the conjugate indicator functional $(I_Q)^* = I_{Q^*}$, by convex dualization $\{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I_{Q^*}}(\{\lambda^*_e\}) \Leftrightarrow \{\lambda^*_e\} \in \boldsymbol{\partial I_Q}(\{\pi_{\Gamma_e} u_e\})$. Thus, under condition $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$, equivalence (4) follows as the equivalence of the variational inequalities of primal inclusions $\{\lambda^*_e\} \in \boldsymbol{\partial I_Q}(\{\pi_{\Gamma_e} u_e\})$ and $\{\pi^T_{\Gamma_e} \lambda^*_e\} \in \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\})$. □

Regarding the primal evolution state system of the theory, related to a given time interval $(0, T)$ with $T > 0$ arbitrary and fixed, the general evolution macro-hybrid mixed variational framework is defined by the primal and dual evolution reflexive Banach spaces $\mathcal{V}_{MH} = L^p(0, T; V_{\{\Omega_e\}}) \equiv \{\{v_e\} : [0, T] \to V_{\{\Omega_e\}} |\ \|\{v_e\}\|_{\mathcal{V}_{\{\Omega_e\}}} = [\int_0^T \|\{v_e\}(t)\|^p_{V_{\{\Omega_e\}}} dt]^{1/p} < \infty\}$, $2 \leq p < \infty$, and $\mathcal{Y}^*_{MH} = L^{q^*}(0, T; Y^*_{\{\Omega_e\}}) \equiv \{\{y^*_e\} : [0, T] \to Y^*_{\{\Omega_e\}} |\ \|\{y^*_e\}\|_{\mathcal{Y}^*_{\{\Omega_e\}}} = [\int_0^T \|\{y^*_e\}(t)\|^{q^*}_{Y^*_{\{\Omega_e\}}} dt]^{1/q^*} < \infty\}$, $q^* = p/(p-1)$, with topological duals $\mathcal{V}^*_{MH} = L^{q^*}(0, T; V^*_{\{\Omega_e\}})$ and $\mathcal{Y}_{MH} = L^p(0, T; Y_{\{\Omega_e\}})$. The primal state solution space is defined by $\mathcal{W}_{MH} = \{\{v_e\} : \{v_e\} \in \mathcal{V}_{MH}, \{dv_e/dt\} \in \mathcal{V}^*_{MH}\}$ with the operator norm $\|\{v_e\}\|_{\mathcal{W}_{MH}} = \|\{v_e\}\|_{\mathcal{V}_{MH}} + \|\{dv_e/dt\}\|_{\mathcal{V}^*_{MH}}$, continuous and densely embedded in the space $C([0, T]; H_{\{\Omega_e\}})$ of $H_{\{\Omega_e\}}$-continuos functions, and with initial values set $\{\{v_e\}(0) : \{v_e\} \in \mathcal{W}_{MH}\} = H_{\{\Omega_e\}}$ (cf. (Lions, J. L., 1969)). Further the corresponding primal evolution boundary space is defined by the reflexive Banach space $\mathcal{B}_{MH} = L^p(0, T; B_{\{\partial\Omega_e\}})$ with dual $\mathcal{B}^*_{MH} = L^{q^*}(0, T; B^*_{\{\partial\Omega_e\}})$. On the other hand, for the dual evolution macro-hybrid governing state system, the solution dual space is given by $\mathcal{X}^*_{MH} = \{\{q^*_e\} : \{q^*_e\} \in \mathcal{Y}^*_{MH}, \{dq^*_e/dt\} \in \mathcal{Y}_{MH}\}$ with the norm $\|\{q^*_e\}\|_{\mathcal{X}^*_{MH}} = \|\{q^*_e\}\|_{\mathcal{Y}^*_{MH}} + \|\{dq^*_e/dt\}\|_{\mathcal{Y}_{MH}}$, similarly continuous and densely embedded in the space $C([0, T]; Z^*_{\{\Omega_e\}})$ of $Z^*_{\{\Omega_e\}}$-continuos functions, and with initial values set $\{\{q^*_e\}(0) : \{q^*_e\} \in \mathcal{X}^*_{MH}\} = Z^*_{\{\Omega_e\}}$.

3. Optimal Control of Evolution Variational Problems

In this section, we begin with the optimal control problem of the theory, governed by primal and dual evolution macro-hybrid mixed variational state inclusions. This will be a macro-hybrid version of our recent mixed optimal control study (Alduncin, G., 2013), whose optimality analysis will be treated in the subsequent section.

Taking into account the macro-hybrid functional frameworks stated in the Preliminaries, for the optimal control minimization problem, let $C_{\mathcal{MH}} = L^2(0, T; U_{\{\Omega_e\}})$ denote the evolution space of $U_{\{\Omega_e\}} \equiv \prod_{e=1}^E U(\Omega_e)$-controls, a reflexive Banach space, and $C_{ad_{MH}}$ be a closed convex subset of admissible controls. Then the optimization problem of the theory reads as follows,

$$(O_{MH}) \begin{cases} \text{Find } \{\kappa_e\} \in C_{ad_{MH}} \subset C_{\mathcal{MH}} : \\ J_{MH}((\{u_{\kappa_e}\}, \{p^*_{\kappa_e}\}, \{\lambda^*_{\kappa_e}\}), \{\kappa_e\}) \\ \qquad \leq J_{MH}((\{u_{\kappa_e}\}, \{p^*_{\kappa_e}\}\{\lambda^*_{\kappa_e}\}), \{\eta_e\}), \ \forall \{\eta_e\} \in C_{\mathcal{MH}}, \end{cases}$$

where $(\{u_{\kappa_e}\}, \{p^*_{\kappa_e}\}, \{\lambda^*_{\kappa_e}\}) \in \mathcal{W}_{MH} \times \mathcal{Y}^*_{MH} \times \mathcal{B}^*_{MH}$ or $(\{u_{\kappa_e}\}, \{p^*_{\kappa_e}\}, \{\lambda^*_{\kappa_e}\}) \in \mathcal{V}_{MH} \times \mathcal{X}^*_{MH} \times \mathcal{B}^*_{MH}$ are corresponding κ-optimal macro-hybrid mixed states of the primal or dual governing evolution macro-hybrid mixed systems, to be denoted by $(\mathcal{MH}_\kappa)$ and $(\mathcal{MH}^*_\kappa)$, respectively. A general cost or objective functional $J_{MH} : (\mathcal{V}_{MH} \times \mathcal{Y}^*_{MH} \times \mathcal{B}^*_{MH}) \times C_{\mathcal{MH}} \to \Re \cup \{+\infty\}$ will be of the macro-hybrid mixed form

$$J_{MH}((\{v_{\kappa_e}\}, \{y^*_{\kappa_e}\}, \{v^*_{\kappa_e}\}), \{\eta_e\}) = \int_0^T (g_1(\{v_{\kappa_e}\}) + g_2(\{y^*_{\kappa_e}\}) + g_3(\{v^*_{\kappa_e}\}) + j(\{\eta_e\})\, dt, \tag{5}$$

whose integrand functional components must satisfy appropriate variational properties.

3.1 Primal Evolution Macro-Hybrid Mixed State System

Following our previous study on evolution mixed inclusions with optimal control, (Alduncin, G., 2012), we consider the abstract primal evolution macro-hybrid mixed governing state system of optimal control problem (O_{MH}) (cf. (Alduncin, G., 2007; Alduncin, G., 2011)).

$$(\boldsymbol{\mathcal{MH}_\kappa})\left\{\begin{array}{ll}\text{Given } \{\widetilde{f_e^*}\} \in \boldsymbol{\mathcal{V}^*_{MH}}, \{g_e\} \in \{\boldsymbol{L^p}(0,T;\mathcal{R}(\Lambda_e))\}, \{u_{e_0}\} \in \boldsymbol{H_{\{\Omega_e\}}}, & \\ \text{find } \{u_e\} \in \boldsymbol{\mathcal{W}_{MH}} \text{ and } \{p_e^*\} \in \boldsymbol{\mathcal{Y}^*_{MH}}: & \\ \{-\Lambda_e^T p_e^*\} - \{\pi_{\Gamma_e}^T \lambda_e^*\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\widetilde{F}_e(u_e)\} + \{\partial(\Psi_{e_C} \circ \pi_{e_C})(u_e)\} & \\ \qquad +\{B_e^*\kappa_e\} - \{\widetilde{f_e^*}\}, & \text{in } \boldsymbol{\mathcal{V}^*_{MH}}, \\ \{\Lambda_e u_e\} \in \{\partial G_e^*(p_e^*)\} + \{g_e\}, & \text{in } \boldsymbol{\mathcal{Y}_{MH}}, \\ \{u_e(0)\} = \{u_{e_0}\}; & \\ \text{and } \{\lambda_e^*\} \in \boldsymbol{\mathcal{B}^*_{MH}} \text{ satisfying the dual synchronizing condition} & \\ \{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}), \ \text{in } \boldsymbol{\mathcal{B}_{MH}}, & \end{array}\right.$$

with $[B_e^*] \in \boldsymbol{\mathcal{L}}(\boldsymbol{C_{MH}}, \boldsymbol{\mathcal{V}^*_{MH}})$ the macro-hybrid coupling optimal control linear continuous operator.

In this macro-hybrid mixed model $[\Lambda_e] \in \boldsymbol{\mathcal{L}}(\boldsymbol{\mathcal{V}_{MH}}, \boldsymbol{\mathcal{Y}_{MH}})$ is the linear continuous coupling operator of the system, with range subspace $\{\mathcal{R}(\Lambda_e)\}$, and the primal subdifferential and right hand side term are given by

$$\begin{aligned}[][\partial\widetilde{F}_e] = [\partial F_e] + \boldsymbol{\partial}(\{I_{\{\widehat{u}_e\}} \circ [\pi_{e_P}]) : \boldsymbol{\mathcal{V}_{MH}} \to \boldsymbol{2}^{\mathcal{V}^*_{MH}}, \\ -\{\widetilde{f_e^*}\} = \{\pi_{e_D}^T \widehat{p}_e^*\} - \{f_e^*\} \in \boldsymbol{\mathcal{V}^*_{MH}},\end{aligned} \tag{6}$$

where $[\partial F_e] : \boldsymbol{\mathcal{V}_{MH}} \to \boldsymbol{2}^{\mathcal{V}^*_{MH}}$ stands for the macro-hybrid primal evolution operator, and $[\partial G_e^*] : \boldsymbol{\mathcal{Y}^*_{MH}} \to \boldsymbol{2}^{\mathcal{Y}_{MH}}$ models the dualized macro-hybrid evolution distributed primal constraint of the problem, both being maximal monotone subdifferentials. Also, considering a disjoint boundary decomposition $\partial\Omega = \partial\Omega_C \cup \partial\Omega_P \cup \partial\Omega_D$, $[\partial\Psi_{e_C}] : \boldsymbol{\mathcal{B}_{MH_C}} \to \boldsymbol{2}^{\mathcal{B}^*_{MH_C}}$ is a maximal monotone subdifferential that models the primal boundary constraints imposed to the system on the local external sub-boundary $\{\partial\Omega_{e_C}\} \subset \partial\Omega_C$. Further $\{\widehat{u}_e\} \in \boldsymbol{\mathcal{B}_{MH_P}}$ and $\{\widehat{p}_e^*\} \in \boldsymbol{\mathcal{B}^*_{MH_D}}$ correspond to the primal and dual boundary values prescribed on the disjoint external sub-boundaries $\{\partial\Omega_{e_P}\} \subset \partial\Omega\backslash\partial\Omega_C$ and $\{\partial\Omega_{e_D}\} \subset \partial\Omega\backslash\partial\Omega_C\backslash\partial\Omega_P$, respectively. Hence, in a mechanical sense, the $\boldsymbol{\mathcal{V}^*_{MH}}$-primal variational equation of problem $(\boldsymbol{\mathcal{MH}_\kappa})$ may correspond to the localized constitutive or balance equation of the system, with localized boundary conditions incorporated variationally (after the application of its macro-hybrid divergence formula), and the $\boldsymbol{\mathcal{Y}_{MH}}$-dual subdifferential equation to a localized variational model of the imposed interior or distributed constraints, dualized for computational purposes.

On the other hand, dual $\boldsymbol{\mathcal{B}_{MH}}$-subdifferential equation of transmission imposes the internal boundary interface continuity constraint $\{\lambda_e^*\} \in \boldsymbol{Q^*}$, of the spatial decomposition, via the indicator functional I_{Q^*} in accordance with Lemma 1. Notice that by convex dualization the primal transmission equation is $\{\lambda_e^*\} \in \boldsymbol{\partial I_Q}(\{\pi_{\Gamma_e} u_e\})$, in $\boldsymbol{\mathcal{B}^*_{MH}}$, for $\{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\mathcal{B}_{MH}}$.

Then, proceeding as in (Alduncin, G., 2012) (cf. (Alduncin, G., 2007)), we introduce the classical primal evolution macro-hybrid compatibility condition

$$(\boldsymbol{C_{\{G_e\},[\Lambda_e]}}) \quad \text{int}\boldsymbol{\mathcal{D}}(\{G_e\}) \cap \boldsymbol{\mathcal{R}}([\Lambda_e]) \neq \boldsymbol{\emptyset},$$

under which the composition duality relation

$$[\Lambda_e^T \partial G_e \circ \Lambda_e] = [\partial(G_e \circ \Lambda_e)] \tag{7}$$

holds true (Ekeland, I. & Temam, R., 1974). Thereby, the primal evolution macro-hybrid mixed duality principle for the state system can be concluded, whose necessity is readily implied by compositional dualization (7) and Lemma 1.

Theorem 2 *Under compatibility conditions* $(\boldsymbol{C_{\{G_e\},[\Lambda_e]}})$ *and* $(\boldsymbol{C_{[\pi_{\Gamma_e}]}})$*, primal evolution macro-hybrid mixed state problem* $(\boldsymbol{\mathcal{MH}_\kappa})$ *is solvable if, and only if, the primal evolution macro-hybridized state problem*

$$(\mathcal{P}_{MH_\kappa})\begin{cases} \text{Find } \{u_e\} \in \mathcal{W}_{MH} : \\ \{0_e\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\widetilde{F}_e(u_e)\} + \{\partial(G_e \circ \Lambda_e)(u_e - v_{g_e})\} \\ \qquad +\{\partial(\Psi_{e_C} \circ \pi_{e_C})(u_e)\} + \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) + \{B_e^*\kappa_e\} \\ \qquad -\{\widetilde{f}_e^*\}, \qquad\qquad \text{in } \mathcal{V}^*_{MH}, \\ \{u_e(0)\} = \{u_{e_0}\}, \end{cases}$$

is solvable, where $\{v_{g_e}\} \in \mathcal{V}_{MH}$ *is a fixed* $[\Lambda_e]$*-preimage of function* $\{g_e\}$.

