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Tripled Coincidence Point Results for (ψ, φ)-weakly Contractive Mappings in Partially Ordered S-matric Spaces

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Abstract

In this paper, the notion of S-metric spaces will be introduced. We present a some tripled coincidence point results for a mixed g-monotone mappings $F : X^3 \to X$ satisfying (ψ, φ)-contractions in partially ordered complete S-metric spaces. Also an application and some example are given to support our results.

Keywords: Tripled coincidence fixed Point, mixed g-monotone, Continuous, Cauchy sequence, Convergent, Partially ordered S-metric, Complete S-metric

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1. Introduction

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partially order have been studied by many authors (Agarwal, R. P. & et al., 2008; Ćirić,& et al., 2009).

One of the remarkable generalization, known as φ-contraction, was given by Boyed and Wong (Boyd, D. W. & Wong, S. W., 1969) in 1969. In 1997, Alber and Guerre-Delabriere (Alber, Ya., & Guerre-Delabriere, I. S., 1997), intriduced the notion of a weak φ-contraction which generalizes Boyed and Wong results, so Banach's result. Very recently, inspired from the notion of weak φ-contractions, a new concept of (ψ, φ)-contractions was introduced (Cherichi, M., & Samet, B., 2012; Dhutta, P. N., & Choudhury, B. S., 2008; Dori, D., 2009; Popescu, O., 2010; Berinde, V., & Borcut, M., 2011).

Throughout the paper, $\mathbb{N}^*$ is the set of positive integers.

First we recall some notions, lemmas, and examples wich will be useful later.

Definition 1.1. (Sedghi, S., Shobe, N., & Aliouche, A., 2012) Let X be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be an S-metric on X, if for each $x, y, z, a \in X$,

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space*.

Example 1.2. (Sedghi, S., Shobe, N., & Aliouche, A., 2012) We can easily check that the following examples are S-metric spaces.

1. Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X. Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.
2. Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X. Then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.
3. Let X be a nonempty set and d be a ordinary metric on X. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

Lemma 1.3. (Sedghi, S. & et al., 2014) Let (X, S) be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$, $x, y \in X$.
Definition 1.4. (Kim, J. K. & et al., 2016) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows, respectively:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$

$$B_S[x,r] = \{y \in X : S(y,y,x) \le r\}.$$

Example 1.5. (Kim, J. K. & et al., 2016) Let $X = \mathbb{R}$. Denote $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Thus

$$\begin{aligned} B_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2). \end{aligned}$$

Definition 1.6. (Sedghi, S. & Dung, N. V., (2014) Let (X,S) be an S-metric space and $A \subset X$.

1. The set A is said to be an open subset of X. If for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$.
2. The set A is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
3. A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, $S(x_n,x_n,x) < \varepsilon$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.
4. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \ge n_0$.
5. The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.
6. Let τ be the set of all $A \subset X$ with $x \in A$ and there exists $r > 0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

Definition 1.7. (Kim, J. K. & et al., 2016) Let (X,S) and (X',S') be two S-metric spaces, and let $f : (X,S) \to (X',S')$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X, $S(x_n,x_n,a) \to 0$ implies $S'(f(x_n),f(x_n),f(a)) \to 0$. A function f is continuous at X if and only if it is continuous at all $a \in X$.

Lemma 1.8. (Sedghi, S. & et al., 2015) Let (X,S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_S(x,r)$ is an open subset of X.

Lemma 1.9. (Sedghi, S. & et al., 2015) Let (X,S) be an S-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.10. (Sedghi, S. & et al., 2012) Let (X,S) be an S-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy.

Lemma 1.11. (Sedghi, S. & Dung, N. V., 2014) Let (X,S) be an S- metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y).$$

Definition 1.12. (Berinde, V. & Borcut, M., 2011) A element $(x,y,z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if $F(x,y,z) = x$, $F(y,x,y) = y$ and $F(z,y,x) = z$.

Definition 1.13. (Aydi, H. & et al., 2012) An element $(x,y,z) \in X^3$ is called a tripled common fixed point of $F : X^3 \to X$ and $g : X \to X$ if $F(x,y,z) = gx = x$, $F(y,x,y) = gy = y$ and $F(z,y,x) = gz = z$.

Definition 1.14. (Cherichi, M. & Samet, B., 2012) Let (X,S) be a non-empty set. We say that the mappings $F : X^3 \to X$ and $g : X \to X$ are commutative if $gF(x,y,z) = F(gx,gy,gz)$, for all $x,y,z \in X$.

Definition 1.15. (Borcut, M., 2012) An element $(x,y,z) \in X^3$ is called a tripled coincidence point of $F : X^3 \to X$ and $g : X \to X$ if $F(x,y,z) = gx$, $F(y,x,y) = gy$ and $F(z,y,x) = gz$.

Definition 1.16. (Berinde, V. & Borcut, M., 2011; Borcut, M., 2012) Let $(X,\le)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$.

We say that F has the mixed g-monotone property if $F(x,y,z)$ is g-nondecreasing in x, g-nonincreasing in y and g-nondecreasing in z, that is if, for any $x,y,z \in X$,

$$x_1,x_2 \in X,\ gx_1 \le gx_2 \Rightarrow F(x_1,y,z) \le F(x_2,y,z),$$
$$y_1,y_2 \in X,\ gy_1 \le gy_2 \Rightarrow F(x,y_1,z) \ge F(x,y_2,z),$$

and

$$z_1,z_2 \in X,\ gz_1 \le gz_2 \Rightarrow F(x,y,z_1) \le F(x,y,z_2).$$

Lemma 1.17. (Aydi, H. & et al., 2012) Consider three non-negetive real sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. Suppose there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} max\{a_n, b_n\} = 0$$

and

$$\lim_{n\to\infty} max\{a_n, b_n, c_n\} = \alpha.$$

Then,

$$\lim_{n\to\infty} supc_n = \alpha.$$

Definition 1.18. (Khan, M S. & et al., 1984) The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing,

(2) $\psi(t) = 0$ if and only if $t = 0$.

Let Ψ be the set of altering distances. Again, we denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that

(i) φ is lower-continuous and non-decreasing,

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

2. Main Results

The notion of a fixed point of N-order was first introduced by Samet and Vetro (Samet, B. & et al., 2012). Later, Berinede and Borcut (Berinde, V. & Borcut, M., 2011) proved some tripled fixed point results (N=3) in partially ordered metric spaces (Abbas, M. & et al., 2011; Aydi, H. &et al., 2012; Aydi, H. & Karapanar, E., 2012; Karapanar, E., 2010).

