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Existence of Solutions for Functional Integral Equation Involving the Henstock-Kurzweil-Stieltjes Integral

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Abstract

In this paper, we apply the method associated with the technique of measure of noncompactness and some generalizations of Darbo fixed points theorem to study the existence of solutions for a class of integral equation involving the Henstock-Kurzweil-Stieltjes integral. Meanwhile, an example is provided to illustrate our results.

Keywords: Henstock-Kurzweil-Stieltjes integral, measure of noncompactness, Darbo fixed point theorem, coupled fixed point

2010 MSC: 47H08, 26A39.

1. Introduction

Existence theorems of coupled fixed points have been considered by several authors (Chang & Cho, 1996; Roshan, 2017). In (Chang & Cho, 1996), the authors proved the existence of coupled fixed points for a class of integral operator:

$$A(u,v)(t) = h(t,u(t),v(t)) + \int_0^t K(t,s)\psi(s,u(s),v(s))ds, \tag{1}$$

where $t \in [0,L], L > 0,\ u,\ v \in C[0,L],\ K \in C([0,L]\times[0,L]),\ h,\ \psi \in C([0,L]\times\mathbb{R}_+\times\mathbb{R}_+),\ \mathbb{R}_+ = [0,\infty),\ C[0,L]$ is the space of all real valued continuous functions on $[0,L]$.

In this paper we establish the existence of solutions for the following integral equation involving the Henstock-Kurzweil-Stieltjes integral:

$$(x,y)(t) = h(t,x(t),y(t)) + \phi\left(t, \int_0^t f(t,s,x(s),y(s))dg(s)\right), \tag{2}$$

where h,ϕ,f are continuous functions, $g : [0,L] \to \mathbb{R}$ is of boundary variation. dg can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of g, that is, the system could be controlled by some impulsive force. The Henstock-Kurzweil-Stieltjes integral, which is a generalization of the Lebesgue-Stieltjes integral (Krejčí, 2006; Kurzweil, 1957; Lee, 1989; Schwabik & Ye, 2005), has been proved useful in the study of ordinary differential equations (Chew, 1988; Chew & Flordeliza, 1991; Heikkilä & Ye, 2012; Ye & Liu, 2016).

To achieve our goal, the approach associated with the technique of measure of noncompactness and some generalizations of Darbo fixed points theorem will be used.

This paper is organized as follows. In Section 2, we recall some basic concepts of the Henstock-Kurzweil-Stieltjes integrals and measure of noncompactness. In Section 3, we verify the existence of solutions for (2) by a coupled fixed point theorem. In Section 4, we give an example to illustrate Theorem 3.2 in this paper.

2. Preliminaries

In this section, so that the paper is self-contained, we provide preliminary material with respect to the Henstock-Kurzweil-Stieltjes integral and measure of noncompactness.

2.1 The Henstock-Kurzweil-Stieltjes Integral

The basic concept in the Henstock-Kurzweil-Stieltjes integration theory is that of a δ-fine partition, we refer the interested reader to (Krejčí, 2006; Kurzweil, 1957; Lee, 1989; Schwabik & Ye, 2005).

Now, we introduce the definition of Henstock-Kurzweil-Stieltjes integrals.

For given functions $f,\ g : [a,b] \to \mathbb{R}$ and a δ-fine partition D, we define

$$\begin{aligned} K_D(f,g) &= \sum_{i=1}^{m} f(\xi_i)(g(t_i) - g(t_{i-1})) \\ &= f(b)g(b) - f(a)g(a) - \sum_{i=1}^{m} (f(\xi_i) - f(\xi_{i-1}))g(t_{i-1}), \end{aligned}$$

where $\xi_0 = a,\ \xi_m = b$.