Proof. For the sufficiency, upon the application of Lemma 1, let $(\{u_e\}, \{p_e^*\}) \in \mathcal{W}_{MH} \times \mathcal{Y}^*_{MH}$ be a solution of the corresponding macro-hybridized state problem

$$(\mathcal{M}_{\kappa_{MH}})\begin{cases} \text{Find } \{u_e\} \in \mathcal{W}_{MH} \text{ and } \{p_e^*\} \in \mathcal{Y}^*_{MH} : \\ \{-\Lambda_e^T p_e^*\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\widetilde{F}_e(u_e)\} + \{\partial(\Psi_{e_C} \circ \pi_{e_C})(u_e)\} \\ \qquad +\boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) + \{B_e^*\kappa_e\} - \{\widetilde{f}_e^*\}, \quad \text{in } \mathcal{V}^*_{MH}, \\ \{\Lambda_e u_e\} \in \{\partial G_e^*(p_e^*)\} + \{g_e\}, \qquad \text{in } \mathcal{Y}_{MH}, \\ \{u_e(0)\} = \{u_{e_0}\}. \end{cases}$$

Then there is a functional $\{w_e^*\} \in \{\partial\widetilde{F}_e(u_e)\} + \{\partial(\Psi_{e_C} \circ \pi_{e_C})(u_e)\} \subset \mathcal{V}^*_{MH}$ such that $-\{w_e^*\} - \{\Lambda_e^T p_e^*\} - \{du_e/dt\} - \{B_e^*\kappa_e\} + \{f_e^*\} \in \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\})$. From the variational inequality of this primal inclusion, taking variations $\{v_e\} = \pm\{v_{e_0}\} + \{u_e\}$, with $\{v_{e_0}\}$ in the kernel $\mathcal{N}([\pi_{\Gamma_e}]) \subset \mathcal{V}_{MH}$, it follows that $-\{w_e^*\} - \{\Lambda_e^T p_e^*\} - \{du_e/dt\} - \{B_e^*\kappa_e\} + \{f_e^*\}$ belongs to the polar subspace $\mathcal{N}([\pi_{\Gamma_e}])^\circ \subset \mathcal{V}^*_{MH}$. Hence, by condition $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$, the Closed Range Theorem states that $\mathcal{N}([\pi_{\Gamma_e}])^\circ = \mathcal{R}([\pi_{\Gamma_e}]^T)$ and consequently there is a $[\pi_{\Gamma_e}]^T$-preimage $\{\lambda_e^*\} \in \mathcal{B}^*_{MH}$ such that $\{w_e^*\} = -\{\Lambda_e^T p_e^*\} - \{\pi_{\Gamma_e}^T \lambda^*\} - \{du_e/dt\} - \{B_e^*\kappa_e\} + \{f_e^*\}$. That is, applying Lemma 1, $(\{u_e\}, \{p_e^*\}, \{\lambda_e^*\})$ conforms to a solution of state problem $(\mathcal{MH}_\kappa)$. □

Furthermore, we may express primal state problem $(\mathcal{P}_{MH_\kappa})$ in a classical maximal monotone macro-hybridized sense, introducing the local primal composition superpotentials, for $e = 1, ..., E$,

$$\widetilde{G}_{e_g}(v) = G_e \circ \Lambda_e(v - v_{e_g}), \ v \in V(\Omega_e), \tag{8}$$

with effective domains and subdifferentials $\mathcal{D}(\widetilde{G}_{e_g}) = \mathcal{D}(G_e \circ \Lambda_e) + v_{e_g}$ and $\partial\widetilde{G}_{e_g} = \partial(G_e \circ \Lambda_e)(\cdot - v_{e_g})$. That is, assuming the localized qualifying Moreau-Rockafellar-Robinson conditions (Ernst, E. & Théra, M., 2009),

$$(\boldsymbol{C}_{\widetilde{F}_e, \widetilde{G}_{e_g}, \Psi_{e_C} \circ \pi_{e_C}}) \quad \begin{cases} \text{int}\mathcal{D}(\widetilde{F}_e) \cap \mathcal{D}(\widetilde{G}_{e_g}) \neq \emptyset, \\ \text{int}\mathcal{D}(\widetilde{F}_e + \widetilde{G}_{e_g}) \cap \mathcal{D}(\Psi_{e_C} \circ \pi_{e_C}) \neq \emptyset, \end{cases}$$

for the validity of the primal subdifferential sums

$$\partial\widetilde{\varphi}_e \equiv \partial(\widetilde{F}_e + \widetilde{G}_{e_g} + \Psi_{e_C} \circ \pi_{e_C}) = \partial\widetilde{F}_e + \partial\widetilde{G}_{e_g} + \partial(\Psi_{e_C} \circ \pi_{e_C}), \tag{9}$$

macro-hybridized primal state problem $(\mathcal{P}_{MH_\kappa})$ is given by

$$(\widetilde{\mathcal{P}}_{MH_\kappa})\begin{cases} \text{Find } \{u_e\} \in \mathcal{W}_{MH} : \\ \{0_e\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\widetilde{\varphi}_e(u_e)\} + \boldsymbol{\partial}(I_Q \circ [\pi_{s_{\Gamma_e}}])(\{u_e\}) + \{B_e^*\kappa_e\} \\ \qquad -\{\widetilde{f}_e^*\}, \qquad\qquad \text{in } \mathcal{V}^*_{MH}, \\ \{u_e(0)\} = \{u_{e_0}\}. \end{cases}$$

Consequently, in accordance with Akagi and Ôtani existence result (Akagi, G. & Ôtani, M., 2004) (cf. (Alduncin, G., 2011)), we can conclude the following.

Theorem 3 *Let the local coercivity and boundedness conditions, for $e = 1, ..., E$,*

$$(\boldsymbol{C1}_{\varphi_e}) \quad \|v_e\|^p_{V(\Omega_e)} - C_1\|v_e\|^2_{H(\Omega_e)} - C_{e_2} \leq C_{e_3}\widetilde{\varphi}_e(v_e), \quad \forall v_e \in \mathcal{D}(\widetilde{\varphi}_e), 2 \leq p < \infty,$$

$$(\boldsymbol{C2}_{\varphi_e}) \quad \|v_e^*\|^{q^*}_{V^*(\Omega_e)} \leq \ell_e(\|v_e\|_{H(\Omega_e)})\{\widetilde{\varphi}_e(v_e) + 1\}, \quad \forall v_e^* \in \partial\widetilde{\varphi}_e(v_e), q^* = p/(p-1),$$

ℓ_e a non-decreasing real function, be satisfied. Then, under conditions ($\boldsymbol{C}_{\widetilde{F}_e,\widetilde{G}_{e_g},\ \Psi_{e_C}\circ\pi_{e_C}}$), equivalent primal evolution macro-hybridized state problems ($\widetilde{\boldsymbol{\mathcal{P}}}_{\boldsymbol{MH}_\kappa}$) and ($\boldsymbol{\mathcal{P}}_{\boldsymbol{MH}_\kappa}$) attain a unique solution.

Remark 4 As usual, the uniqueness of Theorem 3 follows from the monotonicity of the primal subdifferentials $\partial\varphi_e$, $e = 1, 2, ..., E$, utilizing the time integration by parts formula for $0 \leq s < \tau \leq T$, $\{u_e\}, \{v_e\} \in \boldsymbol{\mathcal{W}_{MH}}$,

$$\int_s^\tau \left\langle \left\{\frac{du_e}{dt}\right\}(t), \{v_e\}(t)\right\rangle_{V_{\{\Omega_e\}}} dt = (\{u_e\}(\tau), \{v_e\}(\tau))_{H(\Omega)} - (\{u_e\}(s), \{v_e\}(s))_{H_{\{\Omega_e\}}} - \int_s^\tau \left\langle \left\{\frac{dv_e}{dt}\right\}(t), \{u_e\ (t)\right\rangle_{V_{\{\Omega_e\}}} dt. \tag{10}$$

The existence result follows via Hilbert approximations defined by the classical, uniquely solvable, Cauchy problems (Brézis, H., 1973),

$$(\widetilde{\boldsymbol{\mathcal{P}}}_{\mathcal{H}_\kappa})_n \begin{cases} \text{Find } \{u_{e_n}\} \in \boldsymbol{\mathcal{U}_{MH}} : \\ \{0_e\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\widetilde{\varphi}_{H(\Omega_e)}(u_e)\} + \boldsymbol{\partial}(I_Q \circ [\pi_{s_{\Gamma_e}}])(\{u_e\}) + \{B_e^*\kappa_e\} \\ \qquad -\{\widetilde{f}^*_{e_n}\}, \qquad\qquad \text{in } \boldsymbol{\mathcal{H}_{MH}}, \\ \{u_{e_n}(0)\} = \{u_{e_{n0}}\}, \end{cases}$$

where $\boldsymbol{\mathcal{U}_{MH}} = \{\{v_e\} : \{v_e\} \in \boldsymbol{\mathcal{V}_{MH}}, \{dv_e/dt\} \in \boldsymbol{\mathcal{H}_{MH}} = \boldsymbol{L}^2(0, T; \boldsymbol{H}_{\{\Omega_e\}})\} \subset \boldsymbol{\mathcal{W}_{MH}}$, $\widetilde{\varphi}_{H(\Omega_e)} : H(\Omega_e) \to \Re \cup \{+\infty\}$ is such that $\widetilde{\varphi}_{H(\Omega_e)} = \widetilde{\varphi}_e$ in $V(\Omega_e)$ and $+\infty$ in $H(\Omega_e)\backslash V(\Omega_e)$, a proper lower semicontinuous functional whose $\partial\widetilde{\varphi}_{H(\Omega_e)} \subset \partial\widetilde{\varphi}_e$. Further, $\widetilde{f}^*_{e_n} \to \widetilde{f}^*_e$ in $\boldsymbol{\mathcal{V}_{MH}}$, and $u_{e_n} \to u_{e_{n0}}$ in $H(\Omega_e)$, are strongly convergent sequences as $n \to +\infty$. Then, under conditions ($\boldsymbol{C1}_{\varphi_e}$) and ($\boldsymbol{C2}_{\varphi_e}$), the sequence of problems $\{(\widetilde{\boldsymbol{\mathcal{P}}}_{\mathcal{H}_\kappa})_n\}$ converge weakly to problem ($\widetilde{\boldsymbol{\mathcal{P}}}_{\boldsymbol{MH}_\kappa}$), classical subdifferential version of primal evolution problem $\boldsymbol{\mathcal{P}}_{\boldsymbol{MH}_\kappa}$ (cf. (Akagi, G. & Ôtani, M., 2004), Theorem 3.2).

Therefore, the macro-hybrid mixed solvability result of the present primal theory is concluded from Theorems 2 and 3.

Theorem 5 *Let compatibility conditions ($\boldsymbol{C}_{\{G_e\}[\Lambda_e]}$) and ($\boldsymbol{C}_{[\pi_{\Gamma_e}]}$) be fulfilled. Then, under local conditions ($\boldsymbol{C}_{\widetilde{F}_e,\widetilde{G}_{e_g},\Psi_{e_C}\circ\pi_{e_C}}$), ($\boldsymbol{C1}_{\varphi_e}$) and ($\boldsymbol{C2}_{\ \varphi_e}$), $e = 1, ..., E$, primal evolution macro-hybrid mixed state problem ($\boldsymbol{\mathcal{MH}_\kappa}$) is solvable with a unique primal $\boldsymbol{\mathcal{W}_{MH}}$-component.*

3.2 Dual Evolution Macro-Hybrid Mixed State System

We next proceed with the dual version of the theory, considering as an abstract general dual evolution macro-hybrid mixed state system, governing optimal control problem ($\boldsymbol{\mathcal{O}_{MH}}$), the following (Alduncin, G., 2012),

$$(\mathcal{MH}^*_\kappa)\begin{cases} \text{Given } \{f_e^* \in L^p(0,T;\mathcal{R}(-\Lambda_e^T))\}, \{g_e\} \in \boldsymbol{\mathcal{Y}_{MH}}, \{p^*_{e_0}\} \in \mathbf{Z}^*_{\{\Omega_e\}}, \\ \text{find } \{u_e\} \in \boldsymbol{\mathcal{V}_{MH}} \text{ and } \{p_e^*\} \in \boldsymbol{\mathcal{X}^*_{MH}} : \\ \quad \{-\Lambda_e^T p_e^*\} - \{\pi^T_{\Gamma_e}\lambda_e^*\} \in \{\partial\widetilde{F}_e(u_e)\} + \{\partial(\Psi_{e_C}\circ\pi_{e_C})(u_e)\} \\ \qquad\quad -\{\widetilde{f}_e^*\}, \qquad\qquad\qquad\qquad\qquad \text{in } \boldsymbol{\mathcal{V}^*_{MH}}, \\ \{\Lambda_e u_e\} \in \left\{\dfrac{dp^*}{dt}\right\} + \{\partial G_e^*(p_e^*)\} + \{B_e\kappa_e\} + \{g_e\}, \qquad \text{in } \boldsymbol{\mathcal{Y}_{MH}}, \\ \{p_e^*(0)\} = \{p^*_{e_0}\}; \\ \text{and } \{\lambda_e^*\} \in \boldsymbol{\mathcal{B}^*_{MH}} \text{ satisfying the dual synchronizing condition} \\ \{\pi_{\Gamma_e}u_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}), \ \text{ in } \boldsymbol{\mathcal{B}_{MH}}, \end{cases}$$

where, in a dual sense, $[B_e] \in \boldsymbol{\mathcal{L}}(\boldsymbol{C_{MH}}, \boldsymbol{\mathcal{Y}_{MH}})$ is the macro-hybrid coupling optimal control linear continuous operator.