In this paper, we establish tripled coincidence point results for mapping $F : X^3 \to X$ and $g : X \to X$ involving nonlinear contractions in the setting of ordered S-metric spaces. Also, we precent an application and some examples in support of our results

Theorem 2.8. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$\begin{aligned} \psi(S(F(x, y, z), F(a, b, c), F(u, v, w))) \leq\ & \psi(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}) \\ & -\varphi(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}) \qquad (5) \end{aligned}$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,

(2) F has the mixed g-monotone property,

(3) F is continuous,

(4) g is continuous, non-decreasing and commutes with F.

Suppose there exists $x_0, y_0, z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Proof. Suppose $x_0, y_0, z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$. Since $F(X^3) \subseteq g(X)$, by (v) we can choose $x_1, y_1, z_1 \in X$ such that $g(x_1) = F(x_0, y_0, z_0)$, $g(y_1) = F(y_0, x_0, y_0)$ and $g(z_1) = F(z_0, y_0, x_0)$. Then $g(x_0) \leq g(x_1)$, $g(y_0) \geq g(y_1)$ and $g(z_0) \leq g(z_1)$. Again from $F(X^3) \subseteq g(X)$ we can choose $x_2, y_2, z_2 \in X$ such that $g(x_2) = F(x_1, y_1, z_1)$, $g(y_2) = F(y_1, x_1, y_1)$ and $g(z_2) = F(z_1, y_1, x_1)$. Since F has the mixed g-monotone property, we have $g(x_0) \leq g(x_1) \leq g(x_2)$, $g(y_0) \geq g(y_1) \geq g(y_2)$ and $g(z_0) \leq g(z_1) \leq g(z_2)$. Contiuing this process we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\begin{aligned} g(x_n) = F(x_{n-1}, y_{n-1}, z_{n-1}) \leq g(x_{n+1}) = F(x_n, y_n, z_n), \\ g(y_{n+1}) = F(y_n, x_n, y_n) \geq g(y_n) = F(y_{n-1}, x_{n-1}, y_{n-1}), \end{aligned}$$

and

$$g(z_n) = F(z_{n-1}, y_{n-1}, x_{n-1}) \leq g(z_{n+1}) = F(z_n, y_n, x_n).$$

If, for some integer n_0, we have $(g(x_{n_0+1}), g(y_{n_0+1}), g(z_{n_0+1})) = (g(x_{n_0}), g(y_{n_0}), g(z_{n_0}))$, then $F(x_{n_0}, y_{n_0}, z_{n_0}) = g(x_{n_0})$, $F(y_{n_0}, x_{n_0}, y_{n_0}) = g(y_{n_0})$ and $F(z_{n_0}, y_{n_0}, x_{n_0}) = g(z_{n_0})$, i.e., $(x_{n_0}, y_{n_0}, z_{n_0})$ is a tripled coincidence point of F and g. Thus we shall assume that $(g(x_{n+1}), g(y_{n+1}), g(z_{n+1})) \neq (g(x_n), g(y_n), g(z_n))$ for all $n \in \mathbb{N}$, i.e., we assume that either $g(x_{n+1}) \neq g(x_n)$ or $g(y_{n+1}) \neq g(y_n)$ or $g(z_{n+1}) \neq g(z_n)$.

Since $g(x_n) \leq g(x_{n+1})$, $g(y_n) \geq g(y_{n+1})$ and $g(z_n) \leq g(z_{n+1})$, that is $(x_n, y_n, z_n) \leq g(x_{n+1}, y_{n+1}, z_{n+1})$.

From (5) we have,

$$\begin{aligned} & \psi(S(gx_{n+1}, gx_{n+1}, gx_n)) := \psi(S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ \leq\ & \psi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}) \\ & -\varphi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}) \\ \leq\ & \psi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}), \end{aligned} \tag{6}$$

$$\begin{aligned} & \psi(S(gy_n, gy_n, gy_{n+1})) := \psi(S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n))) \\ \leq\ & \psi(max\{S(gy_{n-1}, gy_{n-1}, gy_n), S(gx_{n-1}, gx_{n-1}, gx_n)\}) \\ & -\varphi(max\{S(gy_{n-1}, gy_{n-1}, gy_n), S(gx_{n-1}, gx_{n-1}, gx_n)\}) \\ \leq\ & \psi(max\{S(gy_n, gy_n, gy_{n-1}), S(gx_n, gx_n, gx_{n-1}), S(gz_n, gz_n, gz_{n-1})\}), \end{aligned} \tag{7}$$

and

$$\begin{aligned} & \psi(S(gz_{n+1}, gz_{n+1}, gz_n)) := \psi(S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\ \leq\ & \psi(max\{S(gz_n, gz_n, gz_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gx_n, gx_n, gx_{n-1})\}) \\ & -\varphi(max\{S(gz_n, gz_n, gz_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gx_n, gx_n, gx_{n-1})\}) \\ \leq\ & \psi(max\{S(gz_n, gz_n, gz_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gx_n, gx_n, gx_{n-1})\}). \end{aligned} \tag{8}$$

Since $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function, for a, b, $c \in [0, \infty)$, we have

$$\psi(max\{a, b, c\}) = max\{\psi(a), \psi(b), \psi(c)\}.$$

Then, from (6), (7) and (8), it follows that

$$\begin{aligned} & \psi(max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\}) \\ & = max(\{\psi(S(gx_{n+1}, gx_{n+1}, gx_n)), \psi(S(gy_n, gy_n, gy_{n+1})), \psi(S(gz_{n+1}, gz_{n+1}, gz_n))\}) \\ \leq\ & \psi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}). \end{aligned}$$

The fact that ψ is non-decreasing yields that

$$\begin{aligned} & max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\}) \\ \leq\ & max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}. \end{aligned} \tag{9}$$

Thus, $max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\}$ is positive non-increasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\} = r. \tag{10}$$

