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Stabilization of Variable Coefficients Euler-Bernoulli Beam with Viscous Damping under a Force Control in Rotation and Velocity Rotation

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Abstract

This paper investigates the problem of exponential stability for a damped Euler-Bernoulli beam with variable coefficients clamped at one end and subjected to a force control in rotation and velocity rotation. We adopt the Riesz basis approach for show that the closed-loop system is a Riesz spectral system. Therefore, the exponential stability and the spectrum-determined growth condition are obtained.

Keywords: Beam equation, boundary feedback control, exponential stability, semigroup theory, Riesz basis

Mathematics Subject Classification 2010. 35B35, 35P20, 93D15.

1. Introduction

In this paper, we study the exponential stability property of a damped Euler-Bernoulli beams with variable coefficients under a force feedback in rotation and velocity rotation. The equations of motion of the system are described as follows

$$m(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} + \gamma(x)w_t(x,t) = 0, \qquad 0 < x < 1,\ t > 0, \tag{1}$$

$$w(0,t) = w_x(0,t) = 0, \qquad t > 0, \tag{2}$$

$$(EI(.)w_{xx})_x(1,t) = 0, \qquad t > 0, \tag{3}$$

$$-EI(1)w_{xx}(1,t) = \alpha w_{xt}(1,t) + \beta w_x(1,t), \qquad t > 0, \tag{4}$$

where the subscripts t and x denote derivatives with respect to the time t and the position x respectively. $w(x,t)$ stands for the transverse displacement of the beam at the position x and time t. The feedbacks α and β are two given positive constants. We assume that the length of the beam is equal to unity. $EI(x)$ and $m(x)$ are, respectively, the flexural rigidity function and the mass density function of the beam along the spatial variable x satisfying the following condition

$$m(x),\quad EI(x) \in C^4(0,1), \qquad m(x),\ EI(x) > 0 \tag{5}$$

for all $x \in [0,1]$. Non-homogeneous materials, in particular smart materials used in engineering, are typical examples of the importance of assuming coefficients as variables (Lee & Li, 1998). Moreover, $\gamma(x)$ is a continuous coefficient function of feedback damping that is assumed to satisfy the condition

$$\int_0^1 \left(\frac{\gamma(x)}{m(x)}\right)\left(\frac{m(x)}{EI(x)}\right)^{1/4} dx > 0. \tag{6}$$

Notice that the condition (6) will allow γ to be indefinite in the interval $[0,1]$.

In the theory of dynamic systems, stability is a matter of great interest to mathematicians and engineers. More particularly, exponential stability is the most desirable stability, especially for damped systems. The study of the case ($\gamma \equiv 0$) was done in (Bomisso, Touré & Yoro, 2017) where the authors have obtained a result according to the authors Wang et al (see e.g. Wang, 2004) to show the exponential stability of system (1)-(4). In (Wang, 2004), a question has been raised and is valid for our system (1)-(4): Due to the nonuniform physical thickness and/or density of the Euler-Bernoulli beam with the variable coefficient damping $\gamma(x)$ in equation (1), what conditions are needed to put onto the damping term to guarantee exponential stability? Here it is very hard to have the exact precise location of the eigenvalues because equation

(1) contains variable coefficients and subject to the boundary conditions (2)-(4). This question is treated when (1) is associated to hinged boundary conditions (see Wang, 2004), and when the equation (1) is subjected to the force control in position and velocity (see Touré, Coulibaly & Kouassi, 2015). Moreover, in order to study the eigenvalues of systems with variable coefficients, we will used the two steps provided by Birkhoff's works (Birkhoff, 1908) and Naimark's works (Naimark, 1967). This approach was used by many authors for study the Euler-Bernoulli beams equations with variable coefficients (see e.g. Guo, 2002; Guo & Wang, 2006; Wang, 2004; Wang, Xu & Yung, 2005). In our case, we rely on idea of Wang et al (see e.g. Wang, 2004; Wang, Xu & Yung, 2005) in order to study the problem with eigenvalues related to problem (1)-(4) in the form of an ordinary differential equation $L(f) = \lambda f$ with boundary conditions λ-polynomials. We establish conditions on the two feedbacks parameters at the boundary α and β to obtain the property of the Riesz basis and the exponential stability of the system (1)-(4).

Our main contribution is to prove the exponential stability of the perturbed system (1)-(4).

The rest of the paper is organized as follows. In section **2**, the system (1)-(4) is formulated as an evolution problem and studied in semigroup framework. In section **3**, a spectral analysis is made and next, we prove that the system operator has Riesz basis property in the corresponding state space. Consequently, in section **4**, we give conditions that ensure the exponential stability of our perturbed system.

2. Semigroup Formulation

We define the following functional spaces:

$$H_E^2(0,1) = \left\{ w \in H^2(0,1) \,|\, w(0) = w_x(0) = 0 \right\} \tag{7}$$

and

$$\mathbb{H} = H_E^2(0,1) \times L^2(0,1), \tag{8}$$

with the inner product

$$\langle w, v \rangle_{\mathbb{H}} = \int_0^1 m(x) f_2(x) \overline{g_2(x)}\, dx + \int_0^1 EI(x) f_1''(x) \overline{g_1''(x)}\, dx + \beta f_1'(1) \overline{g_1'(1)}, \tag{9}$$

where $w = (f_1, f_2)^T \in \mathbb{H}$, $v = (g_1, g_2)^T \in \mathbb{H}$ and $\|.\|_{\mathbb{H}}$ denotes the corresponding norm. We recall also the definitions of the following spaces :

$$L^2(0,1) = \left\{ w : [0,1] \to \mathbb{C} \,\middle|\, \int_0^1 |w|^2\, dx < \infty \right\} \tag{10}$$

$$H^k(0,1) = \left\{ w : [0,1] \to \mathbb{C} \,\middle|\, w, w^{(1)}, \ldots, w^{(k)} \in L^2(0,1) \right\}. \tag{11}$$

Let $A_\gamma : D(A_\gamma) \subset \mathbb{H} \to \mathbb{H}$ an unbounded linear operator with the domain

$$D(A_\gamma) = \left\{ (f,g)^T \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) \,\middle|\, (EI(.)f''(.))'(1) = 0, -EI(1)f''(1) = \alpha g_x(1) + \beta f_x(1) \right\} \tag{12}$$

defined as

$$A_\gamma (f,g)^T = \left(g(x), -\frac{1}{m(x)} \left[(EI(x) f''(x))'' + \gamma(x) g(x) \right] \right)^T. \tag{13}$$

So, we can written (1)−(4) as a first order evolution problem

$$\begin{cases} \dfrac{d}{dt} z(t) = A_\gamma z(t) \\ z(0) = z_0 \in \mathbb{H}, \end{cases} \tag{14}$$

where $z(t) = (w, w_t)^T$, $z(0) = (w_0, v_0)^T$.
Furthermore, notice that

$$\Gamma_\gamma(f,g) = A_\gamma - A_0 = \left(0, -\frac{\gamma(x) g(x)}{m(x)} \right)$$

is a linear operator and bounded on $\mathbb{H}$ with A_0 denotes the operator in the undamped case $\gamma(x) \equiv 0$.