Here $\{\mathcal{R}(-\Lambda_e^T)\}$ denotes the range subspace of the linear continuous coupling transpose operator $[\Lambda_e^T] \in \boldsymbol{\mathcal{L}}(\boldsymbol{\mathcal{Y}^*}_{MH}, \boldsymbol{\mathcal{V}^*}_{MH})$ of the system, and primal definitions (6) are considered in force too. A mechanical interpretation of this dual case is that now $\boldsymbol{\mathcal{V}^*_{MH}}$-primal equation may correspond to localized variational models of imposed interior or distributed constraints, with incorporated local boundary conditions and constraints, and $\boldsymbol{\mathcal{Y}_{MH}}$-dual subdifferential equation to the localized constitutive or balance equation of the system. Further, dual $\boldsymbol{\mathcal{B}_{MH}}$-subdifferential equation is the transmission equation of the macro-hybrid spatial nonoverlapping decomposition, which enforces the internal boundary interface continuity constraint $\{\lambda_e^*\} \in \boldsymbol{Q^*}$ via the indicator functional I_{Q^*}, in accordance with Lemma 1. Recall that the primal transmission equation, $\{\lambda_e^*\} \in \boldsymbol{\partial I_Q}(\{\pi_{\Gamma_e}u_e\})$ in $\boldsymbol{\mathcal{B}^*_{MH}}$, is deduced by convex dualization, for $\{\pi_{\Gamma_e}u_e\} \in \boldsymbol{\mathcal{B}_{MH}}$.

For the analysis of the state problem, we introduce the dual evolution macro-hybrid compatibility condition

$$(\boldsymbol{C}_{(\{\widetilde{F}_{|V(\Omega_e)}+(\Psi_C\circ\pi_C)_{|V(\Omega_e)}\})^*,[-\Lambda_e^T]})$$
$$\operatorname{int}\mathcal{D}((\{\widetilde{F}_{|V(\Omega_e)} + (\Psi_C\circ\pi_C)_{|V(\Omega_e)}\})^*) \cap \boldsymbol{\mathcal{R}}([-\Lambda_e^T]) \neq \boldsymbol{\emptyset},$$

under which the composition duality relation

$$\begin{aligned} &\{-\Lambda_e\partial((\widetilde{F}_{|V(\Omega_e)} + (\Psi_C\circ\pi_C)_{|V(\Omega_e)})^*)\circ(-\Lambda_e^T)\} \\ &\qquad = \{\partial((\widetilde{F}_{|V(\Omega_e)} + (\Psi_C\circ\pi_C)_{|V(\Omega_e)})^*\circ(-\Lambda_e^T))\} \end{aligned} \tag{11}$$

is valid (Ekeland, I. & Temam, R., 1974). Then the dual evolution macro-hybrid mixed duality principle of state system $(\mathcal{MH}^*_\kappa)$ is similarly concluded as for the primal case, whose necessity follows by compositional dualization (11) and Lemma 1. The sufficiency of the principle is established via the same arguments as for Theorem 2.

Theorem 6 *Under compatibility conditions* $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$ *and* $(\boldsymbol{C}_{(\{\widetilde{F}_{|V(\Omega_e)}+(\Psi_C\circ\pi_C)_{|V(\Omega_e)}\})^*,[-\Lambda_e^T]})$ *dual evolution macro-hybrid mixed state problem* $(\mathcal{MH}^*_\kappa)$ *is solvable if, and only if, the dual evolution macro-hybridized state problem*

$$(\mathcal{D}_{MH_\kappa})\begin{cases} \text{Find } \{p_e^*\} \in \boldsymbol{\mathcal{X}^*_{MH}} : \\ \{0_e\} \in \left\{\dfrac{dp_e^*}{dt}\right\} + \{\partial G_e^*(p_e^*)\} + \{\partial((\widetilde{F}_{|V(\Omega_e)} + (\Psi_C\circ\pi_C)_{|V(\Omega_e)})^* \\ \qquad \circ(-\Lambda_e^T))\}(\{p_e^* + q^*_{\widetilde{f}_e^*}\}) + \{B_e\kappa_e\} + \{g_e\}, \qquad \text{in } \boldsymbol{\mathcal{Y}_{MH}}, \\ \{p_e^*(0)\} = \{p^*_{e_0}\}, \end{cases}$$

is solvable, where $\{q^*_{\widetilde{f}_e^*}\} \in \boldsymbol{\mathcal{Y}^*_{MH}}$ *is a fixed* $[-\Lambda_e^T]$*-preimage of function* $\{\widetilde{f}_e^*\}$.

Next, in order to apply Akagi and Ôtani existence Theorem (Akagi, G. & Ôtani, M., 2004), we express the dual problem as a classical subdifferential one. Thus, we introduce the local dual composition superpotentials, for $e = 1, ..., E$,

$$\widetilde{H}_{\widetilde{f}_e^*}(q^*) = (\widetilde{F}_{|V(\Omega_e)} + (\Psi_C\circ\pi_C)_{|V(\Omega_e)})^* \circ (-\Lambda_e^T)(q_e^* + q^*_{\widetilde{f}_e^*}), \ q_e^* \in Y^*(\Omega_e), \tag{12}$$

whose effective domains and subdifferentials are $\mathcal{D}(\widetilde{H}_{\widetilde{f}_e^*}) = \mathcal{D}(\widetilde{F}_{|V(\Omega_e)} + (\Psi_C \circ \pi_C)_{|V(\Omega_e)})^* \circ (-\Lambda_e^T)) - q^*_{\widetilde{f}_e^*}$ and $\partial\widetilde{H}_{\widetilde{f}_e^*}(q^*) = \partial((\widetilde{F}_{|V(\Omega_e)} + (\Psi_C \circ \pi_C)_{|V(\Omega_e)})^* \circ (-\Lambda_e^T))(q^* + q^*_{\widetilde{f}_e^*})$, and assume the localized Moreau-Rockafellar-Robinson conditions (Ernst, E. & Théra, M., 2009).

$$(\boldsymbol{C}_{G_e^*,\widetilde{H}_{\widetilde{f}_e^*}}) \quad \mathrm{int}\mathcal{D}(G_e^*) \cap \mathcal{D}(\widetilde{H}_{\widetilde{f}_e^*}) \neq \emptyset$$

that guarantee the dual subdifferential sum rules

$$\partial\widetilde{\varphi}_e^* \equiv \partial(G_e^* + \widetilde{H}_{e_{\widetilde{f}^*}}) = \partial G_e^* + \partial\widetilde{H}_{\widetilde{f}_e^*},\ e = 1, ..., E. \tag{13}$$

Then problem $(\boldsymbol{\mathcal{D}}_{\boldsymbol{MH}_\kappa})$ has the subdifferential form

$$(\widetilde{\boldsymbol{\mathcal{D}}}_{\boldsymbol{MH}_\kappa}) \begin{cases} \text{Find } \{p_e^*\} \in \boldsymbol{\mathcal{X}}^*_{MH} : \\ \{0_e\} \in \left\{\dfrac{dp_e^*}{dt}\right\} + \partial\{\widetilde{\varphi}_e^*(p_e^*)\} + \{B_e\kappa\} + \{g_e\}, \quad \text{in } \boldsymbol{\mathcal{Y}}_{\boldsymbol{MH}}, \\ \{p_e^*(0)\} = \{p_{e_0}^*\}, \end{cases}$$

and the following dual existence result can be concluded (cf. Remark 4).

Theorem 7 *Let the local coercivity and boundedness dual conditions, for* $e = 1, ..., E$,

$$(\boldsymbol{C1}_{\varphi_e^*}) \quad \|q_e^*\|^{q^*}_{Y^*(\Omega_e)} - C_{e_1}^*\|q_e^*\|^2_{Z(\Omega_e)} - C_{e_2}^* \leq C_3^*\,\widetilde{\varphi}_e^*(q_e^*), \quad \forall q_e^* \in \mathcal{D}(\widetilde{\varphi}_e^*), 2 \leq q^* < \infty,$$

$$(\boldsymbol{C2}_{\varphi_e^*}) \quad \|q_e^*\|^p_{Y(\Omega_e)} \leq \ell_e^*(\|q_e^*\|_{Z(\Omega_e)})\{\widetilde{\varphi}_e^*(q_e^*) + 1\}, \quad \forall q_e^* \in \partial\widetilde{\varphi}_e^*(q_e^*), p = q^*/(q^* - 1),$$

ℓ_e^* *a non-decreasing real function, be fulfilled. Then, under condition* $(\boldsymbol{C}_{G_e^*,\widetilde{H}_{\widetilde{f}_e^*}})$*, dual evolution equivalent state problems* $(\widetilde{\boldsymbol{\mathcal{D}}}_{\boldsymbol{MH}_\kappa})$ *and* $(\boldsymbol{\mathcal{D}}_{\boldsymbol{MH}_\kappa})$ *are uniquely solvable.*

Therefore, the macro-hybrid mixed existence result of the dual theory is achieved from Theorems 6 and 7.

Theorem 8 *Let compatibility conditions* $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$ *and* $(\boldsymbol{C}_{(\{\widetilde{F}_{|V(\Omega_e)}+\ (\Psi_C\circ\pi_C)_{|V(\Omega_e)}\})^*,[-\Lambda_e^T]})$ *be satisfied. Then under local conditions* $(\boldsymbol{C}_{G_e^*\widetilde{H}_{\widetilde{f}_e^*}})$, $(\boldsymbol{C1}_{\varphi_e^*})$ *and* $(\boldsymbol{C2}_{\varphi_e^*})$*, dual evolution macro-hybrid mixed state problem* $(\boldsymbol{\mathcal{MH}}^*_\kappa)$ *is solvable with a unique dual* $\boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}}$*-component.*

3.3 Evolution Macro-hybrid Mixed Optimal Control Problem

Now, we can state the solvability of the optimization problem $(\boldsymbol{O}_{\boldsymbol{MH}})$, governed by primal and dual evolution macro-hybrid mixed state systems $(\boldsymbol{\mathcal{MH}}_\kappa)$ and $(\boldsymbol{\mathcal{MH}}^*_\kappa)$, and conclude the corresponding optimal control solvability.

In accordance with Migórski's analysis (Migórski, S., 2001), we assume that the cost or objective functional of optimization problem $(\boldsymbol{O}_{\boldsymbol{MH}})$, with general macro-hybrid mixed form (5), satisfies the condition

$$(\boldsymbol{C}_{J_{MH}}) \quad \begin{cases} J_{MH} : (\boldsymbol{\mathcal{V}}_{\boldsymbol{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \boldsymbol{\mathcal{B}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C}_{\boldsymbol{MH}} \to \Re \cup \{+\infty\} \text{ is lower} \\ \text{semicontinuous, bounded below, and } \boldsymbol{C}_{\boldsymbol{MH}} - \text{convex.} \end{cases}$$

Then the optimization solvability of the control problem is concluded.

Theorem 9 *Under qualifying condition* $(\boldsymbol{C}_{\boldsymbol{J}_{MH}})$*, macro-hybrid optimization problem* $(\boldsymbol{O}_{\boldsymbol{MH}})$ *attains a solution.*

Thereby, in accordance with Theorems 2 and 3, as well as Theorems 6 and 7, the variational optimal control existence results of the macro-hybrid mixed theory are achieved.

Corollary 10 *Let conditions* $(\boldsymbol{C}_{\boldsymbol{J}_{MH}})$, $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$ *and* $(\boldsymbol{C}_{\{G_e\},\{\Lambda_e\}})$ *be fulfilled. Then under local conditions* $(\boldsymbol{C}_{\widetilde{F}_e,\widetilde{G}_{e_g},\Psi_{e_C}\circ\pi_{e_C}})$, $(\boldsymbol{C1}_{\varphi_e})$ *and* $(\boldsymbol{C2}_{\varphi_e})$*,* $e = 1, ..., E$*, there exists an optimal control pair* $((\{u_e\}, \{p_e^*\}, \{\lambda_e^*\}), \{\kappa_e\}) \in (\boldsymbol{\mathcal{W}}_{\boldsymbol{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \boldsymbol{\mathcal{B}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C}_{\boldsymbol{MH}}$ *to problem* $(\boldsymbol{O}_{\boldsymbol{MH}})$*-*$(\boldsymbol{\mathcal{MH}}_\kappa)$*,* and

Corollary 11 *Let conditions* $(\boldsymbol{C}_{\boldsymbol{J}_{MH}})$, $(\boldsymbol{C}_{[\pi_{\Gamma_e}]})$, $(\boldsymbol{C}_{((\{\widetilde{F}_{|V(\Omega_e)}+(\Psi_C\circ\pi_C)\ _{|V(\Omega_e)}\})^*,[-\Lambda_e^T])})$, $(\boldsymbol{C}_{G^*,\widetilde{H}_{\widetilde{f}^*}})$, $(\boldsymbol{C1}_{\varphi^*})$ *and* $(\boldsymbol{C2}_{\varphi^*})$ *be satisfied. Then there exists an optimal control pair* $((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in (\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}$ *to problem* $(\boldsymbol{O_{MH}})$-$(\boldsymbol{\mathcal{M}}^*_{\boldsymbol{\kappa}})$.

4. Optimality Condition Analysis

In this section, we elaborate on the primal and dual optimality conditions for the constrained evolution macro-hybrid mixed optimal control problem $(\boldsymbol{O_{MH}})$, based on our previous study (Alduncin, G., 2013), via perturbation conjugate duality. In this manner, we extend the mixed optimal control theory to a macro-hybrid version, theory that was motivated by the analysis of Azé and Bolintinéanu on constrained convex parabolic control problems (Azé, D. & Bolintinéanu, S., 2000). We refer to (Alduncin, G., 2013, Section 5), where proximation penalty-duality algorithms for variational mixed optimality conditions are derived and analyzed, issue that will not be pursued here.