Having in mind that φ is non-decreasing, then

$$\begin{aligned} & \varphi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}) \\ \geq\ & \varphi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1})\}), \end{aligned} \tag{11}$$

so from (6)-(8), we get that

$$\begin{aligned} & \psi(max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\}) \\ & = max(\{\psi(S(gx_{n+1}, gx_{n+1}, gx_n)), \psi(S(gy_n, gy_n, gy_{n+1})), \psi(S(gz_{n+1}, gz_{n+1}, gz_n))\}) \\ \leq \; & \psi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1}), S(gz_n, gz_n, gz_{n-1})\}) \\ & -\varphi(max\{S(gx_n, gx_n, gx_{n-1}), S(gy_n, gy_n, gy_{n-1})\}). \qquad (12) \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 \; & \leq \; max\{S(gx_n, gx_n, gx_{n-1}), S(gy_{n-1}, gy_{n-1}, gy_n)\} \\ & \leq \; max\{S(gx_n, gx_n, gx_{n-1}), S(gy_{n-1}, gy_{n-1}, gy_n), S(gz_n, gz_n, gz_{n-1})\}, \qquad (13) \end{aligned}$$

so by (10), the real sequence $max\{S(gx_n, gx_n, gx_{n-1}), S(gy_{n-1}, gy_{n-1}, gy_n)\}$ is bounded. Thus there exists a real number r_1 with $0 \leq r_1 \leq r$ and subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that

$$\lim_{n\to\infty} max\{S(gx_{n(k)+1}, gx_{n(k)+1}, gx_{n(k)}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1})\} = r_1. \qquad (14)$$

We rewite (12)

$$\begin{aligned} & \psi(max\{S(gx_{n(k)+1}, gx_{n(k)+1}, gx_{n(k)}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}), S(gz_{n(k)+1}, gz_{n(k)+1}, gz_{n(k)})\}) \\ \leq \; & \psi(max\{S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1})\}) \\ & -\varphi(max\{S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1})\}). \qquad (15) \end{aligned}$$

Letting $k \to \infty$ in (15), having in mind (10), (14), the continuity of ψ and the lower semi-contiuity of φ, we obtain

$$\begin{aligned} \psi(r) \; & = \; \limsup_{k\to\infty} \psi(max\{S(gx_{n(k)+1}, gx_{n(k)+1}, gx_{n(k)}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}), S(gz_{n(k)+1}, gz_{n(k)+1}, gz_{n(k)})\}) \\ & \leq \; \limsup_{k\to\infty} \psi(max\{S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1})\}) \\ & \quad - \liminf_{k\to\infty} \varphi(max\{S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1})\}) \\ & \leq \; \psi(r) - \varphi(r_1), \end{aligned}$$

which implies that $\varphi(r_1) = 0$, and using a property of φ, we find $r_1 = 0$. Thanks to Lemma (2.6) together with (10) and (14), it yields that

$$\begin{aligned} r \; & =: \; \lim_{k\to\infty} max\{S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1})\}) \\ & = \; \limsup_{k\to\infty} S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}). \qquad (16) \end{aligned}$$

For any $k \in \mathbb{N}$, we rewite (8) as

$$\begin{aligned} & \psi(S(gz_{n(k)+1}, gz_{n(k)+1}, gz_{n(k)})) \\ \leq \; & \psi(max\{S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1})\}) \\ & -\varphi(max\{S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gx_{n(k)}, gx_n, gx_{n(k)-1})\}). \qquad (17) \end{aligned}$$

Again, letting $k \to \infty$ in (17), having in mind (10), (16) and by the properties of ψ, φ, we obtain

$$\begin{aligned} \psi(r) \; & = \; \limsup_{k\to\infty}(S(gz_{n(k)+1}, gz_{n(k)+1}, gz_{n(k)})) \\ & \leq \; \limsup_{k\to\infty} \psi(max\{S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1})\}) \\ & \quad - \liminf_{k\to\infty} \varphi(max\{S(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}), S(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}), S(gx_{n(k)}, gx_n, gx_{n(k)-1})\}) \\ & \leq \; \psi(r) - \varphi(r), \end{aligned}$$

which gives that $\varphi(r) = 0$, i.e., by (10),

$$\lim_{n\to\infty} max\{S(gx_{n+1}, gx_{n+1}, gx_n), S(gy_n, gy_n, gy_{n+1}), S(gz_{n+1}, gz_{n+1}, gz_n)\} = 0. \qquad (18)$$

Our next step is to show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences.

Assume the contrary, i.e., $\{gx_n\}$, $\{gy_n\}$ or $\{gz_n\}$ is not Cauchy sequence, i.e.,

$$\lim_{n,m\to+\infty} S(gx_m, gx_m, gx_n) \neq 0,$$

or

$$\lim_{n,m\to+\infty} S(gy_m, gy_m, gy_n) \neq 0,$$

or

$$\lim_{n,m\to+\infty} S(gz_m, gz_m, gz_n) \neq 0.$$

This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$max\{S(gx_{m_k}, gx_{m_k}, gx_{n_k}), S(gy_{m_k}, gy_{m_k}, gy_{n_k}), S(gz_{m_k}, gz_{m_k}, gz_{n_k})\} \geq \varepsilon. \tag{19}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (19). Then

$$max\{S(gx_{m_k}, gx_{m_k}, gx_{n_k-1}), S(gy_{m_k}, gy_{m_k}, gy_{n_k-1}), S(gz_{m_k}, gz_{m_k}, gz_{n_k-1})\} < \varepsilon. \tag{20}$$

By (S3) and (20), we have

$$\begin{aligned} S(gx_{m_k}, gx_{m_k}, gx_{n_k}) &\leq 2S(gx_{m_k}, gx_{m_k}, gx_{n_k-1}) + S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}) \\ &< 2\varepsilon + S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Thus, by (18) we obtain

$$\lim_{k\to\infty} S(gx_{m_k}, gx_{m_k}, gx_{n_k}) \leq \lim_{k\to\infty} 2S(gx_{m_k}, gx_{m_k}, gx_{n_k-1}) \leq 2\varepsilon. \tag{21}$$

Similarly, we have

$$\lim_{k\to\infty} S(gy_{m_k}, gy_{m_k}, gy_{n_k}) \leq \lim_{k\to\infty} 2S(gy_{m_k}, gy_{m_k}, gy_{n_k-1}) \leq 2\varepsilon. \tag{22}$$

$$\lim_{k\to\infty} S(gz_{m_k}, gz_{m_k}, gz_{n_k}) \leq \lim_{k\to\infty} 2S(gz_{m_k}, gz_{m_k}, gz_{n_k-1}) \leq 2\varepsilon. \tag{23}$$