Two results are immediate and the first one is a consequence of the perturbation theory of semigroups (see e.g. Pazy, 1983).

Theorem 1 *Let operators A_γ and A_0 be defined as before. Thus A_0 is a dissipative operator and generates a C_0-semigroup of contractions on $\mathbb{H}$ denoted by $\{S(t)\}_{t\geq 0}$ and therefore A_γ is a generator of the contraction semi-group $e^{A_\gamma t}$ on $\mathbb{H}$ denoted by $\{T(t)\}_{t\geq 0}$.*

Proof. The first assertion has been proved in (Bomisso, Touré & Yoro, 2017). The authors have used the Lumer-Phillips Theorem (see, e.g., Pazy, 1983, pp.14) and Hille-Yosida-Phillips Theorem in order to show that the dissipative operator A_0 generates a C_0-semigroup $S(t) = e^{A_0 t}$ on H, such that

$$\|S(t)\| \leq Ce^{\omega t}.$$

Hence, the perturbation theory (see Theorem 1.1 of Pazy, 1983) allows us to deduce that $A_\gamma = \Gamma_\gamma + A_0$ generates a C_0-semigroup $T(t) = e^{A_\gamma t}$ such that $\|T(t)\| \leq Ce^{(\omega + C\|\Gamma_\gamma\|)t}$. ■

Theorem 2 *The operator A_γ has compact resolvents and* $0 \in \rho\left(A_\gamma\right)$.

Proof. The first assertion is trivial. It remains only to show that $0 \in \rho\left(A_\gamma\right)$. We prove that A_γ^{-1} exists. For any $\Psi = (g_1,\ g_2)^T \in \mathbb{H}$, it is enough to find a unique $\Phi = (f_1,\ f_2)^T \in D(A)$ such that $A\Phi = \Psi$ which yields

$$\begin{cases} f_2(x) = g_1(x), \quad g_1 \in H_E^2(0,1) \\ \left(EI(x) f_1''(x)\right)'' = -m(x) g_2(x) - \gamma(x) f_2(x), \quad g_2 \in L^2(0,1) \\ f_1(0) = f_1'(0) = (EI(.)f_1''(.))'(1) = 0 \\ -EI(1) f_1''(1) = \alpha f_2'(1) + \beta f_1'(1) = \alpha g_1'(1) + \beta f_1'(1). \end{cases}$$

The solution of above system is obtained straightforward after computation:

$$\begin{cases} f_2(x) = g_1(x) \\ f_1(x) = -\displaystyle\int_0^x \int_0^s \Big[\frac{\beta f_1'(1) + \alpha g_1'(1)}{EI(\xi)} + \frac{1}{EI(\xi)} \int_\eta^1 \int_x^1 m(r) g_2(r) + \gamma(r) g_1(r)\, dr d\eta\Big] d\xi ds. \end{cases}$$

Therefore, $0 \in \rho\left(A_\gamma\right)$. Moreover, using Sobolev's Embedding Theorem, we deduce that A_γ^{-1} is a compact operator on the Hilbert space $\mathbb{H}$. ■

3. Spectral Analysis of Operator and Riesz Basis Property

3.1 Spectral Analysis of Operator A_γ

Spectral analysis is one of the methods used today to determine the behavior of eigenvalues of operators of dynamic systems. In what follows, we will rely on idea of Wang et al (Wang, Xu & Yung, 2005) in order to study the eigenvalues problem associated to system (1)-(4).

But first, we recall the following definitions and notations useful in the sequel.

Let $L(f)$ be an ordinary differential operator of order $n = 2m \in \mathbb{N}$ defined as

$$L(f) = f^{(n)}(x) + \sum_{\nu=1}^{n} f_\nu(x) f^{(n-\nu)}(x),\ 0 < x < 1, \tag{15}$$

under the following boundary conditions

$$B_j(f) = \sum_{\nu=0}^{k_j} \left(\alpha_{j_\nu} f^{(k_j-\nu)}(0) + \beta_{j_\nu} f^{(k_j-\nu)}(1)\right),\ 1 \leq j \leq n, \tag{16}$$

where $k_j \in \mathbb{N}$, $1 \leq k_j \leq n-1$ and $\alpha_{j_\nu}, \beta_{j_\nu} \in \mathbb{C}$, $\left|\alpha_{j_0}\right| + \left|\beta_{j_0}\right| > 0$. Assume that the coefficient functions $f_\nu(x)$ $(1 \leq \nu \leq n)$ in (15) are sufficiently smooth in $(0,1)$, and that the boundary conditions are normalized in the sense that $K = \sum_{j=1}^{n} k_j$ is minimal with respect to all equivalent boundary conditions (see Naimark, 1967).

Let $f_k(x,\rho)$ $(k = 1,2,\ldots,n)$ be the fundamental solutions for the equation:

$$L(f) + \rho^n f + \rho^m \mu(x) f(x) = 0,\ \rho \in \mathbb{C} \tag{17}$$

where $\mu(x)$ being continuous in $[0,1]$. We denote the n-th roots of $\omega^n+1=0$ by ω_k $(k=1,2,\ldots,n)$ and the characteristic determinant of (17) under the boundary conditions (16) by $\Delta(\rho)$ defined as follows

$$\Delta(\rho)=\det\left[B_j(f_k(.,\rho))\right]_{j,k=1,2,\ldots,n}.$$

Moreover, asymptotically, $\Delta(\rho)$ can be rewritten in the following form for $(r\geq 1)$

$$\Delta(\rho)=\rho^k\sum_{\mathbb{k}_k}e^{\rho\mu_{\mathbb{k}_k}}\left[F^{\mathbb{k}_k}\right]_r, \tag{18}$$

whenever ρ is large enough (see Shkalikov, 1986; Naimark, 1967). Here, $\mathbb{k}_k$ is a k-elements subset of $\{1,2,\ldots,n\}$, $\mu_{\mathbb{k}_k}=\sum_{j\in\mathbb{k}_k}\omega_j$,

$$\left[F^{\mathbb{k}_k}\right]_r=F_0^{\mathbb{k}_k}+\rho^{-1}F_1^{\mathbb{k}_k}+\ldots+\rho^{-r+1}F_{r-1}^{\mathbb{k}_k}+O(\rho^{-r}),$$

and the sum runs over all possible selections of $\mathbb{k}_k$. Here and henceforth, $O(\rho^{-r})$ means that $|\rho^r\times O(\rho^{-r})|$ is bounded as $|\rho|\to\infty$.

