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On S-quasi-Dedekind Modules

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Abstract

Let R be a commutative ring and M an unital R-module. A proper submodule L of M is called primary submodule of M, if $rm \in L$, where $r \in R$, $m \in M$, then $m \in L$ or $r^n M \subseteq L$ for some positive integer n. A submodule K of M is called semi-small submodule of M if, $K + L \neq M$ for each primary submodule L of M. An R-module M is called S-quasi-Dedekind module if, for each $f \in End_R(M)$, $f \neq 0$ implies $Kerf$ semi-small in M. In this paper we introduce the concept of S-quasi-Dedekind modules as a generalisation of small quasi-Dedekind modules, and gives some of their properties, characterizations and exemples. Another hand we study the relationships of S-quasi-Dedekind modules with some classes of modules and their endomorphism rings.

Keywords: Primary submodules, semi-small submodules, quasi-Dedekind modules, S-quasi-Dedekind modules

1. Introduction

Throughout all rings are associative, commutative with identity and all modules are unitary R-module. A submodule K of M is small in M if, $K + N \neq M$ for each submodule N of M. A proper submodule L of M is called primary submodule of M, if $rm \in L$, where $r \in R$, $m \in M$, then $m \in L$ or $r^n M \subseteq L$ for some positive integer n. A submodule K of M is called semi-small submodule of M if $K + L \neq M$ for each primary submodule L of M. An R-module M is called quasi-Dedekind module if any nonzero endomorphism of M is a monomorphism. An R-module M is called small quasi-Dedekind module if, for each $f \in End_R(M), f \neq 0$ implies $Kerf$ small in M. An R-module M is called S-quasi-Dedekind module if, for each $f \in End_R(M), f \neq 0$ implies $Kerf$ semi-small in M. Mijbass introduce and study the concept of quasi-Dedekind module (Mijbass, A. S. (1997)). Ghawi study the concept of small quasi-Dedekind module (Ghawi, Th. Y. (2010). In this paper we introduce and study the concept of S-quasi-Dedekind as a generalization of small quasi-Dedekind module.

In the first section, we introduce S-quasi-Dedekind modules and study some basic properties of this concept.

In the second section , we study the relations between S-quasi-Dedekind modules and other related modules.

In third section, we study the endomorphism ring of S-quasi-Dedekind module.

2. Some Properties of S-quasi-Dedekind Modules

In this section, we introduce the concept of S-quasi-Dedekind module as a generalization of quasi-Dedekind module and give some basic properties examples and characterization of this concept.

Definition 1

1. *A proper submodule L of M is called primary submodule of M, if $rm \in L$, where $r \in R$, $m \in M$, then $m \in L$ or $r^n M \subseteq L$ for some positive integer n.*
2. *An ideal I in a ring R is called primary ideal in R, if $xy \in I$, where $x, y \in R$, then either $x^n \in I$ or $y^k \in I$ for some positive integers n and k.*

Definition 2 *Let M be an R-module and $N \leq M$.*

1. *N is called small submodule of M ($N \ll M$, for short) if $N + L \neq M$ for each submodule L of M..*
2. *N is called semi-small submodule of M ($N \ll_s M$, for short) if $N + L \neq M$ for each primary submodule L of M.*
3. *An ideal J in a ring R is called semi-small ideal in R if $I + J \neq R$, for each primary ideal I of R*

Remark 1

1. *Each small submodule is semi-small submodule.*
2. *For each module M, we have $\{0\}$ is a semi-small submodule of M.*
3. *If M is semi-simple module, then $\{0\}$ is the only semi-small submodule.*

Definition 3 *Let M be an R-module.*

1. *M is called small quasi-Dedekind if for all $f \in End_R(M), f \neq 0$ implies $Kerf \ll M$.*
2. *M is called S-quasi-Dedekind if for all $f \in End_R(M), f \neq 0$ implies $Kerf \ll_s M$.*

Example 1

1. *$\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind.*
2. *Let p is a prime integer and $\mathbb{Z}(p^\infty) = \{\frac{a}{p^k} + \mathbb{Z}/\ a,\ k$ are integers and k is positive $\}$. The only submodules of $\mathbb{Z}(p^\infty)$ are $0 \leq \frac{a}{p} + \mathbb{Z} \leq \frac{a}{p^2} + \mathbb{Z} \leq ...$*
   Hence the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is S-quasi-Dedekind.

Remark 2

1. *It is clear that every quasi-Dedekind R-module is a S-quasi-Dedekind R-module. But the converse is not true in general, for example $\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind but it is not quasi-Dedekind.*
2. *Every small quasi-Dedekind R-module is a S-quasi-Dedekind R-module.*
3. *The direct sum of S-quasi-Dedekind modules is not necessary that a S-quasi-Dedekind module, for example each of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind. But $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is not a S-quasi-Dedekind $\mathbb{Z}$-module.*
4. *Every integral domain R is a S-quasi-Dedekind R-module (Mijbass, A. S. (1997)). But the converse need not be in general; for example $\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$-module is S-quasi-Dedekind module, but $\mathbb{Z}/4\mathbb{Z}$ is not an integral domain.*

Proposition 1 *Let M be a semi-simple R-module. Then M is S-quasi-Dedekind if and only if M is quasi-Dedekind.*

Proof. $\Rightarrow$) Let $f \in End_A(M), f \neq 0$. Since M is S-quasi-Dedekind, then $Kerf \ll_s M$. But M is semi-simple, so $Kerf = \{0\}$. Thus M is quasi-Dedekind.
($\Leftarrow$ It is clear.

Proposition 2 *Let M be a finitely generated R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Proof. $\Rightarrow$) Let $f \in End_R(M), f \neq 0$. Suppose that N is a proper submodule of M such that $Kerf + N = M$. Since M is a finitely generated R-module, then there exists a maximal submodule L such that $N \subseteq L$. Thus $Kerf + L = M$. But L is a primary submodule of M and $Kerf \ll_s M$, so $L = M$, a contradiction. Thus $Kerf \ll M$ and M is small quasi-Dedekind.
($\Leftarrow$ It is clear.