4.1 Primal Optimality Conditions

Let us first treat optimal control problem $(\boldsymbol{O_{MH}})$ governed by the primal macro-hybrid mixed state problem $(\boldsymbol{\mathcal{MH}_\kappa})$, with associated state primal problem $(\boldsymbol{\mathcal{P}_{MH_\kappa}})$ (cf. (Alduncin, G., 2013), Section 4.1). From here on, we assume that the conditions of Theorems 2 and 3 hold true. Hence, as the macro-hybrid mixed state-control operator, $\boldsymbol{\mathcal{T}_{MH}} : (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}} \to \boldsymbol{\mathcal{V}}^*_{\boldsymbol{MH}}$, we have

$$\begin{aligned}\boldsymbol{\mathcal{T}_{MH}}&((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\})\\ &= \left\{\frac{dv_e}{dt}\right\} + \{v^*_{v_e}\} + \{\Lambda_e^T y_e^*\} + \{\pi_{\Gamma_e}^T v_e^*\} + \{B_e^*\eta_e\},\\ &\qquad \{v^*_{v_e}\} \in \{\partial(\widetilde{F}_e + \Psi_{e_C}\circ\pi_{e_C})(v_e)\},\end{aligned} \tag{14}$$

and as the closed convex constraint domain of the primal problem

$$\begin{aligned}\boldsymbol{\mathcal{M}_{MH}} &= \{((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) \in (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}} :\\ &\{v_e\} \in \textstyle\prod_{e=1}^E \mathcal{D}(\widetilde{\varphi}_e), \{y_e^*\} \in \prod_{e=1}^E \mathcal{D}(G_e^*), \{v_e^*\} \in \boldsymbol{Q}^*, \{\eta_e\} \in \boldsymbol{C_{ad_{MH}}}\}.\end{aligned} \tag{15}$$

Further, for the constraint qualification condition, it is assumed that there is a closed subspace $\boldsymbol{Q}^*_{\boldsymbol{MH}} \subset \boldsymbol{\mathcal{V}}^*_{\boldsymbol{MH}}$ such that

$$(\boldsymbol{C}_{\mathcal{T}_{MH}}) \quad \boldsymbol{Q}^*_{\boldsymbol{MH}} \subset \Re_+(\boldsymbol{\mathcal{T}_{MH}}(\boldsymbol{\mathcal{M}_{MH}}) - \{\widetilde{f}_e^*\}),$$

and the proper convex and lower semicontinuous perturbation functional, $\mathcal{S}_{MH} : ((\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q}^*_{\boldsymbol{MH}} \to \Re \cup \{+\infty\}$, is then defined by

$$\begin{aligned}&\mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{q_e^*\})\\ &\quad = J_{MH}((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) + I_{\mathcal{K}_{MH}}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{q_e^*\}),\end{aligned} \tag{16}$$

with $\boldsymbol{\mathcal{K}_{MH}} \subset ((\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q}^*_{\boldsymbol{MH}}$, the closed convex subset

$$\begin{aligned}\boldsymbol{\mathcal{K}_{MH}} &= \{(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{q_e^*\}) \in \boldsymbol{\mathcal{M}_{MH}} \times \boldsymbol{Q}^*_{\boldsymbol{MH}} :\\ &\{q_e^*\} \in \boldsymbol{\mathcal{T}_{MH}}(\boldsymbol{\mathcal{M}_{MH}}) - \{\widetilde{f}_e^*\}\}.\end{aligned} \tag{17}$$

Also, we introduce the marginal or infimal value convex functional $\mu^*_{MH} : \boldsymbol{Q}^*_{\boldsymbol{MH}} \to \Re \cup \{+\infty\}$,

$$\begin{aligned}\mu^*_{MH}(\{q_e^*\}) = &\inf_{((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) \in \mathcal{W}_{MH}\times\mathcal{Y}^*_{MH}\times\mathcal{B}^*_{MH})\times C_{MH}}\\ &\mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{q_e^*\}).\end{aligned} \tag{18}$$

Therefore, in accordance with the perturbation duality theory (Ekeland, I. & Temam, R., 1974; Rockafellar, R. T., 1974), we can now proceed to state the primal, dual and mixed optimization problems of the optimal control theory, in a macro-hybrid perturbation sense. Indeed, the macro-hybrid optimal control functional, $J_{MH} : (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}} \times \mathcal{B}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}} \to$

$\Re\cup\{+\infty\}$, is recovered through a zero perturbation; i.e., $J_{MH}((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) = \mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\{0_e\}_{Q^*_{MH}})$, and then the primal evolution macro-hybrid mixed optimal control problem in a perturbation sense is given by

$$(\boldsymbol{OC_{MH}})\begin{cases} \text{Find } ((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \boldsymbol{\mathcal{M}_{MH}}: \\ \mathcal{S}_{MH}(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{0_e\}_{Q^*_{MH}}) \\ \qquad \leq \mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\{0_e\}_{Q^*_{MH}}), \\ \forall((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \in (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_{MH}}) \times \boldsymbol{C_{MH}}. \end{cases}$$

Moreover, denoting the dual subspace of the constraint qualification condition closed subspace $\boldsymbol{Q}^*_{MH}$ by $\boldsymbol{Q}_{MH} \subset \boldsymbol{\mathcal{V}}_{MH}$, and the conjugate of perturbation functional $\boldsymbol{\mathcal{S}}_{MH}$ by $\boldsymbol{\mathcal{S}}^*_{MH} : ((\boldsymbol{\mathcal{W}}^*_{MH} \times \boldsymbol{\mathcal{Y}}_{MH} \times \boldsymbol{\mathcal{B}}_{MH}) \times \boldsymbol{C}^*_{MH}) \times \boldsymbol{Q}_{MH} \to \Re \cup \{+\infty\}$, the perturbed dual convex functional $\pi_{MH} : \boldsymbol{Q}_{MH} \to \Re \cup \{+\infty\}$ is defined by

$$\pi_{MH}(\{q_e\}) = \mathcal{S}^*_{MH}(((\{0_e^*\}_{\mathcal{V}^*_{\{\Omega_e\}}},\{0_e\}_{\mathcal{Y}_{\{\Omega_e\}}},\{0_e\}_{\mathcal{B}_{\{\Gamma_e\}}}),\{0_e^*\}_{C^*_{\{\Omega_e\}}}),\{q_e\}), \tag{19}$$

which, in fact, corresponds to the conjugate of the marginal functional μ^*_{MH}. Then, in a dual perturbation sense, the primal evolution macro-hybrid mixed optimal control problem is the following

$$(\boldsymbol{OC^*_{MH}})\begin{cases} \text{Find } \{p_e\} \in \mathcal{D}(\pi_{MH}): \\ -\mathcal{S}^*_{MH}(((\{0_e^*\}_{\mathcal{V}^*_{\{\Omega_e\}}},\{0_e\}_{\mathcal{Y}_{\{\Omega_e\}}},\{0_e\}_{\mathcal{B}_{\{\Gamma_e\}}}),\{0_e^*\}_{C^*_{\{\Omega_e\}}}),\{p_e\}) \\ \quad \geq -\mathcal{S}^*_{MH}(((\{0_e^*\}_{\mathcal{V}^*_{\{\Omega_e\}}},\{0_e\}_{\mathcal{Y}_{\{\Omega_e\}}},\{0_e\}_{\mathcal{B}_{\{\Gamma_e\}}}),\{0_e^*\}_{C^*_{\{\Omega_e\}}}),\{q_e\}), \\ \qquad\qquad \forall\{q_e\} \in \boldsymbol{Q_{MH}}. \end{cases}$$

Further, the corresponding convex-concave Lagrangian, $\mathcal{L} : ((\boldsymbol{\mathcal{W}}_{MH} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_{MH}}) \times \boldsymbol{C_{\mathcal{MH}}}) \times \boldsymbol{Q_{MH}} \to \Re \cup \{+\infty\}$, is defined by

$$\begin{aligned} &\mathcal{L}_{MH}(((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\{q_e\}) = -\mathcal{S}^*_{MH_{((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\})}}(\{q_e\}) \\ &= \begin{cases} J_{MH}((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) - \langle \boldsymbol{\mathcal{T}_{MH}}((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \\ \quad -\{\widetilde{f_e^*}\},\{q_e\}\rangle_{\mathcal{V}^*_{\{\Omega_e\}},\mathcal{V}_{\{\Omega_e\}}}, \text{ if } ((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \in \boldsymbol{\mathcal{D}_{MH}}, \\ +\infty, \qquad \text{if } ((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \notin \boldsymbol{\mathcal{D}_{MH}}, \end{cases} \\ &\boldsymbol{\mathcal{D}_{MH}} = \{((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \in \boldsymbol{\mathcal{M}_{MH}} : \boldsymbol{\mathcal{T}_{MH}}(\boldsymbol{\mathcal{M}_{MH}}) - \{\widetilde{f_e^*}\}\}. \end{aligned} \tag{20}$$

Notice that $\boldsymbol{\mathcal{D}_{MH}}$ is the projection of set $\boldsymbol{\mathcal{K}_{MH}}$ on $(\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_{MH}}) \times \boldsymbol{C_{MH}}$. Here, $\mathcal{S}^*_{MH_{((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\})}} : \boldsymbol{Q_{MH}} \to \Re \cup \{+\infty\}$ is the conjugate of $\mathcal{S}_{MH_{((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\})}} = \boldsymbol{\mathcal{S}}_{MH}\,(((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\cdot) : \boldsymbol{Q^*_{MH}} \to \Re \cup \{+\infty\}$, for $((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}) \in (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_{MH}}) \times \boldsymbol{C_{MH}}$. Thereby, in a mixed perturbation sense, the primal evolution macro-hybrid mixed optimal control problem turns out to be

$$(\boldsymbol{\mathcal{M}OC_{MH}})\begin{cases} \text{Find } (((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\}) \in \mathcal{D}(\mathcal{L}_{MH}): \\ \mathcal{L}_{MH}(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\}) \\ \qquad \leq \mathcal{L}_{MH}(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{q_e\}) \\ \qquad\qquad \leq \mathcal{L}_{MH}(((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\{q_e\}), \\ \forall\ (((\{v_e\},\{y_e^*\},\{\nu_e^*\}),\{\eta_e\}),\{q_e\}) \\ \qquad \in ((\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q_{MH}}. \end{cases}$$

Lastly, we shall require the classical duality principle of the perturbation duality theory (Rockafellar, R. T., 1974; Ekeland, I. & Temam, R., 1974),

Lemma 12 *In a macro-hybrid sense, the following propositions are equivalent to each other:*

(P1) $((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \mathcal{D}(J_{MH})$ *and* $\{p_e\} \in \mathcal{D}(\pi_{MH})$ *are respective solutions to primal and dual problems* $(\boldsymbol{OC_{MH}})$ *and* $(\boldsymbol{OC^*_{MH}})$*, such that* $inf(\boldsymbol{OC_{MH}}) = sup(\boldsymbol{OC^*_{MH}})$*;*

(P2) $(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\}) \in \mathcal{D}(\mathcal{L}_{MH})$ *is a solution to mixed problem* $(\boldsymbol{\mathcal{M}OC_{MH}})$*.*

Moreover, considering that the marginal domain $\mathcal{D}(\mu^*_{MH})$ is the projection of the perturbation domain $\mathcal{D}(\mathcal{S}_{MH})$ on $\boldsymbol{Q^*_{MH}}$, we shall require the result (Jeyakumar, V., 1990).

Lemma 13 *In a macro-hybrid sense, let the marginal domain* $\mathcal{D}(\mu^*_{MH})$ *be such that* $\Re_+\mathcal{D}(\mu^*_{MH})$ *is a closed subspace, and let* $\mu^*_{MH}(0_{\mathcal{B}^*_{MH}})$ *be finite. Then*

$$\begin{aligned}&\inf_{((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\})\in(\mathcal{W}_{MH}\times\mathcal{Y}^*_{MH}\times\mathcal{B}^*_{MH})\times C_{MH}}\\ &\mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{0_e^*\}_{Q^*_{MH}})\\ &= \max_{\{q_e\}\in Q_{MH}}(-\mathcal{S}^*_{MH}(((\{0_e^*\}_{\mathcal{V}^*_{MH}},\{0_e\}_{\mathcal{Y}_{MH}},\{0_e\}_{\mathcal{B}_{MH}}),\{0_e^*\}_{C^*_{MH}}),\{q_e\}).\end{aligned} \tag{21}$$

Therefore, we can now establish the macro-hybrid mixed optimality condition for optimal control problem $(\boldsymbol{O_{MH}})$, governed by primal evolution macro-hybrid mixed state problem $(\boldsymbol{\mathcal{MH}_\kappa})$ following (Alduncin, G.,2013, Theorem 4.3).

Theorem 14 *A primal macro-hybrid mixed state-control* $((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \boldsymbol{\mathcal{M}_{MH}}$ *is an optimal solution of constrained evolution primal macro-hybrid mixed problem* $(\boldsymbol{O_{MH}}\text{-}\boldsymbol{\mathcal{MH}_\kappa})$ *if, and only if, there exists a primal qualifying perturbation function* $\{p_e\} \in \boldsymbol{Q_{MH}}$ *such that* $(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\})$ *solves the primal macro-hybrid mixed perturbed control variational problem*

$$(\widetilde{\boldsymbol{\mathcal{M}OC_{MH}}})\begin{cases}\text{Find } ((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \boldsymbol{\mathcal{M}_{MH}} \text{ and } \{p_e\} \in \boldsymbol{Q_{MH}}:\\ \boldsymbol{\mathcal{T}^T_{MH}}\{p_e\} \in \boldsymbol{\partial J_{MH}}((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\})\\ \qquad\qquad \text{in } (\boldsymbol{\mathcal{W}^*_{MH}} \times \boldsymbol{\mathcal{Y}_{MH}} \times \boldsymbol{\mathcal{B}_{MH}}) \times \boldsymbol{C^*_{MH}},\\ -\boldsymbol{\mathcal{T}_{MH}}((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \boldsymbol{\partial 0}(\{p_e\}) - \{\widetilde{f_e^*}\},\\ \qquad\qquad \text{in } \boldsymbol{Q^*_{MH}},\end{cases}$$

that determines the optimality conditions of perturbed problem $(\boldsymbol{\mathcal{M}OC_{MH}})$*.*

Proof. Let $\{q_e^*\} \in \boldsymbol{Q^*_{MH}}$ and $\{\lambda_e\} > \{0_e\}$. From constraint qualification condition $(\boldsymbol{C_{\mathcal{T}_{MH}}})$, there is an admissible macro-hybrid mixed state-control $((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) \in \boldsymbol{\mathcal{M}_{MH}}$ such that

$$\{\lambda_e^{-1}q_e^*\} \in \boldsymbol{\mathcal{T}_{MH}}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) - \{\widetilde{f_e^*}\}.$$