The following definition is in (Wang, Xu & Yung, 2005, pp.461).

Definition 3 *The boundary problem* (17) *with* (16) *is said to be regular if the coefficients* $F_0^{\mathbb{k}_k}$ *in* (18) *are nonzero. Furthermore, the regular boundary problem* (17) *with* (16) *is said to be strongly regular if the zeros of* $\Delta(\rho)$ *are asymptotically simple and isolated one from another.*

Let $W_2^m(0,1)$ be the usual Sobolev space of order m and let

$$V_E^m(0,1)=\left\{f(x)\in W_2^m(0,1)\,|\,B_j(f)=0,\quad k_j<m\right\}.$$

Let H be a Hilbert space defined as

$$H=V_E^m(0,1)\times L^2(0,1),$$

with

$$\|(f,g)\|_H^2=\|f\|_{W_2^m}^2+\|g\|_2^2$$

which denotes its corresponding norm and let $\mathbb{A}$ be a operator in H defined by

$$\begin{cases}\mathbb{A}(f,g)=(g,-L(f)-\mu(x)g)\\ D(\mathbb{A})=\{(f,g)\in H\,|\,\mathbb{A}(f,g)\in H,\ B_j(f)=0,\ k_j\geq m\}.\end{cases} \tag{19}$$

The Theorem **4** used in (Wang, Xu & Yung, 2005) was presented in (Wang, 2003). The reader may also refer to chapter 3 of (Wang, 2004).

Theorem 4 *If the ordinary differential system with parameter* $\lambda=\rho^m$

$$\begin{cases}L(f,\lambda)=L(f)+\lambda^2f+\lambda\mu(x)f\\ B_j(f)=0,\quad 1\leq j\leq 2m\end{cases} \tag{20}$$

has strongly regular boundary conditions, then the generalized eigenfunction system of $\mathbb{A}$ *form a Riesz basis in the Hilbert space* H.

According to Theorem **2**, A_γ has a compact resolvent. Thus, the spectrum of A denoted by $\sigma(A_\gamma)$, consists only of isolated eigenvalues, which distribute in conjugate pairs on the complex plane.

Now, the eigenvalue problem of operator A_γ can be investigated. Let λ be an eigenvalue of the spectrum $\sigma(A_\gamma)$ and $\Phi=(\phi,\Psi)$ denoting its corresponding eigenfunction. Thus, $\Psi=\lambda\phi$ with ϕ satisfies the following equations:

$$\begin{cases}\lambda^2m(x)\phi(x)+(EI(x)\phi''(x))''+\lambda\gamma(x)\phi(x)=0,\quad 0<x<1,\\ \phi(0)=\phi'(0)=(EI(.)\phi''(.))'(1)=0\\ \phi''(1)=-\dfrac{1}{EI(1)}(\alpha\lambda+\beta)\phi'(1).\end{cases} \tag{21}$$

In order to solve (21), spatial transformations as introduced in (Guo, 2002) are performed, which convert the first equation of (21) into a more convenient form. For this purpose, for $0 < x < 1$, the system (21) is firstly rewritten as :

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)}\phi(x) + \dfrac{\lambda\gamma(x)}{EI(x)}\phi(x) = 0, \\ \phi(0) = \phi'(0) = \left(EI(.)\phi''(.)\right)'(1) = 0, \\ \phi''(1) = -\dfrac{1}{EI(1)}(\alpha\lambda + \beta)\phi'(1). \end{cases} \tag{22}$$

Moreover, introduce the following space transformation in order to transform the coefficient function appearing with ϕ in the first expression of (22) into a constant. Let

$$f(z) = \phi(x), \quad z = z(x) = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta \tag{23}$$

with

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta. \tag{24}$$

Thus, using again its boundary conditions, the system (22) can be transformed as

$$\begin{cases} f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda^2 h^4 f(z) + \lambda h^4 d(z) f(z) = 0, 0 < z < 1, \\ f(0) = f'(0) = 0, \\ EI(1)z_x^3(1) f'''(1) + \left[EI'(1)z_x^2(1) + 3EI(1)z_{xx}(1)z_x(1)\right] f''(1) + \left[EI(1)z_{xxx}(1) + EI'(1)z_{xx}(1)\right] f'(1) = 0, \\ f''(1) + \left[\dfrac{z_{xx}(1)}{z_x^2(1)} + \dfrac{\alpha\lambda + \beta}{EI(1)z_x(1)}\right] f'(1) = 0, \end{cases} \tag{25}$$

with

$$a(z) = \frac{6z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)} \tag{26}$$

$$b(z) = \frac{3z_{xx}^2}{z_x^4} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI''(x)}{z_x^2 EI(x)} + \frac{4z_{xxx}}{z_x^3} \tag{27}$$

$$c(z) = \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)} \tag{28}$$

$$d(z) = \frac{\gamma(x)}{m(x)} \tag{29}$$

$$z_x = \frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}, \quad z_x^4 = \frac{1}{h^4}\frac{m(x)}{EI(x)} \tag{30}$$

and

$$z_{xx} = \frac{1}{4h}\left(\frac{m(x)}{EI(x)}\right)^{-\frac{3}{4}} \frac{d}{dx}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}. \tag{31}$$

Next, we use the idea of Naimark presented in Chapter 2 of (Naimark, 1967) for solve (25). Then, in order to cancel the third derivative term $a(z) f'''(z)$ in (25), we introduce a new invertible space transformation

$$g(z) = \exp\left(\frac{1}{4}\int_0^z a(\zeta)\, d\zeta\right) f(z), \quad 0 < z < 1.$$