Definition 2.1. (Krejčí, 2006) Let $f,\ g : [a,b] \to \mathbb{R}$ be given. We say that $J \in \mathbb{R}$ is the Henstock-Kurzweil-Stieltjes integral (HKS) over $[a,b]$ of f with respect to g and denote

$$J = (HKS)\int_a^b f(t)dg(t) = (HKS)\int_a^b f dg,$$

if for every $\varepsilon > 0$, there exists positive function $\delta > 0$, such that for every δ-fine D, we have

$$|J - K_D(f,g)| \le \varepsilon.$$

Definition 2.2. (Krejčí, 2006) A function $f : [a,b] \to \mathbb{R}$ is called regulated on $[a,b]$, if the limits

$$\lim_{s\to t-} f(s) = f(t-), t \in (a,b], \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+), t \in [a,b)$$

exist and are finite with the convention

$$f(a-) = f(a), \quad f(b+) = f(b).$$

Denote by $G[a,b]$ the space of all real valued regulated functions on $[a,b]$.

Obviously, both the space $C[a,b]$ of all real valued continuous functions on $[a,b]$ and the space $BV[a,b]$ of all functions of bounded variation on $[a,b]$ are subsets of $G[a,b]$.

Lemma 2.3. (Krejčí, 2006) *If $f \in G[a,b]$, $g \in BV[a,b]$, then both $\int_a^b f dg$ and $\int_a^b g df$ exist, and*

$$\begin{aligned} \left|\int_a^b f(t)dg(t)\right| &\le \|f\| \cdot \operatorname*{Var}_{[a,b]} g, \\ \left|\int_a^b g(t)df(t)\right| &\le \|f\|\left(|g(a)| + |g(b)| + \operatorname*{Var}_{[a,b]} g\right). \end{aligned}$$

Moreover, if $f_n \in G[a,b]$, $g_n \in BV[a,b]$ are such that $\lim_{n\to\infty} \|f_n - f\| = 0$, $\lim_{n\to\infty} \|g_n - g\| = 0$ as $n \to \infty$, and $\operatorname*{Var}_{[a,b]} g_n \le C$ independently of n, then

$$\lim_{n\to\infty} \int_a^b f_n(t)dg_n(t) = \int_a^b f(t)dg(t).$$

2.2 Measure of Noncompactness

In this subsection, we recall some fundamental facts concerning measure of noncompactness (see[Banaś & Goebel(1980)]). Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0 and $B(x,r)$ denote the closed ball in E centered at x with radius r. The symbol B_r stands for the ball $B(0,r)$. Denote by $\overline{X}$, $\overline{conv}X$ the closure and the closed convex hull of a nonempty subset X of E singly. Finally, let us denote by $\mathfrak{m}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{n}_E$ its subfamilies consisting of all relatively compact subsets.

Definition 2.4. (Mursaleen, 2017) Let (E,d) be a metric space and X a bounded subset of E. The Hausdorff measure of noncompactness (μ-measure or ball measure of noncompactness) of the set X, denoted by $\mu(X)$ is defined to be the infimum of the set of all reals $\varepsilon > 0$ such that X can be covered by a finite number of balls of radii $< \varepsilon$, that is,

$$\mu(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}. \tag{3}$$

The function μ is called the Hausdorff measure of noncompactness.

For each $x \in C[0, L]$, we define

$$\omega(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, L], |t - s| \le \varepsilon\}.$$

Obviously, $\omega(x, \varepsilon) \to 0$, as $\varepsilon \to 0$, since x is uniformly continuous on $[0, L]$. Moreover, if this limit relation holds uniformly for x running over some bounded set $X \subset C[0, L]$, then X is equicontinuous, and vice versa. Therefore, we have:

Theorem 2.5. *On the space $C[0, L]$, the measure of noncompactness (3) is equivalent to*

$$\mu(X) = \lim_{\varepsilon \to 0} \sup_{x \in X} \omega(x, \varepsilon) \tag{4}$$

for all bounded sets $X \subset C[0, L]$.