Thus, $(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{\lambda_e^{-1}q_e^*\}) \in \boldsymbol{\mathcal{K}_{MH}}$, $\mathcal{S}_{MH}(((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}),\{\lambda_e^{-1}q_e^*\}) = J_{MH}((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\}) < \infty$, and by marginal functional definition (18), $\{\lambda_e^{-1}q_e^*\} \in \mathcal{D}(\mu^*_{MH})$ and $\mathcal{D}(\mu^*_{MH}) = \boldsymbol{\mathcal{T}_{MH}}(\boldsymbol{\mathcal{K}_{MH}}) - \{\widetilde{f_e^*}\}$. Thereby, $\boldsymbol{Q^*_{MH}} = \Re_+\mathcal{D}(\mu^*_{MH})$, and conjugate duality result (21) holds true. On the other hand, if $((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}) \in \boldsymbol{\mathcal{M}_{MH}}$ is a solution of primal problem $(\boldsymbol{OC_{MH}})$, then there is a solution of dual problem $(\boldsymbol{OC^*_{MH}})$, $\{p_e\} \in \boldsymbol{Q_{MH}}$. Therefore, by Lemma 12, $(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\}) \in (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}}) \times \boldsymbol{Q_{MH}}$ is a solution of macro-hybrid primal-dual problem $(\boldsymbol{OC_{MH}})$-$(\boldsymbol{OC^*_{MH}})$ if, and only if, $(((\{u_e\},\{p_e^*\},\{\lambda_e^*\}),\{\kappa_e\}),\{p_e\}) \in \boldsymbol{\mathcal{D}_{MH}} \times \boldsymbol{Q_{MH}}$ is a solution of minimax problem $(\boldsymbol{\mathcal{M}OC_{MH}})$, for which primal macro-hybrid mixed state-control-perturbation problem $(\widetilde{\boldsymbol{\mathcal{M}OC_{MH}}})$ states its optimality conditions. □

4. 2 Dual Optimality Conditions

We next proceed with the dual optimality part of the theory, for the optimal control problem $(\boldsymbol{O_{MH}})$ governed by the dual macro-hybrid mixed state problem $(\boldsymbol{\mathcal{MH}^*_\kappa})$, with associated state dual problem $(\boldsymbol{\mathcal{D}_{MH_\kappa}})$ (cf. Alduncin, G., 2013, Section 4.2). In this subsection, the solvability conditions of Theorems 6 and 7 will be in force. Now, as the macro-hybrid mixed state-control operator $\boldsymbol{\mathcal{T}^*_{\mathcal{MH}}} : (\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}^*_{MH}}) \times \boldsymbol{C_{MH}} \to \boldsymbol{\mathcal{Y}_{MH}}$, we have

$$\begin{aligned}&\boldsymbol{\mathcal{T}^*_{\mathcal{MH}}}((\{v_e\},\{y_e^*\},\{v_e^*\}),\{\eta_e\})\\ &\qquad = \left\{\frac{dy_e^*}{dt}\right\} + \{y_{y_e^*}\} - \{\Lambda_e v_e\} + \{B_e\eta_e\}, \quad \{y_{y_e^*}\} \in \{G_e^*(y_e^*)\},\end{aligned} \tag{22}$$

and the closed convex constraint domain of the dual problem is

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{*}_{\mathcal{MH}} = \{((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in (\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}} :\\ \{v_e\} \in \textstyle\prod_{e=1}^{E} \mathcal{D}(\widetilde{F}_e + \Psi_{e_C} \circ \pi_{e_C}), \{y^*_e\} \in \prod_{e=1}^{E} \mathcal{D}(\widetilde{\varphi}^*_e),\\ \{\eta_e\} \in \boldsymbol{C_{cd_{MH}}}\}.\end{aligned} \tag{23}$$

For the constraint qualification dual condition, we assumed that there is a closed subspace $\boldsymbol{Q_{MH}} \subset \boldsymbol{\mathcal{Y}_{MH}}$ such that

$$(\boldsymbol{C}_{\boldsymbol{\mathcal{T}}^*_{MH}}) \quad \boldsymbol{Q_{MH}} \subset \Re_+(\boldsymbol{\mathcal{T}}^*_{\mathcal{MH}}(\boldsymbol{\mathcal{M}}^*_{\mathcal{MH}}) + \{g_e\}),$$

and introduce the closed convex subset $\boldsymbol{\mathcal{K}}^*_{\boldsymbol{MH}} \subset ((\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q_{MH}}$,

$$\begin{aligned}\boldsymbol{\mathcal{K}}^*_{\boldsymbol{MH}} = \{(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q_e\}) \in \boldsymbol{\mathcal{M}}^*_{\boldsymbol{MH}} \times \boldsymbol{Q_{MH}} :\\ \{q_e\} \in \boldsymbol{\mathcal{T}}^*_{\boldsymbol{MH}}((\{v_e\},\{y^*_e\}),\{\eta_e\}) + \{g_e\}\}.\end{aligned} \tag{24}$$

Then, we define the proper convex and lower semicontinuous perturbation functional, $\mathcal{S}^*_{MH} : ((\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q_{MH}} \to \Re \cup \{+\infty\}$,

$$\begin{aligned}&\mathcal{S}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q_e\})\\ &\quad = J_{MH}((\{v_e\},\{y^*_e\}),\{\eta_e\}) + I_{\mathcal{K}^*_{MH}}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q_e\}).\end{aligned} \tag{25}$$

Moreover, we introduce the marginal or infimal value convex functional $\mu_{MH} : \boldsymbol{Q_{MH}} \to \Re \cup \{+\infty\}$,

$$\begin{aligned}\mu_{MH}(\{q_e\}) = \inf_{((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in (\mathcal{V}_{MH} \times \mathcal{X}^*_{MH}) \times C_{MH}}\\ \mathcal{S}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q_e\}).\end{aligned} \tag{26}$$

Thereby, following the perturbation duality theory (Ekeland, I. & Temam, R., 1974; Rockafellar, R. T., 1974), we can state the primal, dual and mixed optimization problems of the present dual theory, in a macro-hybrid perturbation sense. That is, the macro-hybrid optimal control functional, $J_{MH} : (\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}} \to \Re \cup \{+\infty\}$, is given in terms of a zero perturbation; i.e., $J_{MH}((\{v_e\},\{y^*_e\}),\{\eta_e\}) = \mathcal{S}_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{0_e\}_{Q_{MH}})$, and then the dual evolution macro-hybrid mixed optimal control problem in a perturbation sense is expressed by

$$(\widetilde{\boldsymbol{OC}_{\boldsymbol{MH}}}) \begin{cases} \text{Find } ((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \in \boldsymbol{\mathcal{M}}^*_{\boldsymbol{MH}} : \\ \mathcal{S}^*_{MH}(((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{0_e\}_{Q_{MH}}) \\ \qquad \le \mathcal{S}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{0_e\}_{Q_{MH}}), \\ \forall((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in (\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}. \end{cases}$$

Also, the perturbed dual convex functional $\pi^*_{MH} : \boldsymbol{Q}^*_{\boldsymbol{MH}} \to \Re \cup \{+\infty\}$ is

$$\pi^*_{MH}(\{q^*_e\}) = \mathcal{S}_{MH}(((\{0_e\}_{\mathcal{V}_{\{\Omega_e\}}},\{0^*_e\}_{\mathcal{Y}^*_{\{\Omega_e\}}}),\{0_e\}_{C_{\{\Omega_e\}}}),\{q^*_e\}), \tag{27}$$

where $\boldsymbol{Q}^*_{\boldsymbol{MH}} \subset \boldsymbol{\mathcal{Y}}^*_{\boldsymbol{MH}}$ is the dual subspace of the perturbation $\boldsymbol{Q_{MH}}$, and $\boldsymbol{\mathcal{S}_{MH}} : ((\boldsymbol{\mathcal{V}_{MH}} \times \boldsymbol{\mathcal{X}}^*_{\boldsymbol{MH}}) \times \boldsymbol{C_{MH}}) \times \boldsymbol{Q}^*_{MH} \to \Re \cup \{+\infty\}$ is the dual perturbation conjugate. Then the dual evolution macro-hybrid mixed optimal control problem in a dual perturbation sense is

$$(\widetilde{\boldsymbol{OC}^*_{\boldsymbol{MH}}}) \begin{cases} \text{Find } \{s^*_e\} \in \mathcal{D}(\pi^*_{MH}) : \\ -\mathcal{S}_{MH}(((\{0_e\}_{\mathcal{V}_{\{\Omega_e\}}},\{0^*_e\}_{\mathcal{Y}^*_{\{\Omega_e\}}}),\{0_e\}_{C_{\{\Omega_e\}}}),\{s^*_e\}) \\ \qquad \ge -\mathcal{S}_{MH}(((\{0_e\}_{\mathcal{V}_{\{\Omega_e\}}},\{0^*_e\}_{\mathcal{Y}^*_{\{\Omega_e\}}}),\{0_e\}_{C_{\{\Omega_e\}}}),\{q^*_e\}), \\ \qquad\qquad \forall\{q^*_e\} \in \boldsymbol{Q}^*_{\boldsymbol{MH}}. \end{cases}$$

Further, the convex-concave dual perturbed Lagrangian, $\mathcal{L}^*_{MH} : ((\mathcal{V}_{MH} \times \mathcal{X}^*_{MH}) \times \mathcal{C}_{MH}) \times \mathcal{Q}^*_{MH} \to \Re \cup \{+\infty\}$, is given by

$$\begin{aligned} \mathcal{L}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q^*_e\}) &= -\mathcal{S}_{MH_{((\{v_e\},\{y^*_e\}),\{\eta_e\})}}(\{q^*_e\}) \\ &= \begin{cases} J_{MH}((\{v_e\},\{y^*_e\}),\{\eta_e\}) - \langle\{q^*_e\}, \mathcal{T}^*_{MH}((\{v_e\},\{y^*_e\}),\{\eta_e\}) \\ \qquad +\{g_e\}\rangle_{\mathcal{Y}^*_{MH},\mathcal{Y}_{MH}}, \text{ if } ((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in \mathcal{D}^*_{MH}, \\ +\infty, \qquad \text{if } ((\{v_e\},\{y^*_e\}),\{\eta_e\}) \notin \mathcal{D}^*_{MH}, \end{cases} \\ \mathcal{D}^*_{MH} &= \{((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in \mathcal{M}^*_{MH} : \mathcal{T}^*_{MH}(\mathcal{M}^*_{MH}) + \{g_e\} \subset \mathcal{Q}_{MH}\}, \end{aligned} \tag{28}$$

the set $\mathcal{D}^*_{MH}$ being the projection of set $\mathcal{K}^*_{MH}$ on $(\mathcal{V}_{MH} \times \mathcal{X}^*_{MH}) \times \mathcal{C}_{MH}$. Here $\mathcal{S}_{MH_{((\{v_e\},\{y^*_e\}),\{\eta_e\})}} : \mathcal{Q}^*_{MH} \to \Re \cup \{+\infty\}$ is the conjugate of functional $\mathcal{S}^*_{MH_{((\{v_e\},\{y^*_e\}),\{\eta_e\})}} = \mathcal{S}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\cdot) : \mathcal{Q}_{MH} \to \Re \cup \{+\infty\}$, $((\{v_e\},\{y^*_e\}),\{\eta_e\}) \in (\mathcal{V}_{MH} \times \mathcal{X}^*_{MH}) \times \mathcal{C}_{MH}$. Consequently, in a macro-hybrid mixed perturbation sense, the dual evolution macro-hybrid mixed optimal control problem becomes

$$(\mathcal{MOC}^*_{MH}) \begin{cases} \text{Find } (((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{s^*_e\}) \in \mathcal{D}(\mathcal{L}^*_{MH}) : \\ \mathcal{L}^*_{MH}(((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{s^*_e\}) \\ \qquad \le \mathcal{L}^*_{MH}(((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{q^*_e\}) \\ \qquad\qquad \le \mathcal{L}^*_{MH}(((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q^*_e\}), \\ \forall\ (((\{v_e\},\{y^*_e\}),\{\eta_e\}),\{q^*_e\}) \\ \qquad \in ((\mathcal{V}_{MH} \times \mathcal{X}^*_{MH}) \times \mathcal{C}_{MH}) \times \mathcal{Q}^*_{MH}. \end{cases}$$

For this dual case, we shall also require the corresponding classical duality principle of the perturbation duality theory (Rockafellar, R. T., 1974; Ekeland, I. & Temam, R., 1974),

Lemma 15 *In a macro-hybrid sense, the following propositions are equivalent to each other:*

(P3) $((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \in \mathcal{D}(J_{MH})$ *and* $\{s^*_e\} \in \mathcal{D}(\pi_{MH})$ *are solutions to primal and dual perturbed problems* $(\widetilde{\mathcal{OC}_{MH}})$ *and* $(\widetilde{\mathcal{OC}^*_{MH}})$*, respectively, such that* $sup(\widetilde{\mathcal{OC}^*_{MH}}) = inf(\widetilde{\mathcal{OC}_{MH}})$*;*

(P4) $(((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{s^*_e\}) \in \mathcal{D}(\mathcal{L}^*_{MH})$ *is a solution to mixed perturbed problem* $(\mathcal{MOC}^*_{MH})$*.*

Moreover, taking into account that the marginal domain $\mathcal{D}(\mu_{MH})$ is the projection of the dual perturbation domain $\mathcal{D}(\mathcal{S}^*_{MH})$ on $\mathcal{Q}_{MH}$, we shall also require the result (Jeyakumar, V., 1990).

Lemma 16 *In a macro-hybrid sense, let the marginal domain* $\mathcal{D}(\mu_{MH})$ *be such that* $\Re_+\mathcal{D}(\mu_{MH})$ *is a closed subspace, and let* $\mu_{MH}(0_{\mathcal{B}_{MH}})$ *be finite. Then*

$$\begin{aligned} &\inf_{((\{v_e\},\{y^*_e\},\{v_e\}))\in(\mathcal{V}_{MH}\times\mathcal{X}^*_{MH})\times\mathcal{C}_{MH}} \mathcal{S}^*_{MH}(((\{v_e\},\{y^*_e\},\{v_e\})),\{0_e\}_{\mathcal{Q}_{MH}}) \\ &\quad = \max_{\{q^*_e\}\in\mathcal{Q}^*_{MH}} (-\mathcal{S}_{MH}(((\{0^*_e\}_{\mathcal{V}^*_{MH}},\{0_e\}_{\mathcal{Y}_{MH}}),\{0^*_e\}_{\mathcal{C}^*_{MH}}),\{q^*_e\}). \end{aligned} \tag{29}$$

Therefore, the macro-hybrid mixed optimality condition for optimal control problem $(\mathcal{O}_{MH})$, governed by dual evolution macro-hybrid mixed problem $(\mathcal{MH}^*_\kappa)$, is stated as follows (cf. Theorem 14).