(25) can be written as follows, for any $0 < z < 1$:

$$\begin{cases} g^{(4)}(z) + b_1(z) g''(z) + c_1(z) g'(z) + d_1(z) g(z) + \lambda^2 h^4 g(z) + \lambda h^4 d(z) g(z) = 0, \\ g(0) = g'(0) = 0, \\ g''(1) + b_{11} g'(1) + b_{12} g(1) = 0, \\ g'''(1) + b_{21} g''(1) + b_{22} g'(1) + b_{23} g(1) = 0, \end{cases} \tag{32}$$

where

$$b_1(z) = -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z) \tag{33}$$

$$c_1(z) = \frac{1}{8}a^3(z) - \frac{1}{2}a(z)b(z) - a''(z) + c(z) \tag{34}$$

$$d_1(z) = \frac{3}{16}a'^2(z) - \frac{1}{4}a'''(z) + \frac{3}{32}a'(z)a^2(z) - \frac{3}{256}a^4(z) + b(z)\left(\frac{1}{16}a^2(z) - \frac{1}{4}a'(z)\right) - \frac{a(z)c(z)}{4} \tag{35}$$

$$b_{11} = -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha\lambda}{EI(1)z_x(1)} + \frac{\beta}{EI(1)z_x(1)} \tag{36}$$

$$b_{12} = -\frac{1}{4}a'(1) + \frac{1}{16}a^2(1) - \frac{1}{4}a(1)\left(\frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha\lambda}{EI(1)z_x(1)} + \frac{\beta}{EI(1)z_x(1)}\right) \tag{37}$$

$$b_{21} = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{EI(1)z_x(1)} \tag{38}$$

$$b_{22} = -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{EI'(1)a(1)}{2EI(1)z_x(1)} - \frac{3z_{xx}(1)a(1)}{2z_x^2(1)} + \frac{z_{xxx}(1)}{z_x^3(1)} + \frac{EI'(1)z_{xx}(1)}{EI(1)z_x^3(1)} \tag{39}$$

$$\begin{aligned} b_{23} = & -\frac{1}{4}a''(1) + \frac{3}{16}a'(1)a(1) - \frac{1}{64}a^3(1) - \frac{a'(1)EI'(1)}{4EI(1)z_x(1)} \\ & -\frac{3a'(1)z_{xx}(1)}{4z_x^2(1)} + \frac{EI'(1)a^2(1)}{16EI(1)z_x(1)} + \frac{3z_{xx}(1)a^2(1)}{16z_x^2(1)} - \frac{a(1)z_{xxx}(1)}{4z_x^3(1)}. \end{aligned} \tag{40}$$

Since the above transformations are invertible, the obtained system (32) is equivalent to the original problem (21).

To further solve the eigenvalue problem (32), the complex plane is divided into eight distinct sectors:

$$S_n = \left\{z \in \mathbb{C} : \frac{n\pi}{4} \leq \arg z \leq \frac{(n+1)\pi}{4}\right\},\ n = 0, 1, 2, \ldots, 7. \tag{41}$$

Moreover, we denote the roots of equation $\theta^4 + 1 = 0$ by ω_1, ω_2, ω_3, ω_4 such that inequalities holds

$$\mathrm{Re}\,(\rho\omega_1) \leq \mathrm{Re}\,(\rho\omega_2) \leq \mathrm{Re}\,(\rho\omega_3) \leq \mathrm{Re}\,(\rho\omega_4)\,, \quad \forall \rho \in S_n. \tag{42}$$

Clearly, the choices in the sector S_1 satisfying (42) are given as follows

$$\omega_1 = \exp\left(i\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 = \exp\left(i\frac{1}{4}\pi\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_3 = \exp\left(i\frac{5}{4}\pi\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 = \exp\left(i\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

Notice that, similary, the choices can be obtained for other sectors. In the following, we study the asymptotic behavior of the eigenvalues only for the sectors S_1 and S_2 because the work done in these two sectors is valid in the other sectors.

Set $\lambda = \frac{\rho^2}{h^2}$, in each sector S_n. In order to analyse the asymptotic fundamental solutions of system (32), we will use the following Lemma (see Naimark, 1967; Wang, Xu & Yung, 2005) :

Lemma 5 *For $\rho \in S_n$ with ρ large enough, the equation:*

$$g^{(4)}(z) + b_1(z)\,g''(z) + c_1(z)\,g'(z) + d_1(z)\,g(z) + \rho^4 g(z) + \rho^2 h^2 d(z) g(z) = 0, \ \ 0 < z < 1,$$

has four linearly independent asymptotic fundamental solutions,

$$\Phi_s(z,\rho) = e^{\rho\omega_s z}\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + O\left(\rho^{-2}\right)\right), \quad s = 1, 2, 3, 4$$

and hence their derivatives for $s = 1, 2, 3, 4$ and $j = 1, 2, 3$ are given by

$$\frac{d^j}{dz^j}\Phi_s(z,\rho) = (\rho\omega_s)^j\, e^{\rho\omega_s z}\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + O\left(\rho^{-2}\right)\right)$$

where

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^z b_1(\zeta)\,d\zeta - \frac{h^2}{4\omega_s^3}\int_0^z d(\zeta)\,d\zeta.$$

Hence, for $s = 1, 2, 3, 4$,

$$\Phi_{s,1}(0) = 0,\;\; \Phi_{s,1}(1) = -\frac{1}{4\omega_s}\int_0^1 b_1(\zeta)\,d\zeta = \frac{\omega_s^2\mu_1 + \mu_2}{\omega_s^3},\; with\; \mu_1 = -\frac{1}{4}\int_0^1 b_1(\zeta)\,d\zeta \; and\; \mu_2 = -\frac{h^2}{4}\int_0^1 d(\zeta)\,d\zeta.$$

Proof. For the proof, the reader can refer to (Touré, Coulibaly & H. Kouassi, 2015). ■

In the following, we will use the notation:

$$[a]_2 = a + O\left(\rho^{-2}\right).$$

We also need the following Lemma:

Lemma 6 *For* $\rho \in S_1$, *if we set* $\delta = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, *then the inequalities holds*

$$Re(\rho\omega_1) \le -|\rho|\,\delta,\; Re(\rho\omega_4) \ge |\rho|\,\delta \; and\; e^{\rho\omega_1} = O\left(\rho^{-2}\right)\; when\;\; |\rho| \to \infty.$$