Remark 2.6. For $X \subset C([0, L] \times [0, L])$, $\mu(X)$ can be defined similarly, see (Kazemi & Ezzati, 2016, Theorem 2.2).

Definition 2.7. (Chang & Cho, 1996) An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Lemma 2.8. (Banaś & Goebel, 1980, Theorem 2) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \le k\mu(X)$$

for any $X \subset \Omega$. Then T has a fixed point.

Denote by Φ the class of all continuous functions $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(1) for all $u_1, v_1, u_2, v_2 \in \mathbb{R}_+$, $\varphi(u_1, v_1) \le \varphi(u_2, v_2)$ if $u_1 \le u_2, v_1 \le v_2$;

(2) $\varphi(u, u) < u$ for all $u > 0$;

(3) $\frac{1}{2}\varphi(u_1, v_1) + \frac{1}{2}\varphi(u_2, v_2) \le \varphi(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2})$ for all $u_1, v_1, u_2, v_2 \in \mathbb{R}_+$.

See details in (Roshan, 2017).

The following generalization of Darbo fixed point theorem will be needed in Section 3.

Lemma 2.9. (Roshan, 2017, Theorem 3.7) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \to \Omega$ be a continuous function satisfying*

$$\mu(T(X_1 \times X_2)) \le \varphi(\mu(X_1), \mu(X_2))$$

for all $X_1, X_2 \subseteq \Omega \times \Omega$, where $\varphi \in \Phi$. Then T has at least a coupled fixed point.

3. Main Results

In this section, we shall prove the existence of solutions of Eq. (2).

Firstly, we give the following assumptions:

(D_1) The function $h : [0, L] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\varphi \in \Phi$ such that

$$|h(t, x, y) - h(t, u, v)| \le \varphi(|x - u|, |y - v|), \ \forall t \in [0, L], \ \forall x, y, u, v \in \mathbb{R},$$

and $M_1 = \sup\{|h(t, 0, 0)| : t \in [0, L]\}$;

(D_2) The function $\phi : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous,

$$|\phi(t, z_1) - \phi(t, z_2)| \le |z_1 - z_2|, \ \forall t \in [0, L], \ \forall z_1, z_2 \in \mathbb{R},$$

and $M_2 = \sup\{|\phi(t, 0)| : t \in [0, L]\}$;

(D_3) The function $g \in BV[0, L] \cap C[0, L]$ is nondecreasing, the function $f : [0, L] \times [0, L] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist functions $m_1 : [0, L] \to \mathbb{R}_+$ and $m_2 : [0, L] \to \mathbb{R}_+$ are continuous, such that

$$|f(t, s, x, y)| \leq m_1(t)m_2(s),$$

and $M_3 = \sup\{m_1(t) \int_0^t m_2(s)dg(s)\}$, for any $t, s \in [0, L]$ such that $s \leq t$, and for each $x \in \mathbb{R}$;

(D_4) There exists $r > 0$ such that

$$M_1 + M_2 + M_3 + \varphi(r, r) \leq r.$$

Let $C[0, L] \times C[0, L]$ be equipped with the norm $\|(x, y)\| = \|x\| + \|y\|$. We define an operator F on $C[0, L] \times C[0, L]$ by

$$F(x, y)(t) = h(t, x(t), y(t)) + \phi\left(t, \int_0^t f(t, s, x(s), y(s))dg(s)\right). \tag{5}$$

Then we have the following statement.