Theorem 17 *A dual macro-hybrid mixed state-control* $((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \in \mathcal{M}^*_{MH}$ *is an optimal solution of constrained evolution dual macro-hybrid mixed problem* $(\mathcal{O}_{MH}\text{-}\mathcal{MH}^*_\kappa)$ *if, and only if, there exists a dual qualifying perturbation function* $\{s^*_e\} \in \mathcal{Q}^*_{MH}$ *such that* $(((\{u_e\},\{p^*_e\}),\{\kappa_e\}),\{s^*_e\})$ *solves the dual macro-hybrid mixed control variational problem*

$$(\widetilde{\mathcal{MOC}^*_{MH}}) \begin{cases} \text{Find } ((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \in \mathcal{M}^*_{MH} \text{ and } \{s^*_e\} \in \mathcal{Q}^*_{MH} : \\ \mathcal{T}^{*T}_{MH}\{s^*_e\} \in \partial J_{MH}((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \\ \qquad\qquad \text{in } (\mathcal{V}^*_{MH} \times \mathcal{X}_{MH}) \times \mathcal{C}^*_{MH}, \\ -\mathcal{T}^*_{MH}((\{u_e\},\{p^*_e\}),\{\kappa_e\}) \in \partial\mathbf{0}^*(\{s^*_e\}) + \{g_e\}, \\ \qquad\qquad \text{in } \mathcal{Q}_{MH}. \end{cases}$$

that gives the optimality conditions of perturbed control problem $(\mathcal{MOC}^*_{MH})$*.*

5. Primal and Dual Macro-Hybrid Variational Applications

In this final section, we proceed to apply the evolution macro-hybrid mixed optimal control theory developed in the previous sections. Here, we shall show how macro-hybridization works out, in particular, for the constrained mechanical evolution state systems treated in (Alduncin, G., 2013), of a primal nonlinear diffusion process and a dual quasistatic elastoviscoplastic deformation phenomena, which exemplified therein the variational optimal control theory in the mixed sense.

We refer to (Alduncin, G., 2015) for a recent application of the mixed theory, (Alduncin, G., 2013), to a constrained transport-flow mechanical state system in a Hilbert functional framework.

5.1 Mixed Distributed Control Nonlinear Diffusion State System

Considering the spatial bounded domain Ω stated in the preliminaries, decomposed into disjoint and connected subdomains $\{\Omega_e\}$ with Lipschitz continuous internal boundaries $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1, 2, ..., E$, and interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \leq e < k \leq E$, we illustrate the primal macro-hybrid mixed optimal control theory considering a mixed distributed intrinsic control nonlinear diffusion problem, modeled by a variational primal evolution state system, with primal conservation, and dual constitutivity, distributed constraint and transmission equations (cf. Alduncin, G., 2013, Section 6.1)

$$(\mathcal{MH}_{\kappa_d})\begin{cases} \text{Find } \{u_e\} \in \mathcal{W}_{MH_d} \text{ and } (\{\boldsymbol{w}_e^*\}, \{b_e^*\}, \{p_e^*\}) \in \mathcal{Y}_{MH_d}^* : \\ -(\{-grad^T \boldsymbol{w}_e^*\} + \mathcal{I}^T_{\mathcal{V}_{MH_d}}\{b_e^*\} + \mathcal{I}^T_{\mathcal{V}_{MH_d}}\{p_e^*\} + \{\pi^T_{\Gamma_{d_e}}\lambda^*_{d_e}\}) \\ \qquad \in \left\{\dfrac{du_e}{dt}\right\} + \partial \boldsymbol{I}_{\{\widehat{u}_e\}} \circ [\pi_{d_{P_e}}](\{u_e\}) + \{\pi^T_{d_D}\widehat{w}_{n_e}\} + \{B^*_{d_e}\kappa_{d_e}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{V}^*_{MH_d}, \\ (\{-\boldsymbol{grad}\, u_e\}, \mathcal{I}_{\mathcal{V}_{MH_d}}\{u_e\}, \mathcal{I}_{\mathcal{V}_{MH_d}}\{u_e\}) \\ \qquad \in (\{\boldsymbol{w}_e^*\}, \{\partial\varphi_e^*(b_e^*)\}, \{\partial\phi_e^*(p_e^*)\}), \qquad \text{in } \mathcal{Y}_{MH_d}, \\ \{u_e(0)\} = \{u_{e_0}\}; \\ \text{and } \{\lambda^*_{d_e}\} \in \mathcal{B}^*_{\Gamma_d} \text{ satisfying the dual synchronizing condition} \\ \{\pi_{\Gamma_{d_e}} u_e\} \in \partial \boldsymbol{I}_{Q_d^*}(\{\lambda^*_{d_e}\}), \text{ in } \mathcal{B}_{\Gamma_d}. \end{cases}$$

Here, the physical dependent fields are: the diffusive field of a transport process $\{u_e\}$ (e.g. of mass concentration or temperature), the linear flux vector field $\{\boldsymbol{w}_e^*\}$, the divergence of the nonlinear flux vector field $\{b_e^*\} = \{div\ \widetilde{\boldsymbol{w}}_e^*\}$, and the distributed intrinsic control source field $\{p_e^*\}$ implemented by a maximal monotone mechanism $\{\partial\phi_e^*\}$ (Duvaut, G. & Lions, J.-L., 1972). The variational operators are the gradient $\boldsymbol{grad} \in \mathcal{L}(\mathcal{V}_{MH_d}, \mathcal{Y}_{MH_d})$ with transpose $\boldsymbol{grad}^T \in \mathcal{L}(\mathcal{Y}^*_{MH_d}, \mathcal{V}^*_{MH_d})$, and the identity operator $\mathcal{I}_{\mathcal{V}_{MH_d}} : \mathcal{V}_{MH_d} \to \mathcal{V}_{MH_d}$ with transpose $\mathcal{I}^T_{\mathcal{V}^*_{MH_d}} : \mathcal{V}^*_{MH_d} \to \mathcal{V}^*_{MH_d}$. Also, the nonlinear diffusion constitutive primal equation of the decomposed system, $\{b_e^*\} \in \{\partial\varphi_e(u_e)\}$ ($\Longleftrightarrow \{u_e\} \in \{\partial\varphi_e^*(b_e^*)\}$, by convex dualization) is determined by the local potentials and differentials, $e = 1, 2, ..., E$,

$$\varphi_e(u_e) = \frac{1}{2}\int_{\Omega_e} \|\boldsymbol{grad}\, u_e\|^2 d\Omega_e + \frac{1}{p}\int_{\Omega_e} \|\boldsymbol{grad}\, u_e\|^p d\Omega_e, \tag{30}$$

$$\partial\varphi_e(u_e) = grad^T(1 + \|\boldsymbol{grad}\, u_e\|^{p-2})\boldsymbol{grad}\, u_e, \tag{31}$$

with nonlinear flux vector fields $\widetilde{\boldsymbol{w}_e^*} = -\|\boldsymbol{grad}\, u_e\|^{p-2}\boldsymbol{grad}\, u_e$. Thus, the local diffusion total flux vector fields turn out to be the sum $\boldsymbol{w_e} + \widetilde{\boldsymbol{w_e}}$.

On the other hand, the diffusion macro-hybrid field $\{\lambda^*_{d_e}\} \in \mathcal{B}^*_{\Gamma_d}$ corresponds to the variational internal boundary normal linear flux trace $\{\delta^*_{\Gamma_{d_e}} \boldsymbol{w_e}\} \in \boldsymbol{Q}^*_d$, which satisfies the dual transmission admissibility of interface continuity of (3). Moreover, from Lemma 1, $\{\pi_{\Gamma_{d_e}} u_e\} \in \mathcal{B}_{\Gamma_d}$ is then the variational internal boundary diffusive trace, belonging to the primal transmission admissibility subspace of interface continuity $\boldsymbol{Q_d}$ of (2). Hence, the mechanical communication of the sub-systems is guaranteed; i.e., the transmission conditions across their interfaces

$$\left.\begin{aligned} u_e &= u_k, \\ \boldsymbol{w_e^*}\cdot\boldsymbol{n_e} &= -\boldsymbol{w_k^*}\cdot\boldsymbol{n_k}, \end{aligned}\right\} \text{ across } \Gamma_{ek} \times (0, T), \tag{32}$$

of primal-diffusive and dual-linear flux trace continuity are variationally satisfied.

5. 1. 1 Primal Evolution Duality Principle

For an existence analysis of primal evolution macro-hybrid mixed nonlinear diffusion state problem $(\mathcal{MH}_{\kappa_d})$, we first introduce, as an appropriate stationary mixed functional framework of primal $\{u_e\}$-diffusive fields, and dual $\{\boldsymbol{w_e}\}$-linear fluxes, $\{b_e^*\}$-nonlinear fluxes divergence and $\{p_e^*\}$-source control fields, the Sobolev reflexive Banach spaces (Adams, R. A. & Fournier, J. J. F., 2003).

$$\begin{aligned} \boldsymbol{V}_{\{\boldsymbol{\Omega}_e\}_d} &= \prod_{e=1}^{E} W^{1,p}(\Omega_e),\ 2 < p < +\infty, \\ \boldsymbol{Y}^*_{\{\boldsymbol{\Omega}_e\}_d} &= \prod_{e=1}^{E} \boldsymbol{L}^{q^*}(\Omega_e) \times (W^{1,p}(\Omega_e))^* \times (W^{1,p}(\Omega_e))^*,\ q^* = p/(p-1). \end{aligned} \tag{33}$$

Here, $\boldsymbol{Y}^*_{\{\boldsymbol{\Omega}_e\}_d}$ corresponds to the topological dual of the space $\boldsymbol{Y}_{\{\boldsymbol{\Omega}_e\}_d} = \prod_{e=1}^{E} \boldsymbol{L}^p(\Omega_e) \times W^{1,p}(\Omega_e) \times W^{1,p}(\Omega_e)$. Then, as boundary spaces we have $\boldsymbol{B}_{\{\partial\boldsymbol{\Omega}_e\}_d} = \prod_{e=1}^{E} W^{1/q^*,p}(\partial\Omega_e)$ and its dual $\boldsymbol{B}^*_{\{\partial\boldsymbol{\Omega}_e\}_d} = \prod_{e=1}^{E} W^{-1/q^*,q^*}(\partial\Omega_e)$, for which the macro-hybrid primal and dual boundary compatibility conditions of the theory, $\boldsymbol{C}_{[\pi_{\Gamma_e}]}$ and $\boldsymbol{C}_{[\delta^*_{\Gamma_e}]}$, are satisfied.

Now, we can apply the primal composition duality principle of Theorem 2, assuming the proper condition for local control mechanisms (Duvaut, G. & Lions, J.-L., 1972)

$$(\boldsymbol{C}_{(\boldsymbol{G_e},\boldsymbol{\Lambda_e})_d})\quad \mathrm{int}\mathcal{D}(\phi_e)\Big(\subset W^{1,p}(\Omega_e)\Big) \neq \emptyset,\ e = 1, 2, ..., E.$$

Indeed, the evolution macro-hybrid compatibility condition $(\boldsymbol{C}_{(\{\boldsymbol{G_e}\},[\boldsymbol{\Lambda_e}])})$ of the primal theory reduces in this case to above conditions, and then nonlinear diffusion state problem $(\mathcal{MH}_{\kappa_d})$ is solvable if, and only if, the primal evolution macro-hybridized state problem

$$(\mathcal{P}_{\kappa_d})\begin{cases} \text{Find } \{u_e\} \in \mathcal{W}_{\boldsymbol{MH_d}} : \\ 0 \in \left\{\dfrac{du_e}{dt}\right\} + \{grad^T \boldsymbol{grad}(u_e)\} + \{\partial\varphi_e(u_e)\} + \{\partial\phi_e(u_e)\} \\ +\boldsymbol{\partial}(I_{\{\widehat{u}_e\}} \circ [\pi_{d_{P_e}}])(\{u_e\})) + \{\pi^T_{d_D}\widehat{w}_{n_e}\} + \boldsymbol{\partial}(I_{Q_d} \circ [\pi_{\Gamma_e}])(\{u_e\}) \\ \qquad +\{B^*_{d_e}\kappa_{d_e}\}, \qquad\qquad \text{in } \mathcal{V}_{\boldsymbol{MH_d}}, \\ \{u_e(0)\} = \{u_{e_0}\}, \end{cases}$$

is solvable.

Hence, assuming that corresponding primal existence conditions $(\boldsymbol{C1}_{\varphi_e})$ and $(\boldsymbol{C2}_{\varphi_e})$, $e = 1, 2, ..., E$, are fulfilled, the variational solvability results of Theorems 2 and 3 apply to nonlinear diffusion state problems $(\mathcal{P}_{\kappa_d})$ and $(\mathcal{M}_{\kappa_d})$. Notice that in this case primal subdifferential sum (9) is valid under the proper conditions $\mathcal{D}(\{\phi_e\}) \neq \emptyset$, for admissible control mechanisms, and that, in the classical sense, the nonlinear diffusion variational operators $\partial\varphi_e : W_0^{1,p}(\Omega_e) \to W^{-1,q^*}(\Omega_e)$ are Lipschitz continuous, bounded, strongly monotone and coercive (Lions, J. L., 1969).

5. 1. 2 Primal Optimality Condition

With respect to primal evolution nonlinear diffusion state system $(\mathcal{MH}_{\kappa_d})$, the cost functional of the macro-hybrid mixed theory, (5), corresponds to $J_{MH_d} : (\mathcal{V}_{\boldsymbol{MH_d}} \times \mathcal{Y}^*_{\boldsymbol{MH_d}} \times \mathcal{B}^*_{\boldsymbol{\Gamma_d}}) \times \boldsymbol{C}_{\boldsymbol{MH_d}} \to \Re \cup \{+\infty\}$, with the specific form

$$\begin{aligned} &J_{MH_d}(\{v_e\}, (\{\boldsymbol{v_e^*}\}, \{d_e^*\}, \{q_e^*\}, \{v^*_{d_e}\}), \{\eta_{d_e}\}) \\ &\qquad = \int_0^T (g_{d_1}(\{v_e\}) + g_{d_2}(\{\boldsymbol{v_e^*}\}, \{d_e^*\}, \{q_e^*\}, \{v^*_{d_e}\})) + j_d(\{\eta_{d_e}\})\, dt, \end{aligned} \tag{34}$$

whose arguments are the primal $\{v_e\}$-diffusive field, the dual $(\{\boldsymbol{v_e^*}\}, \{d_e^*\}, \{q_e^*\})$-linear flux-nonlinear flux divergence-intrinsic source control field, and the dual transmission normal linear flux field $\{v^*_{d_e}\}$ of the governing state system, as well as the own optimal control field $\{\eta_{d_e}\}$.