Set $\kappa = -\frac{\alpha a(1)}{4EI(1)z_x(1)h}$. Using Lemma **5**, the asymptotic expressions for the boundary conditions for large enough $|\rho|$, are obtained for $s = 1, 2, 3, 4$,

$$U_4(\Phi_s,\rho) = \Phi_s(0,\rho) = 1 + O\left(\rho^{-2}\right) = [1]_2,$$
$$U_3(\Phi_s,\rho) = \Phi_s'(0,\rho) = \rho\omega_s\left(1 + O\left(\rho^{-2}\right)\right) = \rho\omega_s[1]_2,$$
$$U_2(\Phi_s,\rho) = \Phi_s''(1,\rho) + b_{11}\Phi_s'(1,\rho) + b_{12}\Phi_s(1,\rho),$$
$$U_2(\Phi_s,\rho) = (\rho\omega_s)^2 e^{\rho\omega_s}\left(1 + \kappa\omega_s^{-2} + (\mu_1\omega_s^2 + \mu_2)\rho^{-1}\omega_s^{-3} + \kappa(\omega_s^2\mu_1 + \mu_2)\omega_s^{-4}\rho^{-1} + O\left(\rho^{-2}\right)\right),$$
$$U_2(\Phi_s,\rho) = (\rho\omega_s)^2 e^{\rho\omega_s}\left[1 + \kappa\omega_s^{-2} + (\mu_1\omega_s^2 + \mu_2)\rho^{-1}\omega_s^{-3} + \kappa(\omega_s^2\mu_1 + \mu_2)\omega_s^{-4}\rho^{-1}\right]_2,$$
$$U_1(\Phi_s,\rho) = \Phi_s'''(1,\rho) + b_{21}\Phi_s''(1,\rho) + b_{22}\Phi_s'(1,\rho) + b_{23}\Phi_s(1,\rho),$$
$$U_1(\Phi_s,\rho) = (\rho\omega_s)^3 e^{\rho\omega_s}\left(1 + \left(\mu_1 + b_{21} + \mu_2\omega_s^{-2}\right)\rho^{-1}\omega_s^{-1} + O\left(\rho^{-2}\right)\right),$$
$$U_1(\Phi_s,\rho) = (\rho\omega_s)^3 e^{\rho\omega_s}\left[1 + \left(\mu_1 + b_{21} + \mu_2\omega_s^{-2}\right)\rho^{-1}\omega_s^{-1}\right]_2.$$

Notice that $\lambda \neq 0$ is the eigenvalue of (32) if and only if ρ satisfies the characteristic equation

$$\Delta(\rho) = \begin{vmatrix} U_4(\Phi_1,\rho) & U_4(\Phi_2,\rho) & U_4(\Phi_3,\rho) & U_4(\Phi_4,\rho) \\ U_3(\Phi_1,\rho) & U_3(\Phi_2,\rho) & U_3(\Phi_3,\rho) & U_3(\Phi_4,\rho) \\ U_2(\Phi_1,\rho) & U_2(\Phi_2,\rho) & U_2(\Phi_3,\rho) & U_2(\Phi_4,\rho) \\ U_1(\Phi_1,\rho) & U_1(\Phi_2,\rho) & U_1(\Phi_3,\rho) & U_1(\Phi_4,\rho) \end{vmatrix} = 0. \tag{43}$$

By substitution, the following expression is obtained

$$\Delta(\rho) = \begin{vmatrix} [1]_2 & [1]_2 \\ \rho\omega_1[1]_2 & \rho\omega_2[1]_2 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2}\left[1 + \kappa\omega_2^{-2} + (\mu_1\omega_2^2 + \mu_2)\rho^{-1}\omega_2^{-3} + \kappa(\omega_2^2\mu_1 + \mu_2)\omega_2^{-4}\rho^{-1}\right]_2 \\ 0 & (\rho\omega_2)^3 e^{\rho\omega_2}\left[1 + \left(\mu_1 + b_{21} + \mu_2\omega_2^{-2}\right)\rho^{-1}\omega_2^{-1}\right]_2 \end{vmatrix}$$

$$\begin{matrix} [1]_2 \\ \rho\omega_3[1]_2 \\ (\rho\omega_3)^2 e^{\rho\omega_3}\left[1 + \kappa\omega_3^{-2} + (\mu_1\omega_3^2 + \mu_2)\rho^{-1}\omega_3^{-3} + \kappa(\omega_3^2\mu_1 + \mu_2)\omega_3^{-4}\rho^{-1}\right]_2 \\ (\rho\omega_3)^3 e^{\rho\omega_3}\left[1 + \left(\mu_1 + b_{21} + \mu_2\omega_3^{-2}\right)\rho^{-1}\omega_3^{-1}\right]_2 \end{matrix}$$

$$\begin{array}{c} 0 \\ 0 \\ (\rho\omega_4)^2 e^{\rho\omega_4}\left[1+\kappa\omega_4^{-2}+(\mu_1\omega_4^2+\mu_2)\rho^{-1}\omega_4^{-3}+\kappa(\omega_4^2\mu_1+\mu_2)\omega_4^{-4}\rho^{-1}\right]_2 \\ (\rho\omega_4)^3 e^{\rho\omega_4}\left[1+\left(\mu_1+b_{21}+\mu_2\omega_4^{-2}\right)\rho^{-1}\omega_4^{-1}\right]_2 \end{array}\Bigg| .$$

Developing the determinant, we obtain:

$$\begin{aligned}\Delta(\rho) = \rho^6 e^{\rho\omega_4}\Big\{&(-1)(\omega_3-\omega_1)[\kappa(\omega_2^{-2}-\omega_4^{-2})+(\mu_1+b_{21}+\mu_2\omega_4^{-2})(\omega_4^{-1}-\omega_2^{-1})\rho^{-1}+\kappa(\mu_1+b_{21}+\mu_2\omega_2^{-2})(\omega_2^{-2}\omega_4^{-1}-\omega_4^{-2}\omega_2^{-1})\rho^{-1}\\ &+(\mu_1+\mu_2\omega_4^{-2})(\omega_2^{-1}-\omega_4^{-1})\rho^{-1}+\kappa(\mu_1+\mu_2\omega_2^{-2})(\omega_2^{-3}-\omega_4^{-3})\rho^{-1}]e^{\rho\omega_2}\\ &+(\omega_2-\omega_1)[\kappa(\omega_3^{-2}-\omega_4^{-2})+(\mu_1+b_{21}+\mu_2\omega_4^{-2})(\omega_4^{-1}-\omega_3^{-1})\rho^{-1}+\kappa(\mu_1+b_{21}+\mu_2\omega_3^{-2})(\omega_3^{-2}\omega_4^{-1}-\omega_4^{-2}\omega_3^{-1})\rho^{-1}\\ &+(\mu_1+\mu_2\omega_4^{-2})(\omega_3^{-1}-\omega_4^{-1})\rho^{-1}+\kappa(\mu_1+\mu_2\omega_3^{-2})(\omega_3^{-3}-\omega_4^{-3})\rho^{-1}]e^{-\rho\omega_2}+O\left(\rho^{-2}\right)\Big\}.\end{aligned}$$

We have the following choices in S_1:

$\omega_1^2=-i,\ \omega_2^2=i,\ \omega_3^2=i,\ \omega_4^2=-i,\ \omega_3^{-1}\omega_4=i,\ \omega_2^{-1}\omega_4=-i,\ \omega_3=-\omega_2, \omega_4-\omega_3=\sqrt{2},\ \omega_1-\omega_3=\sqrt{2}i,$ $\omega_2-\omega_1=\sqrt{2},\ \omega_4-\omega_2=-i\sqrt{2}, \omega_2^{-2}-\omega_4^{-2}=-2i,\ \omega_3^{-2}-\omega_4^{-2}=-2i,\ \omega_3^2\omega_4^2=1,\ \omega_2^2\omega_4^2=1,\ \omega_2^{-3}-\omega_4^{-3}=-(1+i)\omega_2,$ $\omega_3^{-3}-\omega_4^{-3}=(1-i)\omega_2.$

Substituting the previous values into $\Delta(\rho)$, we get:

$$\Delta(\rho)=2\sqrt{2}\,\kappa\rho^6e^{\rho\omega_4}\Big\{e^{\rho\omega_2}-i\,e^{-\rho\omega_2}+[\mu_3e^{\rho\omega_2}+\mu_4e^{-\rho\omega_2}]\rho^{-1}+O\left(\rho^{-2}\right)\Big\}, \tag{44}$$

where

$$\begin{cases}\mu_3=\dfrac{\sqrt{2}}{2}\Big[2\mu_1+2\mu_2+b_{21}-\dfrac{b_{21}}{\gamma}\Big]\\ \mu_4=\dfrac{\sqrt{2}}{2}\Big[2\mu_1-2\mu_2+b_{21}+\dfrac{b_{21}}{\gamma}\Big]\end{cases} \tag{45}$$

Theorem 7 *If $\kappa\neq 0$, the boundary eigenvalue problem* (32) *is strongly regular.*

Proof. We have

$$\Theta_{-1,0}=-2\sqrt{2}i\kappa \quad \Theta_{1,0}=2\sqrt{2}\kappa \quad \Theta_{0,0}=0.$$

Therefore, using the Definition **3**, the eigenvalue problem (32) is strongly regular. ■

Next, we study the asymptotic behavior of λ_n. The equation $\Delta(\rho)=0$ implies that

$$e^{\rho\omega_2}-i\,e^{-\rho\omega_2}+\mu_3\rho^{-1}e^{\rho\omega_2}+\mu_4\rho^{-1}e^{-\rho\omega_2}+O\left(\rho^{-2}\right)=0. \tag{46}$$

(46) can be rewritten as follows

$$e^{\rho\omega_2}-ie^{-\rho\omega_2}+O\left(\rho^{-1}\right)=0. \tag{47}$$

Remark that the solutions of equation

$$e^{\rho\omega_2}-ie^{-\rho\omega_2}=0$$

are in the form

$$\rho_n=\left(\frac{1}{4}-n\right)\frac{\pi i}{\omega_2},\ n=1,2,\ldots \tag{48}$$

Let $\widetilde{\rho_n}$ be the solutions of (47). Using Rouche's Theorem (see e.g. Krantz, 2008), we obtain:

$$\widetilde{\rho_n}=\rho_n+\alpha_n=\left(\frac{1}{4}-n\right)\frac{\pi i}{\omega_2}+\alpha_n,\quad \alpha_n=O\left(n^{-1}\right),\ n=N,N+1,\ldots, \tag{49}$$

where N is a large positive integer. Putting $\widetilde{\rho_n}$ into (46) and using this equality $e^{\rho\omega_2}=ie^{-\rho\omega_2}$, we get

$$e^{\alpha_n\omega_2}-e^{-\alpha_n\omega_2}+\mu_3\widetilde{\rho_n}^{-1}e^{\alpha_n\omega_2}-i\mu_4\widetilde{\rho_n}^{-1}e^{-\alpha_n\omega_2}+O\left(\widetilde{\rho_n}^{-2}\right)=0.$$

Moreover, expanding the exponential function according to its Taylor series, we get

$$\alpha_n = -\frac{\mu_3}{2\omega_2\rho_n} + \frac{\mu_4}{2\omega_2\rho_n} i + O\left(n^{-2}\right),\ n = N, N+1, \ldots$$

Therefore, we have

$$\widetilde{\rho_n} = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2} + \frac{\mu_3}{2\left(\frac{1}{4} - n\right)\pi} i + \frac{\mu_4}{2\left(\frac{1}{4} - n\right)\pi} + O\left(n^{-2}\right),\ n = N, N+1, \ldots$$

Note that $\lambda_n = \dfrac{\widetilde{\rho_n}^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{\pi}{4}}$ and $\omega_2^2 = i$. So we have

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) + \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + O\left(n^{-1}\right), \tag{50}$$

where $n = N, N+1, \ldots$ with N large enough.

We make the same work for the sector S_2 because the eigenvalues of system (32) can be got by a similar computation with the following choices

$$\omega_1 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \quad \omega_2 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i,$$

$$\omega_3 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i, \quad \omega_4 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i,$$

such that the inequality (42) is satisfied

$$\mathrm{Re}\,(\rho\omega_1) \leq \mathrm{Re}\,(\rho\omega_2) \leq \mathrm{Re}\,(\rho\omega_3) \leq \mathrm{Re}\,(\rho\omega_4),\ \forall \rho \in S_2.$$

Hence, in sector S_2, the characteristic determinant $\Delta(\rho)$ of (21) :

$$\Delta(\rho) = 2\sqrt{2}\gamma\rho^6 e^{\rho\omega_4}\left\{e^{\rho\omega_2} + i\, e^{-\rho\omega_2} - [\mu_2 e^{\rho\omega_3} + \mu_4 e^{-\rho\omega_2}]\rho^{-1} + O\left(\rho^{-2}\right)\right\}.$$

After computation, we have

$$\widetilde{\rho_n} = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2} - \frac{\mu_3}{2\left(\frac{1}{4} - n\right)\pi} i + \frac{\mu_4}{2\left(\frac{1}{4} - n\right)\pi} + O\left(n^{-2}\right),\ n = N, N+1, \ldots \tag{51}$$

with N large enough. Again, using $\lambda_n = \dfrac{\widetilde{\rho_n}^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{3\pi}{4}}$ and $\omega_2^2 = -i$, we obtain as follows the conjugate eigenvalues of the problem (32)

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) - \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + O\left(n^{-1}\right), \tag{52}$$

where $n = N, N+1, \ldots$ with N large enough.