Again by (S3) and (20), we have

$$\begin{aligned} S(gx_{m_k}, gx_{m_k}, gx_{n_k}) &\leq 2S(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k}) \\ &\leq 2S(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + 2S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}) + S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}) \\ &\leq 2S(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + 4S(gx_{m_k-1}, gx_{m_k-1}, gx_{m_k}) \\ &\quad +2S(gx_{m_k}, gx_{m_k}, gx_{n_k-1}) + S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}) \\ &< 2S(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + 4S(gx_{m_k-1}, gx_{m_k-1}, gx_{m_k}) \\ &\quad +2\varepsilon + S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Letting $k \to \infty$ and using (18), we get

$$\lim_{k\to\infty} S(gx_{m_k}, gx_{m_k}, gx_{n_k}) \leq \lim_{k\to\infty} S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}) \leq \varepsilon, \tag{24}$$

$$\lim_{k\to\infty} S(gy_{m_k}, gy_{m_k}, gy_{n_k}) \leq \lim_{k\to\infty} S(gy_{m_k-1}, gy_{m_k-1}, gy_{n_k-1}) \leq \varepsilon, \tag{25}$$

$$\lim_{k\to\infty} S(gz_{m_k}, gz_{m_k}, gz_{n_k}) \leq \lim_{k\to\infty} S(gz_{m_k-1}, gz_{m_k-1}, gz_{n_k-1}) \leq \varepsilon. \tag{26}$$

Using (19) and (24)-(26),we have

$$\begin{aligned} &\lim_{k\to\infty} max\{S(gx_{m_k}, gx_{m_k}, gx_{n_k}), S(gy_{m_k}, gy_{m_k}, gy_{n_k}), S(gz_{m_k}, gz_{m_k}, gz_{n_k})\} \\ \leq\ &\lim_{k\to\infty} max\{S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}), S(gy_{m_k-1}, gy_{m_k-1}, gy_{n_k-1}), S(gz_{m_k-1}, gz_{m_k-1}, gz_{n_k-1})\} \\ \leq\ &\varepsilon. \qquad (27) \end{aligned}$$

Now, using inequality (5) we obtain

$$\begin{aligned} \psi(S(gx_{m_k}, gx_{m_k}, gx_{n_k})) &= \psi(S(F(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), F(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}))) \\ &\leq \psi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1})\}) \\ &\quad -\varphi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1})\}) \qquad (28) \end{aligned}$$

$$\begin{aligned} \psi(S(gy_{m_k}, gy_{m_k}, gy_{n_k})) &= \psi(S(F(y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), F(y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}))) \\ &\leq \psi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\}) \\ &\quad -\varphi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\}), \qquad (29) \end{aligned}$$

and

$$\begin{aligned} \psi(S(gz_{m_k}, gz_{m_k}, gz_{n_k})) &= \psi(S(F(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}), F(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}), F(z_{n_k-1}, y_{n_k-1}, x_{n_k-1}))) \\ &\leq \psi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1})\}) \\ &\quad -\varphi(max\{S((x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1})\}).(30) \end{aligned}$$

We deduce from (28)-(30) that

$$\begin{aligned} &\psi(max\{S(gx_{m_k}, gx_{m_k}, gx_{n_k}), S(gy_{m_k}, gy_{m_k}, gy_{n_k}), S(gz_{m_k}, gz_{m_k}, gz_{n_k})\} \\ =\ &max\{\psi(S(gx_{m_k}, gx_{m_k}, gx_{n_k})), \psi(S(gy_{m_k}, gy_{m_k}, gy_{n_k})), \psi(S(gz_{m_k}, gz_{m_k}, gz_{n_k}))\} \\ \leq\ &\psi(max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1}\}) \\ &-\varphi(max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\}). \qquad (31) \end{aligned}$$

On the other hand, since

$$\begin{aligned} &max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\} \\ \leq\ &max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1}\}, \qquad (32) \end{aligned}$$

then from (27),

$$\limsup_{k\to\infty} max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\} \leq \varepsilon.$$

Therefore, the real sequence $\{max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\}\}$ is bounded. Thus, up to a subsequence (still denoted the same), there exists ε_1 with $0 \leq \varepsilon_1 \leq \varepsilon$ such that

$$\lim_{k\to\infty} max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\} = \varepsilon_1. \qquad (33)$$

Inserting this in (31) and using (27), (33) togwther with the properties of ψ, φ, we get that

$$\begin{aligned} \psi(\varepsilon) &= \limsup_{k\to\infty} \psi(max\{S(gx_{m_k}, gx_{m_k}, gx_{n_k}), S(gy_{m_k}, gy_{m_k}, gy_{n_k}), S(gz_{m_k}, gz_{m_k}, gz_{n_k})\} \\ &\leq \limsup_{k\to\infty} \psi(max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}), S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1}\}) \\ &\quad -\liminf_{k\to\infty} \varphi(max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})\}) \\ &\leq \psi(\varepsilon) - \varphi(\varepsilon_1), \end{aligned}$$

which leads to $\varphi(\varepsilon_1) = 0$, so $\varepsilon_1 = 0$. By this and (27), due to Lemma(2.6), we obtain

$$\limsup_{k\to\infty} S(z_{m_k-1}, z_{m_k-1}, z_{n_k-1}) = \varepsilon.$$

Combining this to (19) and (26), we find

$$\limsup_{k\to\infty} S(z_{m_k}, z_{m_k}, z_{n_k}) = \varepsilon.$$

Letting $k \to \infty$ in (30) and using (27), we deduce

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

i.e., $\varepsilon = 0$, it is a contradiction. We conclude that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the S-metric space (X, S), which is complete. Then, there are x, y, $z \in X$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are respectively convergent to x, y and z, i.e., we have

$$\lim_{n\to\infty} S(gx_n, gx_n, x) = \lim_{n\to\infty} S(x, x, gx_n) = 0, \tag{34}$$

$$\lim_{n\to\infty} S(gy_n, gy_n, y) = \lim_{n\to\infty} S(y, y, gy_n) = 0, \tag{35}$$

$$\lim_{n\to\infty} S(gz_n, gz_n, z) = \lim_{n\to\infty} S(z, z, gz_n) = 0. \tag{36}$$

From (34)-(36) and the continuity of g, we get

$$\lim_{n\to\infty} S(g(gx_n), g(gx_n), gx) = \lim_{n\to\infty} S(gx, gx, g(gx_n)) = 0, \tag{37}$$

$$\lim_{n\to\infty} S(g(gy_n), g(gy_n), gy) = \lim_{n\to\infty} S(gy, gy, g(gy_n)) = 0, \tag{38}$$

$$\lim_{n\to\infty} S(g(gz_n), g(gz_n), gz) = \lim_{n\to\infty} S(gz, gz, g(gz_n)) = 0. \tag{39}$$

Since $gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, x_n)$, so the commutativity of F and g yields that

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n), \tag{40}$$

$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n), \tag{41}$$

$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n). \tag{42}$$

Now we show that $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$.
The mapping F is continuous, so since the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are, respectively, convergent to x, y, z, hence using Difinition 1.7, the sequence $\{F(gx_n, gy_n, gz_n)\}$ is convergent to $F(x, y, z)$. Therefore, from (40), $\{g(gx_{n+1})\}$ is convergent to $F(x, y, z)$. By uniquueness of the limit, from (37) we have $F(x, y, z) = gx$.