Corollary 1 *Let R be an Artinian principal ideal ring and let M be a co-Hopfian R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Proof. Since R is an Artinian principal ideal ring and M is a co-Hopfian R-module, then by (Barry & all, (1997)), M is a finitely generated R-module, thus the result is obtained.

Corollary 2 *Let R be an Artinian principal ideal ring and let M be a weakly co-Hopfian R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Proof. Since R is an Artinian principal ideal ring and M is a weakly co-Hopfian R-module, then (Barry & all, (2010)), M is a finitely generated R-module, thus the result is obtained.

Corollary 3 *Let R be an Artinian principal ideal ring and let M be a Dedekind finite R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Proof. Since R is an Artinian principal ideal ring and M is a Dedekind finite R-module, then by (Barry & all, (2011)),, M is a finitely generated R-module, thus the result is obtained.

Definition 4 *An R-module is called a multiplication R-module if every submodule N of M is of the form IM, for some ideal I of R.*

Proposition 3 *Let M be a mutiplication R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Proof. $\Rightarrow$) Let $f \in End_R(M)$ such that $f \neq 0$. Suppose that N is a proper submodule such that $kerf + N = M$. Since M is a multiplication R-module, then by (El-Bast & all, (1988)), there exists a prime submodule L such that $N \subseteq L$. Thus $Kerf + L = M$. But L is a primary submodule of M and $Kerf \ll_s M$, so $L = M$. This is a contradiction. Thus $Kerf \ll M$ and M is small quasi-Dedekind.
($\Leftarrow$. It is clear.

Corollary 4 *Let M a cyclic R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.*

Lemma 1 *Let M be an R-module and let $N \ll_s M$. If $K \leq N$, then $K \ll_s M$.*

Proof. Let $K + L = M$, for some primary submodule L of M. Since $K \leq N$, then $N + L = M$, and because $N \ll_s M, M = L$, a contradiction.

The following theorem is a characterization of S-quasi-Dedekind modules.

Theorem 1 *Let M be an R-module. Then M is S-quasi-Dedekind if and only if $Hom(M/N, M) = \{0\}$, for all $N \not\ll_s M$.*

Proof. $\Rightarrow$) Suppose that there exists $N \not\ll_s M$ such that $Hom(M/N, M) \neq \{0\}$, then there exists $\phi : M/N \longrightarrow M$, $\phi \neq 0$. Hence $\phi \circ \pi \in End_A(M)$, where π is the canonical projection, and $\phi \circ \pi \neq 0$ which implies $Kerf(\phi \circ \pi) \ll_s M$, but $N \subseteq Ker(\phi \circ \pi)$, so $N \ll_s M$ by lemma 1 which implies a contradiction.
$\Leftarrow$) Suppose that there exits $f \in End_R(M), f \neq 0$ such that $Kerf \not\ll_s M$, define
$g : M/Kerf \longrightarrow M$ by $g(m+Kerf) = f(m)$, for all $m \in M$. So g is well-defined and $g \neq 0$. Hence $Hom(M/Kerf, M) \neq \{0\}$ which is a contraction.

Proposition 4 *Let M de an R-module and let $\overline{R} = R/J$, where J is an ideal of R such that $J \subseteq ann_R(M)$. Then M is a S-quasi-Dedekind R-module if and only if M is a S-quasi-Dedekind $\overline{R}$-module.*

Proof. $\Rightarrow$) We have $Hom_R(M/K, M) = Hom_{\overline{R}}(M/K, M)$, for all $K \leq M$ by (Kasch,F. (1982)). Thus, if M is a S-quasi-Dedekind R-module then $Hom_R(M/K, M) = \{0\}$ for all $K \not\ll_s M$, so $Hom_{\overline{R}}(M/K, M) = \{0\}$ for all $K \not\ll_s M$. Thus M is a S-quasi-Dedekind $\overline{R}$-module.
$\Leftarrow$) The proof of the converse is similary.

Definition 5 *Let M be an R-module and let $N \leq M$. N is called quasi-invertible if $Hom(M/N, M) = \{0\}$.*

Lemma 2 *(Mijbass (1997), Proposition 1.14) Let M be an R-module and let $N \leq M$. Then $ann_R(M) = ann_R(N)$.*

Proposition 5 *Let M be a S-quasi-Dedekind R-module. Then $ann_R(M) = ann_R(N)$ for all $N \not\ll_s M$.*

Proof. Since M is a S-quasi-Dedekind R-module, so by theorem 1 $Hom(M/N, M) = \{0\}$ for all $N \not\ll_s M$ which implies N is a quasi-invertible submodule for all $N \not\ll_s M$. Thus by lemma 2 $ann_R(M) = ann_R(N)$ for all $N \not\ll_s M$.

Lemma 3 *(Abdullah & all, (2011), Proposition 1.16)*
Let M and M' be R-modules and let $f : M \longrightarrow M'$ be an R-epimorphism.
If $K \ll_s M$ such that $Kerf \subseteq K$, then $f(K) \ll_s M'$.

Proposition 5 *Let M_1, M_2 be R-modules such that $M_1 \cong M_2$.*
Then M_1 is a S-quasi-Dedekind R-module if and only if M_2 is a S-quasi-Dedekind R-module.

Proof. $\Rightarrow$) Let $f \in End_R(M_2), f \neq 0$. To prove $Kerf \ll_s M_2$. Since $M_1 \cong M_2$, there exists an isomorphism $g : M_1 \longrightarrow M_2$ and $g^{-1} : M_2 \longrightarrow M_1$. We have. Hence $h = g^{-1} \circ f \circ g \in End_R(M_1), h \neq 0$. So $Kerh \ll_s M_1$, then $g(kerh) \ll_s M_2$ by lemma 3. But we can show that $g(Kerh) = Kerf$ as follows; let $y \in g(kerh)$, then $y = g(x), x \in Kerh$. Hence $h(x) = 0$; that is $g^{-1} \circ f \circ g(x) = 0$, then $g^{-1} \circ f(y) = 0$, so $g^{-1}(f(y)) = 0$ and hence $f(y) = 0$, since g^{-1} is a monomorphism, so that $y \in Kerf$. Now let $y \in Kerf$, then $f(y) = 0$, but $y \in M_2$, so there exists an $x \in M_1$ such that $y = g(x)$, since g is surjective. Thus $f(g(x)) = 0$ and so $g^{-1}(f(g(x)) = 0$; that is $h(x) = 0$. Hence $x \in Kerh$. This implies $y = g(x) \in g(Kerh)$, thus $kerf = g(kerh)$, hence $Kerf \ll_s M_2$.
$\Leftarrow$) The proof the converse is similary.