The expressions (50) and (52) give, in the following Theorem, the asymptotic expression on the eigenvalues:

Theorem 8 *Let A be the operator defined by* (12) *and* (13). *If $\kappa \neq 0$, then an asymptotic expression of the eigenvalues of the problem* (32) *is given by*

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) \pm \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + O\left(n^{-1}\right), \tag{53}$$

where $n = N, N+1, \ldots$ with N large enough, and

$$\mu_4 - \mu_3 = \sqrt{2}(-2\mu_2 + \frac{b_{21}}{\gamma}) = -2\sqrt{2}\mu_2 - \frac{1}{\alpha}\left(\sqrt{2}h(m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right) \tag{54}$$

where

$$\begin{aligned}\mu_2 &= -\frac{h^2}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx \\ &= -\frac{h}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx, \end{aligned} \tag{55}$$

and

$$\mu_4 + \mu_3 = \sqrt{2}(2\mu_1 + b_{21}). \tag{56}$$

Furthermore, λ_n $(n = N, N+1, \ldots)$ *with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy*

$$\lim_{n\to+\infty} Re\lambda_n = \frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{1}{\alpha h}\left((m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right). \tag{57}$$

Remark 1 This Theorem is a bit of surprise because although the beam is nonuniform, we obtain an asymptotic uniform rate of decay in terms of the viscous damping function and thus the important question asked in introduction is answered.

Moreover, with reference to (Naimark, 1967), we can say that the eigenvalues generated by the other sectors S_n coincide with those determined in the sectors S_1 and S_2.

3.2 Riesz Basis Property of the Generalized Eigenfunctions of A_γ

In what follows, we follow an idea of Wang in (Wang, Xu & Yung, 2005) in order to discuss the Riesz basis property of the eigenfunctions of operator A_γ of system (14). To begin, we prove that the generalized eigenfunctions of A form an unconditional basis in the Hilbert space H. Thus let us choose a transformation $\mathcal{L}$ such that

$$\mathcal{L}(f, g) = (\phi, \psi)$$

with

$$\phi(x) = f(z),\ \psi(x) = g(z),\ z = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta, \tag{58}$$

and

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta. \tag{59}$$

Notice that $\mathcal{L}$ is invertible and is a bounded operator on $\mathbb{H}$. Also, we define the following ordinary differential operator:

$$\begin{cases} L(f) = f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z), \\ \mu(z) = h^2 d(z), \\ B_1(f) = f(0) = 0,\ \ B_2(f) = f'(0) = 0, \\ B_3(f) = f''(1) + \left[\frac{z_{xx}(1)}{z_x^2(1)} + \frac{(\alpha\lambda+\beta)}{EI(1)z_x(1)}\right]f'(1) = 0, \\ B_4(f) = EI(1)z_x^3(1)f'''(1) + \left[EI'(1)z_x^2 + 3EI(1)z_{xx}(1)z_x(1)\right]f''(1) + \left[EI(1)z_{xxx}(1) + EI'(1)z_{xx}(1)\right]f'(1) = 0, \end{cases} \tag{60}$$

where the coefficients are defined by (25)-(29). Let $\mathbb{A}$ be defined as in (19), $\eta \in \sigma(\mathbb{A})$ be an eigenvalue of $\mathbb{A}$ and (f, g) be an eigenfunction corresponding to η, then we obtain $g = \eta f$ and f satisfies the following equation:

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \eta\mu(z)f(z) + \eta^2 f(z) = 0,$$

with boundary conditions $B_j(f) = 0,\ j = 1, 2, 3, 4$. Now, when we take $\lambda = \dfrac{\eta}{h^2}$ and

$$\mathcal{L}(f, g) = (\phi(x), \psi(x))$$

we obtain that $\psi = \lambda\phi$ and ϕ satisfies

$$\begin{cases} \phi^{(4)}(x) + \frac{2EI'(x)}{EI(x)}\phi'''(x) + \frac{EI''(x)}{EI(x)}\phi''(x) + \lambda\frac{\gamma(x)}{EI(x)}\phi(x) + \frac{\lambda^2 m(x)}{EI(x)}\phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = (EI(.)\phi'')'(1) = 0, \\ \phi''(1) = -\frac{1}{EI(1)}(\alpha\lambda+\beta)\phi'(1). \end{cases} \tag{61}$$

Thus, the following result is obtained: $\eta \in \sigma(\mathbb{A}) \Leftrightarrow \lambda \in \sigma\left(A_\gamma\right)$.

Theorem 9 *Let A_γ be defined by* (12) *and* (13). *Thus, A_γ has the eigenvalues which are all simple except for finitely many of them, and the generalized eigenfunctions of operator A_γ form on $\mathbb{H}$ a Riesz basis.*

Proof. The Theorem **7** underlines that the boundary problem (32) is strongly regular. Then, the first statement follows. Moreover, using Theorem **4**, we obtain that the generalized eigenfunction sequence $F_n = (f_n, \eta_n f_n)$ of operator $\mathbb{A}$ forms a Riesz basis for $\mathbb{H}$. Also, $\mathcal{L}$ being an invertible and bounded operator on $\mathbb{H}$, we can deduce that $\Psi_n = (\phi_n, \lambda_n \phi_n) = \mathcal{L} F_n$ forms on $\mathbb{H}$ a Riesz basis.

Now, we are moving on to the study of the exponential stability of system (14). Referring to (Curtain & Zwart, 1995), we note that the Riesz basis property implies the spectrum-determined growth condition and (57) describes the asymptote of $\sigma\left(A_\gamma\right)$, for any small $\varepsilon > 0$ there are only finitely many eigenvalues of A in the following half-plane:

$$\Sigma : \mathrm{Re}\lambda > \frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{1}{\alpha h}\left((m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right) + \varepsilon. \tag{62}$$

4. Exponential Stability

According to idea of Theorem 2.4 of (Guo, 2002), all the properties of operator A_γ obtained previously allow us to say that for the semigroup $e^{A_\gamma t}$ generated by A_γ, the spectrum-determined growth condition is satisfied:

$$\omega\left(A_\gamma\right) = s\left(A_\gamma\right),$$

with

$$\omega\left(A_\gamma\right) = \lim_{t\to+\infty}\frac{1}{t}\left\|e^{A_\gamma t}\right\|_H$$

which is called the growth order of $e^{A_\gamma t}$ and

$$s\left(A_\gamma\right) = \sup\left\{\mathrm{Re}\lambda | \lambda \in \sigma\left(A_\gamma\right)\right\}$$

which is called the spectral bound of A_γ.

We will use the Theorem **9** which is fundamental property to conclude the exponential stability of the system (1)−(4).