Similarly, one finds $F(y, x, y) = gy$ and $F(z, y, x) = gz$, and this makes end to the proof.

Corollary 2.9. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0, 1)$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$S(F(x, y, z), F(a, b, c), F(u, v, w)) \leq k(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}). \tag{43}$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,

(2) F has the mixed g-monotone property,

(3) F is continuous,

(4) g is continuous, non-decreasing and commutes with F.

Suppose there exist $x_0, y_0, z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Proof. It follows by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \geq 0$.

Corollary 2.10. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0, 1)$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$S(F(x,y,z), F(a,b,c), F(u,v,w)) \leq \frac{k}{3}(S(gx, ga, gu) + S(gy, gb, gv) + S(gz, gc, gw)).$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,

(2) F has the mixed g-monotone property,

(3) F is continuous,

(4) g is continuous, non-decreasing and commutes with F.

Suppose there exist $x_0, y_0, z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Proof. It suffices to remark that

$$\frac{k}{3}(S(gx, ga, gu) + S(gy, gb, gv) + S(gz, gc, gw)) \leq k(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}).$$

In the next theorem, we omit the continuoity hypothesis of F. We need the following definition.

Definition 2.11. Let $(X, \leq)$ be a partially ordered set and S be a S-metric on X. We say that $(X, S, \leq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all n.

Theorem 2.12. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$\begin{aligned} \psi(S(F(x,y,z), F(a,b,c), F(u,v,w))) \leq\ & \psi(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}) \\ & -\varphi(max\{S(gx, ga, gu), S(gy, gb, gv), S(gz, gc, gw)\}). \end{aligned} \tag{44}$$

Assume that $(X, S, \leq)$ is regular. Suppose that $(g(X), S)$ is complete, F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$.Also assume there exist $x_0, y_0, z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x, y, z),\ g(y) = F(y, x, y) \text{ and } g(z) = F(z, y, x).$$

Proof. Proceeding exactly as in (Theorem 2.8), we have that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the complete S-metric space $(g(X), S)$. Then, there exist $x, y, z \in X$ such that $gx_n \to gx$, $gy_n \to gy$ and $gz_n \to gz$. Since $\{gx_n\}$ and $\{gz_n\}$

are non-decreasing and $\{gy_n\}$ non-increasing, using the regularity of $(X, S, \leq)$, we have $gx_n \leq gx$, $gz_n \leq gz$ and $gy \leq gy_n$ for all $n \geq 0$. Using (5), we get

$$\begin{aligned}\psi(S(F(x,y,z),F(x,y,z),gx_{n+1})) &= \psi(S(F(x,y,z),F(x,y,z),F(x_n,y_n,z_n)) \\ &\leq \psi(max\{S(gx,gx,gx_n),S(gy,gy,gy_n),S(g,gz,gz_n)\}) \\ &\quad -\varphi(max\{S(gx,gx,gx_n),S(gy,gy,gy_n),S(g,gz,gz_n)\}). \qquad (45)\end{aligned}$$

Letting $n \to \infty$ in inequality, we obtain that

$$\psi(S(F(x,y,z),F(x,y,z),gx)) \leq \psi(0) - \varphi(0) = 0,$$

which implies that $S(F(x,y,z),F(x,y,z),gx) = 0$, i.e., $gx = F(x,y,z)$.

Similarly, one can show that $gy = F(y,x,y)$ and $gz = F(z,y,x)$. Thus we proved that (x,y,z) is a tripled coincidence point of F and g.

Similarly, we can state the following corollary.

Corollary 2.13. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0, 1)$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$S(F(x,y,z),F(a,b,c),F(u,v,w)) \quad \leq \quad k(max\{S(gx,ga,gu),S(gy,gb,gv),S(gz,gc,gw)\}).$$

Assume that $(X, S, \leq)$ is regular. Suppose that $(g(X), S)$ is complete, F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$.Also assume there exist x_0, y_0, $z_0 \in X$ such that $g(x_0) \leq F(x_0,y_0,z_0)$, $g(y_0) \geq F(y_0,x_0,y_0)$ and $g(z_0) \leq F(z_0,y_0,x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

Corollary 2.14. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0, 1)$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$S(F(x,y,z),F(a,b,c),F(u,v,w)) \quad \leq \quad \frac{k}{3}(S(gx,ga,gu)+S(gy,gb,gv)+S(gz,gc,gw)).$$

Suppose that $(g(X), S)$ is complete, F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$. Also assume there exists x_0, y_0, $z_0 \in X$ such that $g(x_0) \leq F(x_0,y_0,z_0)$, $g(y_0) \geq F(y_0,x_0,y_0)$ and $g(z_0) \leq F(z_0,y_0,x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

Remark 2.15. Other corollaries could be derived from Theorem (2.8) and (2.12) by taking $g = I_x$. Where I is identity map.

Now, from previous obtained results, we will deduce some tripled coincidence point results for mappings satisfying a contraction of integral type in S-metric space. Let us introduce first some notions from (Aydi, H. & et al., 2012).

We denote by Γ the set of functions $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) α is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$,

(ii)for all $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \alpha(s)ds > 0.$$

Let $N \in \mathbb{N}^*$ be fixed. Let $\{\alpha_i\}_{1 \leq i \leq N}$ be a family of N functions that belong to Γ. For all $t \geq 0$, we denote $(I_i)_{i=1,2,...,N}$ as follows:

$$\begin{aligned} I_1(t) &= \int_0^t \alpha_1(s)ds, \\ I_2(t) &= \int_0^{I_1(t)} \alpha_2(s)ds = \int_0^{\int_0^t \alpha_1(s)ds} \alpha_2(s)ds, \\ &\vdots \\ I_N(t) &= \int_0^{I_{N-1}(t)} \alpha_N(s)ds. \end{aligned}$$

We have the following result.

Theorem 2.16. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$\begin{aligned} I_N(\psi(S(F(x,y,z), F(a,b,c), F(u,v,w))) &\leq I_N(\psi(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\})) \\ &\quad -I_N(\varphi(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\})). \end{aligned} \tag{46}$$

Assume that F and $g$satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,

(2) F has the mixed g-monotone property,

(3) F is continuous,

(4) g is continuous, non-decreasing and commutes with F.