Assuming as an optimal control purpose, for the nonlinear diffusion state system, to resemble desired target profiles like given primal diffusive field $\{\widetilde{v}_e\} \in \mathcal{V}_{\boldsymbol{MH_d}} = L^p(0, T; \boldsymbol{V}_{\{\boldsymbol{\Omega}_e\}_d})$, dual linear flux, nonlinear flux divergence and intrinsic source control fields $(\{\widetilde{\boldsymbol{v}}_e^*\}, \{\widetilde{d_e^*}\}, \{\widetilde{q_e^*}\}) \in \mathcal{Y}^*_{\boldsymbol{MH_d}} = L^{q^*}(0, T; \boldsymbol{Y}^*_{\{\boldsymbol{\Omega}_e\}_d})$ and dual transmission field $\{\widetilde{v}^*_{d_e}\} \in \mathcal{B}^*_{\boldsymbol{\Gamma_d}} = L^{q^*}(0, T; \boldsymbol{B}^*_{\{\boldsymbol{\Gamma_e}\}_d})$, the natural cost functions would be

$$
\begin{aligned}
&g_{d_1}(t;\{v_e(t)\}) = w_{g_{d_1}}(t)\frac{1}{p}\|\{v_e(t)\} - \{\widetilde{v}_e(t)\}\|^p_{V_{\{\Omega_e\}_d}},\\
&g_{d_2}(t;(\{\boldsymbol{v}^*_e(t)\},\{d^*_e(t)\},\{q^*_e(t)\}),\{v^*_e(t)\})\\
&\quad = w_{g_{d_2}}(t)\frac{1}{q^*}\|(\{\boldsymbol{v}^*_e(t)\},\{d^*_e(t)\},\{q^*_e(t)\}) - (\{\widetilde{\boldsymbol{v}}^*_e(t)\},\{\widetilde{d}^*_e(t)\}\{\widetilde{q}^*_e(t)\})\|^{q^*}_{Y^*_{\{\Omega_e\}_d}}\\
&\quad + w_{g_{d_3}}(t)\frac{1}{q^*}\|\{v^*_e(t)\} - \{\widetilde{v}^*_{d_e}(t)\}\|^{q^*}_{B^*_{\{\Gamma_e\}_d}}.
\end{aligned}
\tag{35}
$$

Similarly, considering the optimal control Hilbert space $\boldsymbol{C_{MH_d}} = \boldsymbol{\mathcal{H}_{MH_d}} \equiv L^2(0,T);\boldsymbol{L^2_{\{\Omega_e\}}})$,

$$
j_d(t;\{\eta_{d_e}(t)\}) = w_{j_d}(t)\frac{1}{2}\|\{\eta_{d_e}(t)\}\|^2_{L^2_{\{\Omega_e\}}},\ \{\eta_{d_e}\} \in \boldsymbol{C_{ad_{MH_d}}} \subset \boldsymbol{C_{MH_d}}. \tag{36}
$$

Here, $w_{g_{d_1}}, w_{g_{d_2}}, w_{g_{d_3}}, w_{j_d}$ are given bounded and strictly positive weight coefficients belonging to $L^\infty(0,T)$. Furthermore, the admissible controls set $\boldsymbol{C_{ad_{MH_d}}}$ could be defined according to technological constraints as obstacle models (Duvaut, G. & Lions, J.-L., 1972).

Therefore, cost functional (34) turns out to be

$$
\begin{aligned}
\widetilde{J}_d((\{v_{d_e}\},(\{\boldsymbol{v}^*_e\},\{d^*_e\},\{q^*_e\}),\{v^*_{d_e}\}),\{\eta_{d_e}\}) &= \frac{1}{p}\|w_{g_{d_1}}(\{v_e\} - \{\widetilde{v}_e\})\|^p_{\mathcal{V}_{MH_d}}\\
&+\frac{1}{q^*}\|w_{g_{d_2}}((\{\boldsymbol{v}^*_e\},\{d^*_e\},\{q^*_e\}) - (\{\widetilde{\boldsymbol{v}}^*_e\},\{\widetilde{d}^*_e\},\{\widetilde{q}^*_e\}))\|^{q^*}_{\mathcal{Y}^*_{MH_d}}\\
&+\frac{1}{q^*}\|w_{g_{d_3}}(\{v^*_{d_e}\} - \{\widetilde{v}^*_{d_e}\})\|^{q^*}_{\mathcal{B}^*_{\Gamma_d}} + \frac{1}{2}\|w_{j_d}\{\eta_{d_e}\}\|^2_{\mathcal{H}_{MH_d}}
\end{aligned}
\tag{37}
$$

that clearly satisfies the qualifying condition $(\boldsymbol{C_{J_{MH}}})$ of Theorem 9, and the optimization problem $(\boldsymbol{O_{MH}})$ governed by the nonlinear diffusion process attains a solution.

Notice that in this case the macro-hybrid mixed state-control-perturbation variational problem $(\widetilde{\boldsymbol{\mathcal{M}OC_{MH}}})$ of Theorem 14, determines the optimality condition of primal optimal control problem $(\boldsymbol{O_{MH}})$-$(\boldsymbol{\mathcal{MH}_{\kappa_d}})$, governed by the nonlinear diffusion state system. Also the macro-hybrid mixed state-control operator $\boldsymbol{\mathcal{T}_{MH}} : (\boldsymbol{\mathcal{W}_{MH}} \times \boldsymbol{\mathcal{Y}^*_{MH}} \times \boldsymbol{\mathcal{B}^*_\Gamma}) \times \boldsymbol{C_{MH}} \to \boldsymbol{\mathcal{V}^*_{MH}}$, (14), is given by

$$
\begin{aligned}
&\boldsymbol{\mathcal{T}_{MH_d}}((\{v_e\},(\{\boldsymbol{v}^*_e\},\{d^*_e\},\{q^*_e\},\{v^*_{d_e}\}),\{\eta_{d_e}\})\\
&\quad = \left\{\frac{dv_e}{dt}\right\} + \{v^*_{e_{v_e}}\} + \{-grad^T\boldsymbol{v}^*_e\} + \{d^*_e\} + \{q^*_e\}\\
&\quad\quad +\{\pi^T_{\Gamma_{d_e}}v^*_{d_e}\} + \{B^*_e\eta_{d_e}\},\ \ \{v^*_{e_{v_e}}\} \in \{\partial I_{\{\widehat{u}_e\}} \circ \pi_{d_P})(v_e)\},
\end{aligned}
\tag{38}
$$

with respect to the identifications $\{\Lambda^T y^*_e\} \sim \{-grad^T\boldsymbol{u}^*_e + s^*_e + x^*_e\}$, $\{\partial(\widetilde{F_e} + \Psi_{C_e} \circ\pi_{C_e})(\{v_e\}) \sim \{\partial(I_{\{\widehat{u_e}\}} \circ \pi_{e_P})(\{v_e\})$, and the right hand side term $-\{\widetilde{f_e}^*\} \sim \{\pi^T_{e_D}\widehat{s}^*_e\}$.

5.2 Boundary Constrained Quasistatic Elastoviscoplastic State System

We next apply the dual macro-hybrid mixed optimal control theory to a boundary constrained quasistatic elastoviscoplastic problem in the domain $\Omega \subset \Re^n$, $n \in \{1,2,3\}$, decomposed into disjoint and connected subdomains $\{\Omega_e\}$ with Lipschitz continuous internal boundaries $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1,2,...,E$, and interfaces $\Gamma_{ek} = \Gamma_e \cap \Gamma_k$, $1 \le e < k \le E$.

Let us consider a solid body occupying Ω, with boundary $\partial\Omega$ and unit outward normal vector $\boldsymbol{\nu}$, undergoing small strains and displacements along the time interval $(0,T)$, with an elastoviscoplastic constitutive material (Le Tallec, P., 1990), and a bilateral boundary contact constraint modeled by Tresca's law of dry friction. The dual macro-hybrid mixed variational quasistatic contact problem is given by the variational dual evolution state problem, with local primal quasistatic equilibrium $\{\boldsymbol{w_e}\}$-velocity equation, and local dual evolution elastoviscoplastic constitutivity $\{\boldsymbol{S^*_e}\}$-stress and internal boundary $\{\lambda^*_e\}$-transmission equations (cf. (Alduncin, G., 2013), Section 6.2)

$$(\mathcal{M}^*_{\kappa_c})\begin{cases}\text{Find } \{\boldsymbol{w_e}\}\in\mathcal{V}_{MH_c} \text{ and } \{\boldsymbol{S^*_e}\}\in\mathcal{X}^*_{MH_c}:\\ -\{\boldsymbol{H}^T_e\boldsymbol{S_e}\}-\{\pi^T_{\Gamma_e}\lambda^*_{c_e}\}\in\{\boldsymbol{\partial\widetilde{F}_e}(\{\boldsymbol{w_e}\})-\{\widetilde{\boldsymbol{b}}^*_e\}, \qquad \text{in } \mathcal{V}^*_{MH_c},\\ \{\boldsymbol{H_e w_e}\}\in\left\{\boldsymbol{A_e}\dfrac{d\boldsymbol{S^*_e}}{dt}\right\}+\{\boldsymbol{\partial\Phi^*_e}(\boldsymbol{S^*_e})\}+\{B_{c_e}\kappa_{c_e}\}, \quad \text{in } \mathcal{Y}_{MH_c},\\ \{\boldsymbol{S^*_e}(0)\}=\{\boldsymbol{S^*_{e_0}}\},\\ \text{and } \{\lambda^*_{c_e}\}\in\mathcal{B}^*_{\Gamma_c} \text{ satisfying the dual synchronizing condition}\\ \{\boldsymbol{\pi_{\Gamma_e}w_e}\}\in\boldsymbol{\partial I_{Q^*_c}}(\{\lambda^*_{c_e}\}), \qquad \text{in } \mathcal{B}_{\Gamma_c}.\end{cases}$$

Here, for a.e. $t\in(0,T)$, the local primal subdifferential $\boldsymbol{\partial\widetilde{F}_e}:\boldsymbol{V}(\Omega_e)\to\boldsymbol{2}^{V^*(\Omega_e)}$ is such that (Ernst, E., Théra, M., 2009)

$$\begin{aligned}\boldsymbol{\partial\widetilde{F}_e}&=\boldsymbol{\partial}(I_{\{\widehat{w_e}(\cdot,t)\}}\circ\pi_{D_{\nu_e}})+\boldsymbol{\partial}(I_{\{0_e\}}\circ\pi_{C_{\nu_e}})+\boldsymbol{\partial}(\Psi_{C_{\tau_e},f^*_e(\cdot,t)}\circ\boldsymbol{\pi}_{C_{\tau_e}}),\\ \widetilde{F}_e&=I_{\{\widehat{w_e}(\cdot,t)\}}\circ\pi_{D_{\nu_e}}+I_{\{0_e\}}\circ\pi_{C_{\nu_e}}+\Psi_{C_{\tau_e},f^*_e(\cdot,t)}\circ\boldsymbol{\pi}_{C_{\tau_e}},\\ \mathcal{D}(\widetilde{F}_e)&=\boldsymbol{K}_{\widehat{w_e}(\cdot,t),0}=\{\boldsymbol{v_e}\in\boldsymbol{V}(\Omega_e):\pi_{C_{\nu_e}}\boldsymbol{v_e}=\widehat{w}_e(\cdot,t)\text{ in }\boldsymbol{B}(\partial\Omega_{D_e}),\\ &\qquad\qquad\qquad\qquad\pi_{C_{\nu_e}}\boldsymbol{v_e}=0\text{ in }B(\partial\Omega_{C_e})\}.\end{aligned}\tag{39}$$

Also, as dual variational operators we have the elastic compliance operator $[\boldsymbol{A_e}]\in\mathcal{L}(\boldsymbol{Y_{\{\Omega_e\}}},\boldsymbol{Y_{\{\Omega_e\}}})$ (symmetric and positive definite), and the dual dissipation subdifferential $[\boldsymbol{\partial\Phi^*_e}]:\boldsymbol{Y^*_{\{\Omega_e\}}}\to\boldsymbol{2}^{Y_{\{\Omega_e\}}}$ of the conjugate superpotentials $\Phi^*_e:\boldsymbol{Y}^*(\Omega_e)\to\Re\cup\{+\infty\}$, $e=1,2,...,E$, the yield functionals.