Theorem 10 *If $\gamma(x) > 0$, the system* (1)−(4) *is exponential stable for all $\beta > 0$ and $\alpha > 0$. Thus, there exists the constants $M > 0$ and $\omega > 0$ such that the energy $E(t)$ of system* (1)−(4) *satisfies*

$$E(t) = \frac{1}{2}\int_0^1 EIw_{xx}^2\,dx + \frac{1}{2}\int_0^1 mw_t^2\,dx + \frac{\beta}{2}(w_x(1,t))^2 \le ME(0)e^{-\omega t},\ \forall t \ge 0,$$

for all initial condition $(w(x,0), w_t(x,0)) \in \mathbb{H}$.

Proof. To begin, A_γ is a dissipative operator. Indeed, for all $w = (f,g)^T \in D(A)$,

$$< A_\gamma w, w >= \left\langle \left(g(x), -\frac{1}{m(x)}(EI(x)f''(x))'' + \gamma(x)g(x)\right), (f,g)\right\rangle$$

$$< A_\gamma w, w >= -\int_0^1 \left[(EI(x)f''(x))''\overline{g(x)} + \gamma(x)|g(x)|^2\right]dx + \int_0^1 EI(x)g''(x)\overline{f''(x)}\,dx + \beta g'(1)\overline{f'(1)}.$$

Integrating by parts, we obtain

$$< A_\gamma w, w >= -\int_0^1 EI(x)[g''(x)\overline{f''(x)} - f''(x)\overline{g''(x)}]\,dx + \beta(g'(1)\overline{f'(1)} - f'(1)\overline{g'(1)}) - \alpha|g'(1)|^2 - \int_0^1 \gamma(x)|g(x)|^2\,dx.$$

Taking real parts, we obtain $Re < A_\gamma w, w >= -\alpha|g'(1)|^2 - \int_0^1 \gamma(x)|g(x)|^2\,dx < 0$. Thus, A_γ is a dissipative operator and $e^{A_\gamma t}$ is a semigroup of contraction on $\mathbb{H}$. Furthermore, there exists an asymptote

$$\mathrm{Re}\lambda \sim \frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{1}{\alpha h}\left((m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right).$$

for the spectrum of A_γ. In order to conclude the exponential stability, it remains to show that there is no eigenvalue on the imaginary axis. Let $\lambda = i\tau$ where $\tau \in \mathbb{R}^*$ is an eigenvalue of operator A_γ on the imaginary axis. Let $\Psi = (\phi, \psi)^T$ be the corresponding eigenfunction. So, $\psi = \lambda\phi$. Then we have

$$0 = \mathrm{Re}\left(\left\langle A_\gamma \Psi, \Psi \right\rangle_{\mathbb{H}}\right) = -\alpha \left|\psi'(1)\right|^2 - \int_0^1 \gamma(x)|\psi(x)|^2\, dx,$$

$$0 = \|\Psi\|^2_{\mathbb{H}} \,\mathrm{Re}(\lambda) = \mathrm{Re}\left(\left\langle A_\gamma \Psi, \Psi \right\rangle_{\mathbb{H}}\right) = -\alpha \left|\psi'(1)\right|^2 - \int_0^1 \gamma(x)|\psi(x)|^2\, dx,$$

since $\gamma(x) > 0$ and $\psi(x)$ are continuous with $\alpha > 0$, we get

$$\psi'(1) = 0$$

and

$$\gamma(x)|\psi(x)|^2 = 0, \quad \forall x \in [0, 1].$$

Since $\gamma > 0$, we have $\psi \equiv 0$ which implies that Ψ of A_γ is zero because $\psi = \lambda\phi$ and the following differential equation is satisfied by $\phi(x)$:

$$\begin{cases} \lambda^2 m(x)\phi(x) + (EI(x)\,\phi''(x))'' + \lambda\gamma(x)\phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = \phi''(1) = \phi'''(1) = 0. \end{cases} \tag{63}$$

The above equation has a zero solution only (see Touré, Coulibaly & Kouassi, 2015). However, if $\Psi \equiv 0$ then we have a contradiction because Ψ is an eigenfunction and thus there exists no eigenvalue on the imaginary axis. Consequently, $\mathrm{Re}(\lambda) < 0$.

Due to the spectrum-determined growth condition and the Theorem **9**, we can conclude the exponential stability of system (1)−(4). ■

Next, using an idea of (Wang, 2004), we study the situation where $\gamma(x)$ is continuous and indefinite in $[0, 1]$. We have the following Theorem:

Theorem 11 *Let*

$$\gamma_+(x) = \max\{\gamma(x), 0\}, \quad \gamma_-(x) = \max\{-\gamma(x), 0\},$$

and let

$$A_{\gamma_+}(f, g) = \left(g(x), -\frac{1}{m(x)}((EI(x)\,f''(x))'' + \gamma_+(x)\,g(x))\right)^T, \ \forall (f, g) \in D\left(A_{\gamma_+}\right) = D\left(A_\gamma\right)$$

and

$$\Gamma_-(f, g) = \left(0, \frac{\gamma_-(x)}{m(x)} g(x)\right)^T, \ \forall (f, g) \in H.$$

Hence A_γ can be written as $A_\gamma = A_{\gamma_+} + \Gamma_-$. Let $s\left(A_{\gamma_+}\right) = \sup\left\{Re\lambda | \lambda \in \sigma\left(A_{\gamma+}\right)\right\}$. If

$$\max_{x\in[0,1]} \left\{\frac{\gamma_-(x)}{m(x)}\right\} < \left|s\left(A_{\gamma+}\right)\right|,$$

then we obtain the exponential stablility of system (14).

Proof. It is easily to see that Γ_- is self-adjoint operator and

$$\|\Gamma_-\| = \max_{x\in[0,1]} \left\{\frac{\gamma_-(x)}{m(x)}\right\}. \tag{64}$$

According to the Theorem **10** and definition of operator A_{γ_+}, $e^{A_{\gamma_+} t}$ is a semigroup of contraction and $s\left(A_{\gamma_+}\right) < 0$. Furthermore, due to perturbation theory of linear operators semigroup, for example in the Theorem 1.1 pp. 76 of (Pazy, 1983), we obtain $\lambda \in \rho\left(A_\gamma\right)$ whenever $\mathrm{Re}\lambda > s\left(A_{\gamma_+}\right) + \|\Gamma_-\| < 0$. We have again by the Theorem **9** the following important result

$$\omega\left(A_\gamma\right) = s\left(A_\gamma\right) \leq s\left(A_{\gamma_+}\right) + \|\Gamma_-\| < 0.$$

Thus, we can conclude the exponential stability of system (14). ■

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