Lemma 4 *Let M, M' be injective R-modules that can be embedded in each other. Then $M \cong M'$.*

Proof. Since M' is injective, we may assume that $M = M' \oplus X$ and that there exists a monomorphism $f : M \longrightarrow M'$. Note first that if $x_0 + f(x_1) + ... + f(x_n) = 0$, where $x_i \in X$, then all $x_i = 0$. In fact, $x_0 \in Imf \subseteq M'$ implies $x_0 = 0$, and so $x_1 + f(x_2) + ... + f^{n-1}(x_n) = 0$, since f is a monomorphism. By induction, we see that all $x_i = 0$. Therefore, we have $M" = X \oplus f(X) \oplus f^2(X) \oplus .... \subseteq M$. Let $E = E(f(M")) \subseteq M'$, and write $M' = E \oplus Y$. Since $M" = X \oplus f(M"), E(M") = E(X \oplus f(M")) \cong E(X) \oplus E(f(M")) = X \oplus E$. On the other hand $E(M") \cong E(f(M")) = E$, so $X \oplus E \cong E$. From this, we deduce that $M = X \oplus M' = X \oplus E \oplus Y \cong E \oplus Y = M'$

Proposition 6 *Let M, M' be R-modules that can be embedded in each other. Then $E(M)$ is a S-quasi-Dedekind R-module if and only if $E(M')$ is a S-quasi-Dedekind R-module, where $E(M)$ is an injective hull of M.*

Proof. Fix an embedding $f : M \longrightarrow M'$. Then $f(M) \subseteq M' \subseteq E(M')$, so $E(M')$ contains a copy of $E(f(M)) \cong E(M)$. By symmetric $E(M)$ also contains a copy of $E(M'')$. Since $E(M), E(M')$ are injective, then by lemma 4, $E(M) \cong E(M')$. Hence by Proposition 5, the result is obtained.

Definition 6 *Let S be submodule of an R-module M. A submodule C of M is said to be a complement to S in M if C is maximal with respect to the property that $C \cap S = \{0\}$.*

Remark 3

1. *By Zorn's lemma, any submodule S of an R-module has a complement; in fact, any submodule C_0 with $C_0 \cap S = \{0\}$ can be enlarged into a complement to S in M.*
2. *If C is a complement to S, then we have $C \oplus S \leq_e M$*

Proposition 7 *Let M be any R-module and let $g : M \longrightarrow E(M)$. If g is an injective endomorphism of M, then the following assertions are verified.*

1. $E(M)$ is a S-quasi-Dedekind R-module.
2. If $N \leq_e M$, then $E(N)$ is a S-quasi-Dedekind R-module.
3. For any $N \leq M$, there exists $K \leq M$ such that $E(N) \oplus E(K)$ is a S-quasi-Dedekind R-module.
4. If M and M' are R-modules that can be embedded in each other for any injective R-module M', then M' is a S-quasi-Dedekind R-module.

Proof.

1. Let $f \in End_R(E(M))$ such that $f \neq 0$ and $g = f_{|_M}$. Since is $f_{|_M}$injective, we have $M \cap Kerf = \{0\}$. Therefore $M \leq_e E(M)$ implies that $Kerf = \{0\}$,
so $Kerf \ll_s E(M)$. Thus $E(M)$ is a S-quasi-Dedekind R-module.
2. Since $M \leq_e E(M)$, if $N \leq_e M$, then $N \leq_e E(M)$ and $E(M)$ is injective, so the inclusion $N \longrightarrow E(M)$ is an injective enveloppe of M. Thus $E(M) = E(N)$, and so the result is obtained.
3. By Zorn's lemma, there exists a maximal submodule K of M with respect
$N \cap K = \{0\}$. Then $N \oplus K \leq_e M$ and so by the proof of (2) $E(M) \cong E(N \oplus K) \cong E(N) \oplus E(K)$. Thus $E(N) \oplus E(K)$ is a S-quasi-Dedekind R-module.
4. Since M' is an R-module injective, then $E(M') = M'$. By the proposition 6, $E(M) \cong E(M') = M'$, so M' is a S-quasi-Dedekind R-module.

Lemma 5 *(Lam,T. Y. (1999), P. 213)*
Let R be a quasi-Frobenius ring. Then any right R-module M can be embedded in a free module.

Proposition 8 *Let R be a quasi-Frobenius ring and let M be a finitely generated R-module. Then $E(M)$ is a S-quasi-Dedekind R-module if and only $E(M)$ is a small quasi-Dedekind R-module.*

Proof. By lemma 5, we have $M \subseteq F$ for some free module F. Since M is finitely generated, we have $M \subseteq F_0 \subseteq F$ for some free module F_0 of finte rank. Thus by (Lam,T. Y. (1999), P.412), F_0 is an injective R-module, so can be found inside F_0. Then $E(M)$ is a direct summand of F_0 and so is also finitely generated. Thus by proposition 2, the result is obtained.

Lemma 6 *(Lam,T. Y. (1999), P.412-413)*
For any ring, the following are equivalent:
1-R is quasi-Frobenius.
2-A right R-module is projective if and only if it is injective.

Proposition 9 *Let R be a quasi-Frobenius ring and let M a projective R-module. If M is a S-quasi-Dedekind R-module, then $E(M)$ is a S-quasi-Dedekind R-module.*

Proof. By lemma 6, M is injective and so $E(M)$ is a S-quasi-Dedekind R-module.