On the other hand, the local right-hand side term is

$$\{\widetilde{\boldsymbol{b}}^*_e\}=-\{\boldsymbol{\pi}^T_{N_c}\widehat{\boldsymbol{s}}_e\}+\{\boldsymbol{b^*_e}\}\in\prod_e^E\boldsymbol{L}^p(0,T;\mathcal{R}(-\boldsymbol{H}^T_e)),\tag{40}$$

where $\{\boldsymbol{b^*_e}\}$ corresponds to the local body force field, and $\mathcal{R}(-\boldsymbol{H}^T_e)$ is the range of the variational symmetric total small strain rate transpose $-\boldsymbol{H^T_e}=-1/2(\boldsymbol{\nabla w_e}+\boldsymbol{\nabla w_e}^T)\in\mathcal{L}(\boldsymbol{Y}^*(\Omega_e),\boldsymbol{V}^*(\Omega_e))$. Further $\{\widehat{\boldsymbol{s}}_e\}$ and $\{\widehat{\boldsymbol{w}}_e\}$ are the local Neumann and Dirichlet conditions of negative tractions and velocities prescribed on disjoint sub-boundaries $(\partial\Omega_{N_e}\cap\partial\Omega)\times(0,T)$ and $(\partial\Omega_{D_e}\cap\partial\Omega)\times(0,T)$, respectively. Notice that the Tresca's law constraint of the problem on the complementary disjoint contact sub-boundaries $\partial\Omega_{C_e}\cap\partial\Omega$, with prescribed shear bound functions $f^*_e>0$, $e=1,2,...,E$, locally defined by

$$\begin{aligned}w_{\nu_e}&\in\partial\Psi^*_{C_e,0_e}(s_{\nu_e})=\{0_e\},\\ \boldsymbol{w}_{\tau_e}\in\boldsymbol{\partial\Psi^*}_{C_e,f^*_e(\cdot,t)}(\boldsymbol{s}_{\tau_e})&=\begin{cases}\{\boldsymbol{0_e}\}, & \|\boldsymbol{s}_{\tau_e}\|<f^*_e(\cdot,t),\\ \{\xi_e\boldsymbol{s}_{\tau_e}:\xi_e\le 0\}, & \|\boldsymbol{s}_{\tau_e}\|=f^*_e(\cdot,t),\\ \boldsymbol{\emptyset}, & \text{otherwise},\end{cases}\end{aligned}\tag{41}$$

is variationally incorporated by convex compositional dualization, of the primal subdifferentials $\boldsymbol{\partial}(I_{\{0_e\}}\circ\pi_{C_{\nu_e}})$ of normal impenetrability, and tangential friction dual subdifferentials $\boldsymbol{\partial}(\Psi_{C_{\tau_e},f^*_e(\cdot,t)}\circ\boldsymbol{\pi}_{C_{\tau_e}})$, $(39)_1$. Here $w_{\nu_e}=\boldsymbol{w_e\cdot\nu_e}$ and $\boldsymbol{w}_{\tau_e}=(\boldsymbol{I}-\boldsymbol{\nu_e}\otimes\boldsymbol{\nu_e})\boldsymbol{w_e}$ are the normal and tangential velocity components, and $s_{\nu_e}=-\boldsymbol{S_e\nu_e\cdot\nu_e}$ and $\boldsymbol{s}_{\tau_e}=-(\boldsymbol{I}-\boldsymbol{\nu_e}\otimes\boldsymbol{\nu_e})\boldsymbol{S_e\nu_e}$ the contact pressure and tangential negative tractions.

Furthermore, in this deformation case the macro-hybrid field $\{\lambda^*_{c_e}\}\in\mathcal{B}^*_{\Gamma_c}$ results to be the variational internal boundary traction trace $\{\delta^*_{\Gamma_{c_e}}\boldsymbol{S_e}\}\in\boldsymbol{Q^*_c}$, satisfying the dual transmission admissibility of interface continuity (3). In addition, from Lemma 1, $\{\pi_{\Gamma_{c_e}}\boldsymbol{w_e}\}\in\mathcal{B}_{MH_c}$ is the variational internal boundary velocity trace, belonging to the primal transmission admissibility subspace of interface continuity $\boldsymbol{Q_c}$ of (2). In this manner the mechanical communication of the sub-systems is guaranteed; i.e. the transmission conditions across their interfaces, of primal-velocity and dual-traction trace continuity,

$$\left.\begin{aligned}\boldsymbol{w_e}&=\boldsymbol{w_k},\\ -\boldsymbol{S^*_e\nu_e\cdot\nu_e}&=\boldsymbol{S^*_k\nu_k\cdot\nu_k},\end{aligned}\right\}\text{ across }\Gamma_{ek}\times(0,T),\tag{42}$$

are variationally fulfilled.

5. 2. 1 Dual Evolution Duality Principle

For a duality principle of problem $(\boldsymbol{\mathcal{M}}^*_{\kappa_c})$, we apply Theorem 6 assuming the deformation macro-hybrid classical compatibility conditions

$$(\boldsymbol{C}_{(\widetilde{F}^*_e,-\boldsymbol{H}^T_e)})\quad \mathrm{int}\mathcal{D}(\widetilde{F}^*_e)\cap\mathcal{R}(-\boldsymbol{H}^T_e)\neq\emptyset,\ e=1,2,...,E,$$

under which the corresponding compositional operator equalities $-\Lambda\partial(\widetilde{F}_e+\Psi_{C_e}\circ\pi_{C_e})^*\circ(-\Lambda^T_e)=\partial((\widetilde{F}_e+\Psi_{C_e}\circ\pi_{C_e})^*\circ(-\Lambda^T_e))$ hold true. Then the solvability of dual macro-hybrid mixed state problem $(\boldsymbol{\mathcal{M}}^*_{\kappa_c})$ is equivalent to the solvability of its evolution dual macro-hybridized state problem

$$(\widetilde{\boldsymbol{\mathcal{D}}}_{\kappa_c})\begin{cases}\text{Find } \{\boldsymbol{S}^*_e\}\in\boldsymbol{\mathcal{X}}^*_{MH_c}:\\ \{\boldsymbol{0}_e\}\in\left\{\boldsymbol{A}_e\dfrac{d\boldsymbol{S}^*_e}{dt}\right\}+\{\boldsymbol{\partial\Phi}^*_e(\boldsymbol{S}^*_e)\}+\{\partial(\widetilde{F}^*_e\circ(-\boldsymbol{H}^T_e))(\boldsymbol{S}^*_e+\boldsymbol{R}^*_{\widetilde{b}^*_e})\}\\ \qquad+\{B_{c_e}\kappa_{c_e}\},\qquad\qquad \text{in } \boldsymbol{\mathcal{Y}}_{MH_c},\\ \{\boldsymbol{S}_e(0)\}=\{\boldsymbol{S}_{e_0}\}.\end{cases}$$

Furthermore, assuming the local Moreau-Rockafellar-Robinson condition

$$(\boldsymbol{C}_{(\Phi^*_e,\widetilde{H}_{\widetilde{b}^*_e})})\quad \mathrm{int}\mathcal{D}(\Phi^*_e)\cap\mathcal{D}(\widetilde{H}_{\widetilde{b}^*_e})\neq\emptyset,\ e=1,2,...,E,$$

where $\widetilde{H}_{\widetilde{b}^*_e}(\cdot)=(\widetilde{F}^*_e\circ(-\boldsymbol{H}^T_e))(\cdot+\boldsymbol{R}^*_{\widetilde{b}^*_e})$, with $\boldsymbol{R}^*_{\widetilde{b}^*_e}$ a fixed $-\boldsymbol{H}^T_e$-preimage of $\widetilde{\boldsymbol{b}}^*_e$, the dual subdifferential sum rule $\boldsymbol{\partial\varphi}^*_e\equiv\partial(\Phi^*_e+\widetilde{H}_{\widetilde{b}^*_e})=\boldsymbol{\partial\Phi}^*_e+\boldsymbol{\partial}\widetilde{\boldsymbol{H}}_{\widetilde{b}^*_e}$ holds true, and we can apply dual existence Theorems 6 and 7 to dual state problem $(\widetilde{\boldsymbol{\mathcal{D}}}_{\kappa_c})$, once it is expressed as the classical evolution subdifferential inclusion

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}}_{\kappa_c})\begin{cases}\text{Find } \{\boldsymbol{S}^*_e\}\in\boldsymbol{\mathcal{X}}^*_{MH_c}:\\ \{\boldsymbol{0}_e\}\in\left\{\boldsymbol{A}_e\dfrac{d\boldsymbol{S}_e}{dt}\right\}+\{\boldsymbol{\partial\varphi}^*_e(\boldsymbol{S}^*_e)\}+\{\boldsymbol{B}_{c_e}\kappa_{c_e}\},\quad \text{in } \boldsymbol{\mathcal{Y}}_{MH_c},\\ \{\boldsymbol{S}^*_e(0)\}=\{\boldsymbol{S}^*_{e_0}\}.\end{cases}$$

Therefore, under the corresponding coercivity and boundedness dual conditions, dual evolution macro-hybrid mixed state control problem $(\boldsymbol{\mathcal{M}}^*_{\kappa_c})$ has a solution if, and only if, dual evolution state problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}}_{\kappa_c})$ has a solution. We refer to (Le Tallec, P., 1990; Temam, R., 1986; Perzyna, P., 1966; Sofonea, M., Renon, N., Shillor, M., 2004; Alduncin, G., 2011) for some representative elastoviscoplastic constitutive models.

5. 2. 2 Dual Optimality Condition

Regarding the optimal control problem of the theory, governed by the variational dual evolution macro-hybrid mixed deformation contact process $(\boldsymbol{\mathcal{M}}^*_{\kappa_c})$, the general cost functional (5) corresponds to $J_{MH_c}:(\boldsymbol{\mathcal{V}}_{MH_c}\times\boldsymbol{\mathcal{Y}}^*_{MH_c})\times\boldsymbol{C}_{MH_c}\to\mathfrak{R}\cup\{+\infty\}$, of the specific form

$$J_{MHc}((\{\boldsymbol{v}_e\},\{\boldsymbol{T}^*_e\}),\{\eta_{c_e}\})=\int_0^T(g_1(\{\boldsymbol{v}_e\})+g_2(\{\boldsymbol{T}^*_e\})+j(\{\eta_{c_e}\}))\,dt,\tag{43}$$

for the primal $\{\boldsymbol{v}_e\}$-velocity and dual $\{\boldsymbol{T}^*_e\}$-stress fields. Further, in order to exemplify the cost functions, we assume that the control purpose is now to drive the state of the system as close as possible to some desired target profiles. For example, to a given primal velocity target field $\{\widetilde{\boldsymbol{v}}_e\}\in\boldsymbol{\mathcal{V}}_{MH_c}=L^p(0,T;\boldsymbol{V}_{\{\Omega_e\}_c})$ and a dual stress target field $\{\widetilde{\boldsymbol{T}}^*_e\}\in\boldsymbol{\mathcal{Y}}^*_{MH_c}=L^{q^*}(0,T;\boldsymbol{Y}^*_{\{\Omega_e\}_c})$, introducing the natural common cost functions, for $\{\boldsymbol{v}_e\}\in\boldsymbol{\mathcal{V}}_{MH_c}$ and $\{\boldsymbol{T}^*_e\}\in\boldsymbol{\mathcal{Y}}^*_{MH_c}$,

$$\begin{aligned}g_1(t;\{\boldsymbol{v}_e(t)\})&=w_{g_1}(t)\frac{1}{p}\|\{\boldsymbol{v}_e(t)\}-\{\widetilde{\boldsymbol{v}}_e(t)\}\|^p_{V_{\{\Omega_e\}_c}},\\ g_2(t;\{\boldsymbol{T}^*_e(t)\})&=w_{g_2}(t)\frac{1}{q^*}\|\{\boldsymbol{T}^*_e(t)\}-\{\widetilde{\boldsymbol{T}}^*_e(t)\}\|^{q^*}_{Y^*_{\{\Omega_e\}_c}},\end{aligned}\tag{44}$$

and for the Hilbert control space $\boldsymbol{C_{\mathcal{MH}_c}} = \boldsymbol{\mathcal{H}_{MH_c}} \equiv L^2(0,T;\boldsymbol{L^2_{\{\Omega_e\}}})$ the function

$$j(t;\{\eta_e(t)\}) = w_j(t)\frac{1}{2}\|\{\eta_e(t)\}\|^2_{L^2_{\{\Omega_e\}}},\ \{\eta_e\} \in \boldsymbol{C_{ad_{MH_c}}} \subset \boldsymbol{C_{\mathcal{MH}_c}}. \tag{45}$$

Here, $w_{g_1}, w_{g_2}, w_j \in L^\infty(0,T)$ are given bounded and strictly positive weight coefficients. Also, as mentioned in the previous example, the set of admissible controls, $\boldsymbol{C_{ad_{MH_c}}}$, could respond to technological constraints via obstacle models (Duvaut, G. & Lions, J.-L., 1972). Thereby, cost functional (43) turns out to be

$$\begin{aligned}\widetilde{J}_{MH_c}((\{\boldsymbol{v_e}\},\{\boldsymbol{T^*_e}\}),\{\eta_e\}) &= \frac{1}{p}\|w_{g_1}(\{\boldsymbol{v_e}\} - \{\widetilde{\boldsymbol{v}}_{\boldsymbol{e}}\})\|^p_{\mathcal{V}_{MH_c}} \\ &+\frac{1}{q^*}\|w_{g_2}(\{\boldsymbol{T_e}\} - \{\widetilde{\boldsymbol{T}}_{\boldsymbol{e}}\})\|^{q^*}_{\mathcal{Y}^*_{MH_c}} + \frac{1}{2}\|w_j\{\eta_e\}\|^2_{\mathcal{H}^*_{MH_c}},\end{aligned} \tag{46}$$

which satisfies qualifying condition ($\boldsymbol{C_{J_{MH}}}$), and then optimization problem ($\boldsymbol{O_{MH}}$) governed by this elastoviscoplastic deformation contact process has a solution in accordance with Theorem 9.

Lastly, with respect to the optimality condition of this deformation governing process, as established by Theorem 7, the dual optimal control problem ($\boldsymbol{O_{MH}}$)-($\boldsymbol{\mathcal{M}^*_{\kappa_c}}$) has the optimality condition ($\boldsymbol{\mathcal{M}OC^*_{MH}}$), stated by the macro-hybrid mixed state-control-perturbation problem ($\widetilde{\boldsymbol{\mathcal{M}OC^*_{MH}}}$). For this example, the dual macro-hybrid mixed state-control operator $\boldsymbol{\mathcal{T}^*_{\mathcal{MH}_c}} : (\boldsymbol{\mathcal{V}_{MH_c}} \times \boldsymbol{X^*_{MH_c}}) \times \boldsymbol{C_{MH_c}} \to \boldsymbol{\mathcal{Y}_{MH_c}}$, (22), is such that

$$\begin{aligned}&\boldsymbol{\mathcal{T}^*_{\mathcal{MH}_c}}((\{\boldsymbol{v_e}\},\{\boldsymbol{T^*_e}\}),\{\eta_{c_e}\}) \\ &= \left\{\boldsymbol{A_e}\frac{d\boldsymbol{T^*_e}}{dt}\right\} + \{\boldsymbol{v^*_{e\{v_e\}}}\} - \{\boldsymbol{H_e v_e}\} + \{B_{c_e}\eta_{c_e}\},\ \ \{\boldsymbol{v^*_{e\{v_e\}}}\} \in \{\boldsymbol{\partial\Phi^*_e}(\boldsymbol{T^*_e})\},\end{aligned} \tag{47}$$

according to the theoretical relations $\{-\Lambda_e u_e\} \sim \{-\boldsymbol{H_e v_e}\}$ and $\{\partial G^*_e(p^*_e)\} \sim \{\boldsymbol{\partial\Phi^*_e}(\boldsymbol{T^*_e})\}$, and with the right hand side term identification $\{g_e\} \sim \{0_e\}$.

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