Suppose there exists x_0, y_0, $z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

Proof. Take

$$\tilde{\varphi} = I_{N^0\varphi}$$

and

$$\tilde{\psi} = I_{N^0\psi}.$$

It is easy to show that $\tilde{\psi} \in \Psi$ and $\tilde{\varphi} \in \Phi$. From (46), we have

$$\begin{aligned} \tilde{\psi}(S(F(x,y,z), F(a,b,c), F(u,v,w)) &\leq \tilde{\psi}(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\}) \\ &\quad -\tilde{\varphi}(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\}). \end{aligned} \tag{47}$$

Now, applying Theorem (2.8), we obtain the desired result.

Similarly, we have

Theorem 2.17. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have:

$$\begin{aligned} I_N(\psi(S(F(x,y,z), F(a,b,c), F(u,v,w))) &\leq I_N(\psi(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\})) \\ &\quad -I_N(\varphi(max\{S(gx,ga,gu), S(gy,gb,gv), S(gz,gc,gw)\})). \end{aligned}$$

Assume that $(X, S, \leq)$ is regular. Suppose that $(g(X), S)$ is complete, F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$.Also assume there exists x_0, y_0, $z_0 \in X$ such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, y_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$.

Then F and g have a tripled coincidence point. That is there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z),\ g(y) = F(y,x,y) \text{ and } g(z) = F(z,y,x).$$

3. Application to Integral Equations

We in (Gholidahneh, A., & Sedghi, S., 2017) proved coupled common fixed point theorems of integral type contraction in ordered S-metric spaces.

In this section, we study the existence of solutions to nonlinear integral equations using the results proved in section "Main results".

Consider the integral equations in the following system

$$\begin{aligned} x(t) &= \rho(t) + \int_0^T M(t,m)[f(m,x(m)) + k(m,y(m)) + h(m,z(m))]dm \\ y(t) &= \rho(t) + \int_0^T M(t,m)[f(m,y(m)) + k(m,x(m)) + h(m,y(m))]dm \\ z(t) &= \rho(t) + \int_0^T M(t,m)[f(m,z(m)) + k(m,y(m)) + h(m,x(m))]dm. \end{aligned} \tag{48}$$

We will analyze the system (48) under the following assummptions:

(i) $f, k, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous,

(ii) $\rho : [0,T] \to \mathbb{R}$ is continuous,

(iii) $M : [0,T] \times \mathbb{R} \to [0,+\infty)$ is continuous,

(iv) there exists $q > 0$ such that for all $x, y \in \mathbb{R}$, $y \geq x$,

$$\begin{aligned} 0 &\leq f(m,y) - f(m,x) \leq q(y-x) \\ 0 &\leq k(m,x) - k(m,y) \leq q(y-x) \\ 0 &\leq h(m,y) - h(m,x) \leq q(y-x). \end{aligned}$$

(v) We suppose that

$$3q \sup_{t\in[0,T]} \int_0^T M(t,m)dm < 1.$$

(vi) There exist continuous functions $\alpha, \beta, \gamma : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} \alpha(t) &\leq \rho(t) + \int_0^T M(t,m)[f(m,\alpha(m)) + k(m,\beta(m)) + h(m,\gamma(m))]dm \\ \beta(t) &\geq \rho(t) + \int_0^T M(t,m)[f(m,\beta(m)) + k(m,\alpha(m)) + h(m,\beta(m))]dm \\ \gamma(t) &\leq \rho(t) + \int_0^T M(t,m)[f(m,\gamma(m)) + k(m,\beta(m)) + h(m,\alpha(m))]dm. \end{aligned}$$

We consider the space $X = C([0,T],\mathbb{R})$ of continuous functions defined on $[0,T]$ endowed with the (S-complete) S-metric given by

$$S(u,v,w) = \max_{t\in[0,T]} |u(t) - w(t)| + \max_{t\in[0,T]} |v(t) - w(t)|,$$

for all $u, v, w \in X$. We endowed X with the partial ordered $\leq$ given by: $x, y \in X$, $x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in [0,T]$.

On the other hand, we may adjust as in (Nieto, J. J., & Rodriguez-Lopez, R., 2005) to prove that $(X, S, \preceq)$ is regular.

Our result is the following.

Theorem 3.1. Under assumption (i)-(iv), the system (48) has a solution in $X^3 = (C([0,T]),\mathbb{R})^3$.

Proof. We consider the operators $F : X^3 \to X$ and $g : X \to X$ defined dy

$$F(x_1,x_2,x_3)(t) = \rho(t) + \int_0^T M(t,m)[f(m,x_1(m)) + k(m,x_2(m)) + h(m,x_3(m))]dm, \qquad g(x) = x \qquad t \in [0,T],$$

for all $x_1, x_2, x_3, x \in X$.

First, we will prove that F has the mixed monotone property (since g is the identity on X).

In the fact, for $x_1 \leq y_1$ and $t \in [0, T]$, we have

$$F(y_1, x_2, x_3)(t) - F(x_1, x_2, x_3)(t) = \int_0^T M(t, m)[f(m, y_1(m)) - f(m, x_1(m))]dm.$$

Taking into account that $x_1(t) \leq y_1(t)$ and $t \in [0, T]$, so by (iv), $f(m, y_1(m)) \geq f(m, x_1(m))$. Then $F(y_1, x_2, x_3)(t) \geq F(x_1, x_2, x_3)(t)$ for all $t \in [0, T]$, i.e.,

$$F(x_1, x_2, x_3) \leq F(y_1, x_2, x_3).$$

Similarly, for $x_2 \leq y_2$ and $t \in [0, T]$, we have

$$F(x_1, x_2, x_3)(t) - F(x_1, y_2, x_3)(t) = \int_0^T M(t, m)[k(m, x_2(m)) - k(m, y_2(m))]dm.$$

Having $x_2(t) \leq y_2(t)$, so by (iv), $k(m, x_2(m)) \geq k(m, y_2(m)))$. Then $F(x_1, x_2, x_3)(t) \geq F(x_1, y_2, x_3)(t)$ for all $t \in [0, T]$, i.e.,

$$F(x_1, x_2, x_3) \geq F(x_1, y_2, x_3).$$

Now, for $x_3 \leq y_3$ and $t \in [0, T]$, we have

$$F(x_1, x_2, x_3)(t) - F(x_1, x_2, y_3)(t) = \int_0^T M(t, m)[h(m, x_3(m)) - h(m, y_3(m))]dm.$$

Taking into account that $x_3(t) \leq y_3(t)$ and $t \in [0, T]$, so by (iv), $h(m, x_3(m)) \geq f(m, y_3(m))$. Then $F(y_1, x_2, x_3)(t) \geq F(x_1, x_2, y_3)(t)$ for all $t \in [0, T]$, i.e.,

$$F(y_1, x_2, x_3)(t) \leq F(x_1, x_2, y_3)(t).$$

Therefore, F has the mixed monotone property.