Proposition 10 *Let M be a quasi-injective R-module, $T = End_R(M)$ and $m \in M$. If mR is a simple R-module, then $T.m$ is a S-quasi-Dedekind S-module.*

Proof. Let $t \in T$ such that $tm \neq 0$. Consider the R-epimorphism: $\phi : mR \longrightarrow tmR$ given by left multiplication by t. Since mR is simple, ϕ is an isomorphism. Let $\psi = \phi^{-1}$ and extend ψ to an endomorphism $g \in T$. Now $gtm = \psi(tm) = \phi^{-1}(tm) = m$, so $m \in T.m$. Thus $T.m$ is a simple T-module. We have $\forall f \in End_T(T.m), f \neq 0$, $Kerf \ll_s T.m$. Hence $T.m$ is a S-quasi-Dedekind S-module.

Proposition 11 *Let M be a S-quasi-Dedekind and quasi-injective R-module,*
let $N \leq M$ such that for all $U \leq N$, $U \ll_s M$ implies $U \ll_s N$.
Then N is a S-quasi-Dedekind R-module.

Proof. Let $f \in End_R(N), f \neq 0$. To prove that $Kerf \ll_s N$. Since M is a quasi-injectif R-module, there exists $g \in End_R(M)$ such that $g \circ i = i \circ f$, where i is the inclusion mapping. Then $g(N) = f(N) \neq 0$. So $Kerg \ll_s M$. But $Kerf \subseteq Kerg$, hence $Kerf \ll_s M$. On the other hand $Kerf \leq N$, so by hypothesis $Kerf \ll_s N$. Thus N is a S-quasi-Dedekind R-module.

Proposition 12 *Every direct summand of a finitely generated S-quasi-Dedekind module is a S-quasi-Dedekind module.*

Proof. Let $M = N \oplus K$ such that M is a S-quasi-Dedekind R-module. Let $f : K \longrightarrow K, f \neq 0$. We have $h = i \circ f \circ p \in End_R(M), h \neq 0$, where p is the natural projection and i is the inclusion mapping. Hence $Kerh \ll_s M$, so $Kerh \ll M$ since M is finitely generated. But $Kerf \subseteq Kerh$, so $Kerf \ll M$. On the other hand $Kerf \leq K$ implies $Kerf \ll K$ by (Ali, A. H. (2010.), Prop. 1.12). Thus K is a S-quasi-Dedekind R-module.

Remark 4 *If M is a S-quasi-Dedekind R-module, $N \leq M$. Then it is not necessary that M/N is a S-quasi-Dedekind R-module; for example the $\mathbb{Z}$-module $M = \mathbb{Z}$ is S-quasi-Dedekind. Let $N = 12\mathbb{Z} \leq \mathbb{Z}$, then $M/N = \mathbb{Z}/12\mathbb{Z}$ is not a S-quasi-Dedekind R-module.*

Remark 5 *The homomorphic image of S-quasi-Dedekind module is not necessary S-quasi-Dedekind; for example $\mathbb{Z}$ as $\mathbb{Z}$-module S-quasi-Dedekind. But $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z}$, where π is the natural projection. However $\mathbb{Z}/12\mathbb{Z}$ as $\mathbb{Z}$-module is not S-quasi-Dedekind.*

Lemma 7 *(Abdullah & all, (2011), Prop. 1.18)*
Let N and K are submodules of an R-module M such that $N \subseteq K$ and $N \subseteq L$ for each primary submodule L of M, if $N \ll_s M$, then $K/N \ll_S M/N$ if and only if $K \ll_s M$.

Proposition 13 *Let M be a S-quasi-Dedekind R-module such that M/U is projectif for all $U \not\ll_s M$. Let $N \ll_s M$ such that $N \subseteq L$, for each primary submodule L of M. Then M/N is a S-quasi-Dedekind R-module for all $N \leq M$.*

Proof. Let $K/N \not\ll_s M/N$, so by lemma 7, $K \not\ll_s M$.
Suppose that $Hom((M/N)/(K/N), M/N) \neq \{0\}$, but $Hom((M/N)/(K/N), M/N) \cong Hom(M/K, M/N)$, so there exists $f : M/K \longrightarrow M/N, f \neq 0$. Since M/K is projective, then there exists
$g : M/K \longrightarrow M$ such that $\pi \circ g = f$, where π is the canonical projection.
Hence $\pi \circ g(M/K) = f(M/K) \neq 0$, so $g \neq 0$. But $g \in Hom(M/K, M), K \not\ll_s M$. Thus $Hom(M/K, M) \neq \{0\}, K \not\ll_s M$; that is M is not S-quasi-Dedekind, which is a contradiction. Thus M/N is a S-quasi-Dedekind R-module.

Proposition 14 *Let M be a quasi-projective R-module and let $N \ll_s M$ such that $g^{-1}(N) \ll_s M$, for each $g \in End_R(M)$. If $N \subseteq L$, for each primary submodule L of M, then M/N is a S-quasi-Dedekind R-module.*

Proof. Let $f \in End_R(M/N)$ such that $f \neq 0$. Since M is quasi-projective, there exists $g \in End_R(M)$ such that $\pi \circ g = f \circ \pi$ where π is the canonical projection.
Let $Kerf = L/N = \{x + N : f(x + N) = N\} = \{x + N : f \circ \pi(x) = N\} = \{x + N : \pi \circ g(x) = N\} = x + N : g(x) + N = N\} = \{x + N : g(x) \in N\} = \{x + N : x \in g^{-1}(N)\} = g^{-1}(N)/N$. Thus $Kerf = g^{-1}(N)/N$. But $g^{-1}(N) \ll_s M$, so by lemma 7, $g^{-1}(N)/N \ll_s M/N$. That is $Kerf \ll_s M/N$.

3. S-quasi-Dedekind Modules and Other Related Modules

In this section, we study the relations between S-quasi-Dedekind modules and other related modules.

Definition 7

1. *An R-module M is called indecomposable if $M \neq \{0\}$ and it is not a direct sum of two nonzero submodules.*
2. *A left principal indecomposable module of a ring R is a left submodule of R, that is a direct summand of R and is an indecomposable module.*

Proposition 15 *Let R be an Artinian ring which is quasi-Frobenius. Then every principal indecomposable R-module has a S-quasi-Dedekind socle.*

Proof. For any primitive idempotent e, consider the principal indecomposable R-module eR. Since eR is projective, then by lemma 6, it is also injective. Let M be simple submodule of eR. Clearly $eR = E(M)$, so $M \leq_e eR$. In particular $Soc(eR) = M$ is S-quasi-Dedekind.

Proposition 16 *Let R be quasi-Frobenius ring and two principal indecomposable R-modules M, M' such that $M \cong M'$. Then there exists two S-quasi-Dedekind R-modules M_1, M_2 such that $M_1 \cong M_2$.*

Proof. Let $M_1 = Soc(M)$ and $M_2 = Soc(M')$. Then by (Lam,T. Y. (1999), P.423), M_1, M_2 are simple R-modules. If $M \cong M'$, then $M_1 \cong M_2$ and M_1, M_2 are S-quasi-Dedekind R-modules.

Proposition 17 *Let M be an R-module such that every nonzero factor module of M is indecomposable. Then M is a S-quasi-Dedekind module R-module.*

Proof. Let L be a proper submodule of M. Suppose that $M = L+K$, where $K \leq M$. We have $M/L\cap K \cong M/L \oplus M/K$. But $M/L \cap K$ is indecomposable so $M/L \neq \{0\}$ and $M/K = \{0\}$. Hence $M = K$. Thus $L \ll M$ and so M is a S-quasi-Dedekind module R-module.

Proposition 18 *Let M be an indecomposable R-module with finite lengh such that $\forall f \in End_R(M)$, f is not nilpotent. Then M is a S-quasi-Dedekind module R-module.*

Proof. Let $f \in End_R(M)$ such that $f \neq 0$. Since f is not nilpotent, then by (Anderson, F.W.& all (1973), P.138) $Kerf = \{0\}$. Thus M is a S-quasi-Dedekind module R-module.

Definition 8 *An R-module M is said to have the direct summand intersection property (briefly SIP) if the intersection of any two direct summands is again a direct summand.*

Lemma 8 *Let M be an indecomposable R-module and N be any R-module. If $M \oplus N$ has the SIP, then every nonzero R-homomorphism from M to N is a monomorphism.*

Proof. Assume $Hom(M, N) \neq \{0\}$ and let f be a nonzero R-homomorphism from M to N. Since $M \oplus N$ has the SIP, then $Kerf$ is a direct summand of M. But M is indecomposable so $Kerf = \{0\}$ and f is a monomorphism.

Proposition 19 *Let M an indecomposable R-module and let N be any R-module such that $Hom(M, N) \neq \{0\}$. If $M \oplus N$ has the SIP, then M is S-quasi-Dedekind. In particular, if $M \oplus M$ has the SIP, then M is S-quasi-Dedekind.*

Proof. By lemma 8, there is a monomorphism f from M to N. Let $g \in End_R(M)$ such that $g \neq 0$. We claim that $Kerg \ll_s M$. Assume that $Kerg \not\ll_s M$, then $Kerg \neq \{0\}$. Since f is a monomorphism, then $Kerf \circ g = Kerg \neq \{0\}$. This is a contradiction. Thus $Kerg \ll_s M$. Hence M is S-quasi-Dedekind.

Definition 9 *Let M be an R-module.*

1. *M is called local if it has exactly one maximal submodule that contains all proper submodules of M.*
2. *M is called hollow if $M \neq \{0\}$ and every proper submodule of M is small in M.*

Remark 6

1. *Every proper submodule of a local module M is semi-small in M.*
2. *Every Hollow R-module is S-quasi-Dedekind. But the converse is not true in general; for example $\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind, but it is not Hollow.*

Proposition 20 *Every local module M is a S-quasi-Dedekind module.*

Proposition 21 *Let M be a hollow R-module. Then M/N is a S-quasi-Dedekind R-module, for all proper submodule N of M.*

Proof. Suppose that M is a hollow R-module, then M/N is a hollow R-module, for all proper submodule N of M. Thus M/N is a S-quasi-Dedekind R-module, for all proper submodule N of M.

Proposition 22 *Let M be an R-module such that for some proper submodule N of M, M/N is Hollow and $N \ll M$. Then M is a S-quasi-Dedekind R-module.*

Proof. Let L be a proper submodule of M. Then $L + N \neq M$, so $(L + N)/N \ll M/N$. Let $M = L + K$, where $K \leq M$, then $M/N = (L + K)/N = (L + N)/N + (K + N)/N$. But $(L + N)/N \ll M/N$ therefore $M = K + N$. Since $N \ll M$, then $M = K$. Thus M is a S-quasi-Dedekind R-module.

Definition 10 *An R-module M is called faithful if $ann_R(M) = \{0\}$.*

Definition 11 *An R-module M is said to have finite uniform dimension if it does not contain a direct sum of an infinite number of non-zero submodules.*

Definition 12 *An R-module M is scalar if, for all $f \in End_R(M)$ then there exists $r \in R$ such that $f(x) = rx$ for all $x \in M$.*

Remark 7 *Let M be an R-module. Then*

1. *If M has finite uniform dimension, then M is weakly co-hopfian.*
2. *If M is scalar, then by (Mohamed-Ali, E. A. (2006), lemma 6.2), $End_R(M) \cong R/ann_R(M)$.*

Proposition 23 *Let M be a semisimple R-module with finite uniform dimension. Then M is a finite direct sum of S-quasi-Dedekind R-modules.*

Proof. Since M is a semisimple R-modules with finite uniform dimension, then M is finitely generated. Thus M is a finite direct sum of simples R-modules, and so M is a finite direct sum of S-quasi-Dedekind R-modules.