In the what follows we estimate the quantity $S(F(x, y, z), F(a, b, c), F(u, v, w))$ for all $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$ and $z \geq c \geq w$. Since F has the mixed monotone property, we have:

$$F(u, v, w) \leq F(a, b, c) \leq F(x, y, z).$$

We obtain

$$\begin{aligned} & S(F(x, y, z), F(a, b, c), F(u, v, w)) \\ = & \max_{t \in [0,T]} |F(x, y, z)(t) - F(u, v, w)(t)| + \max_{t \in [0,T]} |F(a, b, c)(t) - F(u, v, w)(t)| \\ = & \max_{t \in [0,T]} (F(x, y, z)(t) - F(u, v, w)(t)) + \max_{t \in [0,T]} (F(a, b, c)(t) - F(u, v, w)(t)). \end{aligned}$$

Note that for all $t \in [0, T]$, from (iv), we have

$$\begin{aligned} F(x, y, z)(t) - F(u, v, w)(t) &= \int_0^T M(t, m)[f(m, x(m) - f(m, u(m))]dm \\ &\quad + \int_0^T M(t, m)[k(m, x, y(m)) - k(m, v(m))]dm \\ &\quad + \int_0^T M(t, m)[h(m, z(m)) - h(m, w(m))]dm \\ &\leq q[\max_{t \in [0,T]} |x(m) - u(m)| + \max_{t \in [0,T]} |y(m) - v(m)| \\ &\quad + \max_{t \in [0,T]} |z(m) - w(m)|](\int_0^T M(t, m)dm). \end{aligned}$$

Thus,

$$\begin{aligned}&\max_{t\in[0,T]}(F(x,y,z)(t)-F(u,v,w)(t))\\ \leq\;& q[\max_{t\in[0,T]}|x(m)-u(m)|+\max_{t\in[0,T]}|y(m)-v(m)|+\max_{t\in[0,T]}|z(m)-w(m)|](\sup_{t\in[0,T]}\int_0^T M(t,m)dm). \qquad (49)\end{aligned}$$

Repeating this idea, we may get using definition of the S-metric S

$$\begin{aligned}&\max_{t\in[0,T]}(F(x,y,z)(t)-F(u,v,w)(t))+\max_{t\in[0,T]}(F(a,b,c)(t)-F(u,v,w)(t))\\ \leq\;& q[S(x,a,u)+S(y,b,v)+S(z,c,w)](\sup_{t\in[0,T]}\int_0^T M(t,m)dm)\\ \leq\;& 3q(\sup_{t\in[0,T]}\int_0^T M(t,m)dm)\max\{S(x,a,u)+S(y,b,v)+S(z,c,w)\}.\end{aligned}$$

From (v), we have $3q(\sup_{t\in[0,T]}\int_0^T M(t,m)dm) < 1$. This proves that the operator F satisfies the contractive condition appearing in Corollary (2.13).

Let α, β, γ be the functions in assumption (vi), then by (vi), we get

$$\alpha \geq F(\alpha,\beta,\gamma), \qquad \beta \geq F(\beta,\alpha,\beta), \qquad \gamma \leq F(\gamma,\beta,\alpha).$$

Applying orollary (2.13), we deduce the existence of $x_1, x_2, x_3 \in X$ such that

$$x_1 = F(x_1,x_2,x_3), \qquad x_2 = F(x_2,,x_1,x_2), \qquad x_3 = F(x_3,x_2,x_1),$$

i.e., (x_1, x_2, x_3) is a solution of the system (48).

4. Example

In this section, we state one example to support the usability of our results for S-metric spaces. Before we present our example we worth to mantion the following remark.

Remark 4.1. All our results still valid if $(u, v, w) = (a, b, c)$.

Example 4.2. Let $X = [0, 1]$ with usual order. Define $S : X^3 \to X$ by

$$S(x,y,z) = max\{|x-z|, |y-z|\}.$$

Define $F : X^3 \to X$ by

$$F(x,y,z) = \begin{cases} 0, & \text{if } y \geq min\{x,z\},\\ \dfrac{z-y}{4}, & \text{if } x \geq z \geq y,\\ \dfrac{x-y}{4}, & \text{if } z \geq x \geq y.\end{cases}$$

Also, define $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$ and $\varphi(t) = \frac{1}{2}t$. Then

a. $(X, S, \leq)$ is a complete regular S-metric space.

b. For $x, y, z, u, v, w \in X$ with $x \geq u \geq u$, $y \leq v \leq v$ and $z \geq w \geq w$, we have

$$\begin{aligned}\psi(S(F(x,y,z),F(x,y,z),F(u,v,w))) \leq\;& \psi(max\{S(x,x,u),S(y,y,v),S(z,z,w)\})\\ &-\varphi(max\{S(x,x,u),S(y,y,v),S(z,z,w)\}).\end{aligned}$$

c.F has the mixed monotone property.

Proof. To prove (b), given $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$ and $z \geq w$. Then:

Case 1: $y > min\{x, z\}$ and $v \geq min\{u, w\}$. Here, we have

$$\begin{aligned}\psi(S(F(x,y,z),F(x,y,z),F(u,v,w))) =\;& 0\\ \leq\;& \psi(max\{S(x,x,u),S(y,y,v),S(z,z,w)\})\\ &-\varphi(max\{S(x,x,u),S(y,y,v),S(z,z,w)\})\end{aligned}$$

Case 2: $y \geq min\{x, z\}$ and $u \geq w \geq v$. Here, we have $y \leq v \leq w \leq u \leq x$ and $y \leq v \leq w \leq z$. Hence $y = v = w = u = x$ or $y = v = w = z$. Therefore

$$\begin{aligned} \psi(S(F(x,y,z), F(x,y,z), F(u,v,w))) &= 0 \\ &\leq \psi(max\{S(x,x,u), S(y,y,v), S(z,z,w)\}) \\ &\quad -\varphi(max\{S(x,x,u), S(y,y,v), S(z,z,w)\}) \end{aligned}$$

Cace 3: $y \geq min\{x, z\}$ and $w \geq u \geq v$. Here, we have $y \leq v \leq u \leq w \leq z$ and $y \leq v \leq u \leq x$. Thus $y = u = v = w = z$ or $y = v = u = x$. Therefore

$$\begin{aligned} \psi(S(F(x,y,z), F(x,y,z), F(u,v,w))) &= 0 \\ &\leq \psi(max\{S(x,x,u), S(y,y,v), S(z,z,w)\}) \\ &\quad -\varphi(max\{S(x,x,u), S(y,y,v), S(z,z,w)\}) \end{aligned}$$

Case 4: $x \geq z \geq y$ and $v \geq min\{u, w\}$.