Lemma 9 *Let M be a faithful multiplication R-module, then $ann_M(r) = ann_R(r).M$.*

Proof. We have $ann_M(r) \subseteq M$. Since M is multiplication R-module, so
$ann_M(r) = (ann_M(r) : M).M$. We claim that $ann_R(r) = (ann_M(r) : M)$. To prove our assertion: Let $a \in ann_R(r)$, then $ar = 0$ and $arM = \{0\}$; that is $aM \subseteq ann_M(r)$, so that $a \in (ann_M(r) : M)$. Thus $ann_R(r) \subseteq (ann_M(r) : M)$. Now, if $a \in (ann_M(r) : M)$, then $aM \subseteq ann_M(r)$, so $raM = \{0\}$, this implies $ra \in ann_R(M) = \{0\}$. Thus $a \in (ann_R(r)$, so $(ann_M(r) : M) \subseteq ann_R(r)$. Then $ann_R(r) = (ann_M(r) : M)$ and hence $ann_M(r) = ann_R(r).M$.

Lemma 10 *(Abdullah & all, (2011), theorem 2.2) Let M be a finitely generated faithful multiplication R-module and let $N = IM$ be a proper submodule of M. Then $I \ll_s R$ if and only if $N \ll_s M$.*

Lemma 11 *Let M be a local R-module. Then M is a Hollow and cyclic R-module.*

Proof. Suppose that M is a local R-module, then M is a hollow and cyclic R-module. Show first that M is cyclic. Since M is local, then it has a unique maximal submodule N which contains all proper submodules of M. Let $n \in M$ et $n \notin N$. If $Rm \neq M$, this implies $Rm \subseteq N$ which is a contradiction. To show that M is Hollow, let L be a submodule of M with $L + K = M$ for some $K \leq M$. If $K \neq M$, then both of L and K are proper submodules of M. Thus L and K are contained in M, which implies $L = K + L \subseteq N$, hence $N = M$, a contradiction. Thus $M = K$ and so M is a Hollow module.

Theorem 2 *Let M be a finitely generated faithful multiplication R-module. Then M is a S-quasi-Dedekind R-module if and only if R is a S-quasi-Dedekind R-module.*

Proof. $\Rightarrow$) Let $f : R \longrightarrow R$ be a nonzero R-homomorphism. Then for each $a \in R$, $f(a) = ar$ for some $0 \neq r \in R$. Define $g : M \longrightarrow M$ by $g(m) = rm$ for all $m \in M$. It follows that $g \neq 0$, since if $g = 0$, then $rM = \{0\}$ and so $r \in ann_R(M) = \{0\}$, which is a contradiction.
Since M is S-quasi-Dedekind, then $Kerg \ll_s M$. But $Kerg = \{m \in M : g(m) = rm = 0\} = ann_M(r)$ and by lemma 9 $ann_M(r) = ann_R(r).M$, hence by lemma 10 $ann_M(r) \ll_s M$ and so $ann_R(r) \ll_s R$.
However it is easy to see that $Kerf = ann_R(r)$. Hence $kerf \ll_s R$ and hence R is a S-quasi-Dedekind R-module.
$\Leftarrow$) Let $f : M \longrightarrow M$ such that $f \neq 0$. To prove $Kerf \ll_s M$. Since M is a finitely generated multiplication R-module so by (Naoum, A.G. (1990), theorem 3.2), there exists $0 \neq r \in R$ such that $f(m) = rm$ for $m \in M$ and $Kerf = \{m \in M : f(m) = rm = 0\} = ann_M(r)$.
Now define $g : R \longrightarrow R$ by $g(a) = ra$ for all $a \in R$, hence $g \neq 0$, since if $g = 0$, then $rR = \{0\}$ and so $r = 0$ which is a contradiction. Thus $Kerg \ll_s R$, since R is S-quasi-Dedekind. But $Kerg = \{a \in R : g(a) = ra = 0\} = ann_R(r)$ and so

$ann_R(r) \ll_s R$. On the other hand by lemma 9 $ann_M(r) = ann_R(r).M$, so by lemma 10 $ann_M(r) \ll_s M$. Thus $Kerf \ll_s M$ and M is a S-quasi-Dedekind R-module.

Corollary 5 *Let M an R-module. If M is a local faithful R-module. Then R is a S-quasi-Dedekind R-module.*

Proof. Suppose that M is a local R-module, then by lemma 11, M is a hollow and cyclic R-module. But M is a faithful R-module, thus by theorem 2, R is a S-quasi-Dedekind.

Corollary 6 *Let R be an Artinian principal ideal ring and let M be an R-module module with finite uniform dimension. If M is a faithful multiplication R-module, then R is a S-quasi-Dedekind R-module.*

Proof. Since M is an R-module module with finite uniform dimension, then M is a weakly co-Hopfian R-module, so M is a finitely generated R-module. But M is a faithful multipication R-module, thus by theorem 2, R is a S-quasi-Dedekind.

Definition 13 *An R-module M is called monoform if for each nonzero submodule N of M and for each* $f \in Hom(N, M)$, $f \neq 0$ *implies* $Kerf = \{0\}$.

Proposition 24 *Every monoform R-module is a S-quasi-Dedekind R-module.*

Remark 8 *The converse of proposition 24 is not true in general; for example* $\mathbb{Z}/4\mathbb{Z}$ *as* $\mathbb{Z}$*-module is S-quasi-Dedekind, but it is not monoform.*

Definition 14 *An R-module M is called anti-Hopfian if M is not simple and every nonzero factor module of M is isomorphic to M.*

Definition 15 *Let M be an R-module. M is called generalized Hopfian (gH, for short), if for each* $f \in End_R(M), f$ *surjective implies* $Kerf \ll M$.

Proposition 25 *Let M be an anti-Hopfian R-module. If M is a gH R-module, then M is a S-quasi-Dedekind R-module.*

Proof. Let $f \in End_R(M)$ such that $f \neq 0$. Since M is anti-Hopfian R-module, so by (Hirano & all (1986)), f is surjective. But M is gH R-module implies $Kerf \ll M$. Thus $Kerf \ll_s M$ and so M is a S-quasi-Dedekind R-module.

Proposition 26 *Let M be an anti-Hopfian quasi-projective R-module. If M is Dedekind finite module, then M is a S-quasi-Dedekind R-module.*

Proof. Since M is Dedekind finite quasi-projective, then by (Ghorbani & all (2002) P.327), M is a gH R-module. Moreover M is an anti-Hopfian and gH R-module, thus by proposition 25, M is a S-quasi-Dedekind R-module.