Suppose $w \leq v$, then $w - y \leq v - y$ and hence

$$\begin{aligned} z - y = z - w + w - y &\leq z - w + v - y = \frac{1}{2}[S(z,z,w) + S(v,v,y)] \\ &\leq max\{S(x,x,u), S(y,y,v), S(z,z,w)\}. \end{aligned}$$

Then

$$\begin{aligned} S(F(x,y,z), F(x,y,z), F(u,v,w)) &= S(\frac{z-y}{4}, \frac{z-y}{4}, 0) = \frac{z-y}{2} \\ &\leq \frac{1}{2} max\{S(x,x,u), S(y,y,v), S(z,z,w)\} \\ &= max\{S(x,x,u), S(y,y,v), S(z,z,w)\} \\ &\quad -\frac{1}{2} max\{S(x,x,u), S(y,y,v), S(z,z,w)\}. \end{aligned}$$

Suppose $v < w$, then $u \leq v < w$ and hence $u \leq v \leq w \leq z \leq x$. So

$$\begin{aligned} z - y &\leq x - y = x - u + u - y \\ &\leq (x-u) + (v-y) = \frac{1}{2}[S(x,x,u) + S(v,v,y)] \\ &\leq max\{S(x,x,u), S(y,y,v), S(z,z,w)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} S(F(x,y,z), F(x,y,z), F(u,v,w)) &= S(\frac{z-y}{4}, \frac{z-y}{4}, 0) = \frac{z-y}{2} \\ &\leq \frac{1}{2} max\{S(x,x,u), S(y,y,v), S(z,z,w)\} \\ &= max\{S(x,x,u), S(y,y,v), S(z,z,w)\} \\ &\quad -\frac{1}{2} max\{S(x,x,u), S(y,y,v), S(z,z,w)\}. \end{aligned}$$

Case 5: $z \geq x \geq y$ and $v \geq min\{u, w\}$. Suppose $u \leq v$, then $u - y \leq v - y$ and hence

$$\begin{aligned} x - y = x - u + u - y &\leq (x-u) + (v-y) = \frac{1}{2}[S(x,x,u) + S(v,v,y)] \\ &\leq max\{S(x,x,u), S(y,y,v), S(z,z,w)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{x-y}{4},\frac{x-y}{4},0) \\ &= \frac{x-y}{2} \\ &\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}. \end{aligned}$$

Suppose $v < u$, then $w \leq v < u$ and hence $w \leq v < u \leq x \leq z$. So

$$\begin{aligned} x-y \leq z-y = z-w+w-y &\leq (z-w)+(v-y) = \frac{1}{2}[S(z,z,w)+S(v,v,y)] \\ &\leq max\{S(x,x,u),S(y,y,v),S(z,z,w)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{x-y}{4},\frac{x-y}{4},0) \\ &= \frac{x-y}{2} \\ &\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}. \end{aligned}$$

Case 6: $x \geq z \geq y$ and $u \geq w \geq v$. Here, we have

$$\begin{aligned} S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{z-y}{4},\frac{z-y}{4},\frac{w-v}{4}) \\ &= \frac{1}{2}|(z-w)+(v-y)| \\ &\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}. \end{aligned}$$

Case 7: $x \geq z \geq y$ and $w \geq u \geq v$. Here, we have $y \leq v \leq u \leq w \leq z \leq x$. Thus,

$$\begin{aligned} S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{z-y}{4},\frac{z-y}{4},\frac{u-v}{4}) \\ &= \frac{1}{2}|(z-u)+(v-y)| \\ &\leq \frac{1}{2}[(x-u)+(v-y)] \\ &\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\ &\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}. \end{aligned}$$

Case 8: $z \geq x \geq y$ and $u \geq w \geq v$. Here, we have $y \leq v \leq w \leq u \leq x \leq z$. Therefore, we have

$$\begin{aligned}
S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{x-y}{4},\frac{x-y}{4},\frac{w-v}{4}) \\
&= \frac{1}{2}|(x-w)+(v-y)| \\
&\leq \frac{1}{2}(|z-w|+|v-y|) \\
&= \frac{1}{2}[S(z,z,w)+S(v,v,y)] \\
&\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\
&= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\
&\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}.
\end{aligned}$$

Case 9: $z \geq x \geq y$, $w \geq u \geq v$. Here, we have $y \leq v \leq u \leq w \leq z$. Therefore, we have

$$\begin{aligned}
S(F(x,y,z),F(x,y,z),F(u,v,w)) &= S(\frac{x-y}{4},\frac{x-y}{4},\frac{u-v}{4}) \\
&\leq \frac{1}{2}(|x-u|+|v-y|) \\
&= \frac{1}{2}[S(x,x,u)+S(v,v,y)] \\
&\leq \frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\
&= max\{S(x,x,u),S(y,y,v),S(z,z,w)\} \\
&\quad -\frac{1}{2}max\{S(x,x,u),S(y,y,v),S(z,z,w)\}.
\end{aligned}$$

To prove (c), let $x, y, z \in X$. To show that $F(x,y,z)$ is monotone non-decreasing in x, let $x_1, x_2 \in X$ with $x_1 \leq x_2$. If $y \geq min\{x_1, z\}$, then $F(x_1,y,z) = 0 \leq F(x_2,y,z)$.

If $y < min\{x_1, z\}$, then

$$F(x_1,y,z) = \frac{min(x_1,z)-y}{4} \leq \frac{min(x_2,z)-y}{4} = F(x_2,y,z).$$

Therefore, $F(x,y,z)$ is monotone non-decreasing in x. Similaly, we may show that $F(x,y,z)$ is monotone non-decreasing in z and monotone non-increasing in y. Thus, by Theorem (2.12) and Remark (4.1), F has a tripled fixed point. Here, (0,0,0) is the unique tripled fixed point of F.

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