Definition 16 *An R-module M is called special generalized Hopfian (sgH, for short), if whenever f is a left regular element of* $End_R(M)$*; that is if f is not a left zero divisor, then* $Kerf \ll M$.

Theorem 3 *Let M be a scalar R-module such that* $ann_R(M)$ *is prime. If M is a sgH R-module, then M is a S-quasi-Dedekind R-module.*

Proof. Since M is a scalar R-module, thus by remark 7 $End_R(M) \cong R/ann_R(M)$. Thus $End_R(M)$ is an integral domain. Hence for each $f \in End_R(M), f \neq 0, f$ is nonzero divisor and since M is sgH, so we get $Kerf \ll M$. Thus $Kerf \ll_s M$ and so M is a S-quasi-Dedekind R-module.

Proposition 27 *Let M be an anti-Hopfian R-module. If M is a sgH R-module, then M is a S-quasi-Dedekind R-module.*

Proof. Since M is anti-Hopfian, then by ((Hirano & all (1986)), Theorem 14 P.129) $End_R(M)$ is an integral domain, so that for each $f \in End_R(M), f \neq 0$ implies f is nonzero divisor. Hence
$Kerf \ll M$, since M is sgH. Thus $Kerf \ll_s M$ and so M is a S-quasi-Dedekind R-module.

Definition 18 *Let M be an R-module, put* $\mathbb{Z}(M) = \{m \in M : ann_R(m) \leq_e R\}$*. M is called nonsingular if* $\mathbb{Z}(M) = \{0\}$*, and singular if* $\mathbb{Z}(M) = M$.

Lemma 12 *Let* $f : M \longrightarrow M'$ *of homomorphism of right R-modules. If* $N \leq_e M'$, $f^{-1}(N) \leq_e M$.

Proof. Consider any $e \in M\backslash f^{-1}(N)$. Then $f(e) \neq 0$, so there exists $r \in R$ such that $f(e)r \in N\backslash\{0\}$. Then cleary $er \in f^{-1}(N)\backslash\{0\}$. Thus $f^{-1}(N) \leq_e M$.

Remark 9 *Given* $N \leq_e M'$ *and any element* $y \in M'$*, let* $f : R_R \longrightarrow M'$ *be defined by* $f(r) = yr$*. Then the lemma 12 implies* $f^{-1}(N) = y^{-1}N = \{r \in R : yr \in N\} \leq_e R_R$.

Proposition 28 *Let M be a nonsingular uniform R-module. Then M is a S-quasi-Dedekind R-module.*

Proof. Let $f \in End_R(M)$ such that $f \neq 0$. Then $Kerf = \{0\}$. If $Kerf \neq \{0\}$, then $Kerf \leq_e M$. For any $y \in M, y^{-1}Kerf \leq_e R_R$ by remark 9.
Now $f(y).y^{-1}Kerf \subseteq f(y.y^{-1}Kerf) \subseteq f(Kerf) = \{0\}$, so $f(y) \in \mathbb{Z}(M) = \{0\}$, that is $f = 0$, a contradiction. Then

$Kerf = \{0\}$, and so $Kerf \ll_s M$. Thus M is S-quasi- Dedekind.

Corollary 7 *Let M be a nonsingular uniform R-module. If M is injective, then $E(M)$ is a S-quasi-Dedekind R-module.*

Proof. Since M is injectif, then $E(M) = M$. By proposition 28, $E(M)$ is a S-quasi-Dedekind R-module.

Remark 10 *If M is a nonsingular module, then by (Lam,T. Y. (1999), P.277) $\overline{E}(M) = E(M)$, where $\overline{E}(M)$ is the rational hull of M.*

Corollary 8 *Let M be a nonsingular uniform R-module. If M is injectif, then $\overline{E}(M)$ is a S-quasi-Dedekind R-module.*

Proof. We have $\overline{E}(M) = E(M) = M$. Thus $\overline{E}(M)$ is a S-quasi-Dedekind R-module.

Proposition 29 *Let M be an R-module such that $Hom_R(S, M) = \{0\}$ for any singular module S. If M is uniform, then M is a S-quasi-Dedekind R-module.*

Proof. Let S be a singular R-module such that $Hom_R(S, M) = \{0\}$. Then M is nonsingular R-module. If M is not nonsingular, then $S = \mathbb{Z}(M)$ is a nonzero singular module, and the inclusion map $S \longrightarrow M$ is a nonzero element in $Hom_R(S, M)$, a contradiction. Thus M is a nonsingular uniform R-module, and so by proposition 28 M is a S-quasi-Dedekind R-module.

Proposition 30 *Let M be a nonsingular R-module. If M is quasi-injectve and indecomposable, then M is a S-quasi-Dedekind R-module.*

Proof. We have M is uniform. If M is not uniform, then $N \cap K \neq \{0\}$, where $N \neq \{0\} \neq K$ in M. Upon taking $E(N)$ and $E(K)$ inside $E(M)$, we have $E(N) + E(K) = E(N) \oplus E(K)$ is injective, we may write $E(M) = E(N) \oplus E(K) \oplus X$, for some $X \subseteq E(M)$. By (Lam,T. Y. (1999), P.239) $M = (M \cap E(N)) \oplus (M \oplus E(K)) \oplus (M \cap X)$. Since $M \cap E(N) \neq \{0\} \neq M \cap E(K)$, then M is decomposable, which is a contradiction. Now, M is nonsingular uniform R-module, and by proposition 28, M is a S-quasi-Dedekind R-module.

Lemma 13

1. *If $f : M \longrightarrow M'$ is any R-homomorphism, then $f(\mathbb{Z}(M)) \subseteq \mathbb{Z}(M')$.*
2. *If $M \subseteq M'$, then $\mathbb{Z}(M) = M \cap \mathbb{Z}(M')$.*

Proof.

1. Follows from the fact $ann_R(m) \subseteq ann_R(f(m))$ for any $m \in M$.
2. Follows directly from the definition.

Proposition 31 *Let M be an R-module and let $0 \neq N \leq M$ such that N and M/N are both nonsingular. If M is uniform, then M and N are both S-quasi-Dedekind R-modules.*

Proof. First show that M is a S-quasi-Dedekind R-module. By lemma 13, we have $\mathbb{Z}(M) \cap N = \mathbb{Z}(N) = \{0\}$. Therefore the projection map from M to M/N induces an injective homomorphism $\pi : \mathbb{Z}(M) \longrightarrow M/N$. Thus by lemma 13, we have $\pi(\mathbb{Z}(M)) \subseteq \mathbb{Z}(M/N) = \{0\}$, so $\pi = 0$. This implies that $\mathbb{Z}(M) = \{0\}$. Then M is a nonsingular uniform R-module, and so by proposition 28, M is a S-quasi-Dedekind R-module. It is clear that N is a nonsingular uniform R-module. Then N is a S-quasi-Dedekind R-module.

Proposition 32 *Let M be an R-module all of whose nonzero quotients have minimal submodules such that $Soc(M)$ is nonsingular. If M is uniform, then M is a S-quasi-Dedekind R-module.*

Proof. Assume that $Soc(M)$ is nonsingular. Then $Soc(M) \cap \mathbb{Z}(M) = \{0\}$, so $Soc(M)$ can be enlarged to a complement Q of $\mathbb{Z}(M)$. We have M is nonsingular R-module. If $\mathbb{Z}(M) \neq \{0\}$, then, by given assumption on M, there exists $T \supseteq Q$ such that T/Q is simple, $T \cap \mathbb{Z}(M) \subseteq Soc(M)$, a contradiction. Then M is nonsingular. Thus M is a nonsingular uniform R-module, and so by proposition 28, M is a S-quasi-Dedekind R-module.

4. Some Properties of the Endomorphism Ring of S-quasi-Dedekind Module

Proposition 33 *Let M be a simple R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring.*

Proof. By Schur's lemma $End_R(M)$ is a division ring. Thus $End_R(M)$ is a S-quasi-Dedekind ring.

Proposition 34 *Let M be an anti-Hopfian R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring.* *Proof.* Since M is anti-Hopfian, then by (Hirano, Y. & all (1986), Theorem 14, P.129), $End_R(M)$ is an integral domain. Thus $End_R(M)$ is a S-quasi-Dedekind ring.

Proposition 35 *Let M be a nonsingular uniform R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring.*

Proof. Let $f \neq 0 \neq g \in End_R(M)$, then by the proposition 28, f, g are injectives and so $fg \neq 0$. Thus $End_R(M)$ is an integral domain. Hence $End_R(M)$ is a S-quasi-Dedekind ring.

Proposition 36 *Let M be a scalar R-module with $ann_R(M)$ is a prime ideal of R, then $End_R(M)$ is a S-quasi-Dedekind ring.*

Proof. Since M is a scalar R-module, then by remark 7, $End_R(M) \cong R/ann_R(M)$, so $End_R(M)$ is an integral domain. Hence $End_R(M)$ is a S-quasi-Dedekind ring.

Corollary 9 *If M is scalar and prime R-module, then $End_R(M)$ is a S-quasi-Dedekind ring.*

Proposition 37 *Let M be a scalar faithful R-module. $End_R(M)$ is a S-quasi-Dedekind ring if and only if R is a S-quasi-Dedekind ring.*

Proof. Suppose that M is scalar R-module, so by remark 7, $End_R(M) \cong R/ann_R(M)$. But M is faithful, thus $R/ann_R(M) \cong R$, so $End_R(M) \cong R$. Hence we have on the result.

Proposition 38 *Let R be an Artinian principal ideal ring and let M be a weakly co-Hopfian multiplication faithful R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring if and only if R is a S-quasi-Dedekind ring.*

Proof. Suppose that M is a weakly co-Hopfian R-module, so M is a finitely generated R-module. Thus by (Naoum, A.G. (1990), theorem 3.2), M is scalar R-module; that is M is scalar faithful R-module. Thus by proposition 37, the result is obtained.

Proposition 39 *Let R be an Artinian principal ideal ring and let M be a co-Hopfian multiplication faithful R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring if and only if R is a S-quasi-Dedekind ring.*

Proof. Suppose that M is co-Hopfian R-module, so M is a finitely generated R-module. Thus M is scalar R-module; that is M is scalar faithful R-module. Thus by proposition 37, the result is obtained.

Proposition 40 *Let R be an Artinian principal ideal ring and let M be a Dedekind finite multiplication faithful R-module. Then $End_R(M)$ is a S-quasi-Dedekind ring if and only if R is a S-quasi-Dedekind ring.*

Proof. Suppose that M is a Dedekind finite R-module, so M is a finitely generated R-module. Thus M is scalar R-module; that is M is scalar faithful R-module. Thus by proposition 37, the result is obtained.

Definition 18 *Let M be an R-module. M is said quasi-prime if $ann_R(N)$ is a prime ideal of R.*

Proposition 41 *Let M be a quasi-injective scalar and quasi-prime R-module. Then $End_R(N)$ is a S-quasi-Dedekind ring for all $0 \neq N \leq M$.*

Proof. Assume that $0 \neq N \leq M$. Since M is a quasi-injective scalar R-module, then by (Shibab, B.N. (2004), Prop. 1.1.16), N is a scalar R-module. Thus by remark 7, $End_R(N) \cong R/ann_R(N)$. But M is a quasi-prime R-module, so $ann_R(N)$ is a prime ideal of R; that is $End_R(N)$ is an integral domain. Hence $End_R(N)$ is a S-quasi-Dedekind ring.

Corollary 10 *Let M be an injective scalar and quasi-prime R-module. Then $End_R(N)$ is a S-quasi-Dedekind ring for all $0 \neq N \leq M$.*

Corollary 11 *Let M be a quasi-injective scalar R-module and let $0 \neq N \leq M$ be a faithful R-module. Then $End_R(N)$ is a S-quasi-Dedekind ring for all $0 \neq N \leq M$.*

Proof. It follows by (Shibab, B.N. (2004), Prop. 1.1.16) and proposition 37.

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