Theorem 3.1. *Under the assumptions $(D_1) - (D_4)$, the operator F given in (5) has at least one coupled fixed point in the space $C[0, L] \times C[0, L]$.*

Proof. (i) For any $(x, y) \in C[0, L] \times C[0, L]$, and $\|x\|, \|y\| \leq r$,

$$\begin{aligned}
|F(x, y)(t)| &\leq |h(t, x(t), y(t))| + \left|\phi\left(t, \int_0^t f(t, s, x(s), y(s))dg(s)\right)\right| \\
&\leq |h(t, x(t), y(t)) - h(t, 0, 0)| + |h(t, 0, 0)| \\
&\quad + \left|\phi\left(t, \int_0^t f(t, s, x(s), y(s))dg(s)\right) - \phi(t, 0)\right| + |\phi(t, 0)| \\
&\leq \varphi(|x|, |y|) + M_1 + \left|\int_0^t f(t, s, x(s), y(s))dg(s)\right| + M_2 \\
&\leq M_1 + M_2 + m_1(t)\int_0^t m_2(s)dg(s) + \varphi(\|x\|, \|y\|) \\
&\leq M_1 + M_2 + M_3 + \varphi(\|x\|, \|y\|) \\
&\leq r.
\end{aligned}$$

This implies that F maps the space $B_r \times B_r$ into B_r, where $B_r = \{x, y \in C[0, L] : \|x\| \leq r,\ \|y\| \leq r\}$, r is a constant arising in assumption (D_4).

(ii) We prove that the operator F is continuous on $B_r \times B_r$.

For arbitrary $(x, y) \in B_r \times B_r$, $\varepsilon > 0$, now let $(u, v) \in B_r \times B_r$ with $\|(x, y) - (u, v)\| < \varepsilon$, then we have

$$\begin{aligned}
&|F(x, y)(t) - F(u, v)(t)| \\
&\leq |h(t, x(t), y(t)) - h(t, u(t), v(t))| \\
&\quad + \left|\phi\left(t, \int_0^t f(t, s, x(s), y(s))dg(s)\right) - \phi\left(t, \int_0^t f(t, s, u(s), v(s))dg(s)\right)\right| \\
&\leq \varphi(|x - u|, |y - v|) + \left|\int_0^t f(t, s, x(s), y(s))dg(s) - \int_0^t f(t, s, u(s), v(s))dg(s)\right| \\
&\leq \varphi(\|x - u\|, \|y - v\|) + \int_0^t |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))|\, dg(s) \\
&\leq \varphi(\varepsilon, \varepsilon) + \int_0^L \omega_1(f, \varepsilon)dg(s) \\
&\leq \varphi(\varepsilon, \varepsilon) + +\|\omega_1(f, \varepsilon)\| \cdot \operatorname*{Var}_{[0,L]} g,
\end{aligned} \tag{6}$$

where

$$\omega_1(f, \varepsilon) = \sup\left\{|f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))|, t, s \in [0, L], \|(x, y) - (u, v)\| \leq \varepsilon\right\}.$$

Since uniformly continuity of the function $t \mapsto f(t, s, x, y)$ on the set $[0, L]$, we infer that $\omega_1(f, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus, taking into account the property of the function φ and linking (6), for each $t \in [0, L]$ we get

$$|F(x, y)(t) - F(u, v)(t)| \leq \varepsilon. \tag{7}$$

Hence, the operator F is continuous on $B_r \times B_r$.

(iii) Taking arbitrary nonempty subsets X_1, X_2 of the ball B_r. Fix $\varepsilon > 0$, choose arbitrarily $t_1, t_2 \in [0, L]$ such that $|t_1 - t_2| \leq \varepsilon$. Without loss of generality, assuming that $t_2 < t_1$. Then, for arbitrary $(x, y) \in X_1 \times X_2$, we get

$$\begin{aligned}
&|F(x, y)(t_1) - F(x, y)(t_2)| \\
&\leq |h(t_1, x(t_1), y(t_1)) - h(t_2, x(t_2), y(t_2))| \\
&\quad + \left|\phi\left(t_1, \int_0^{t_1} f(t_1, s, x(s), y(s))dg(s)\right) - \phi\left(t_2, \int_0^{t_2} f(t_2, s, x(s), y(s))dg(s)\right)\right| \\
&\leq |h(t_1, x(t_1), y(t_1)) - h(t_1, x(t_2), y(t_2))| + |h(t_1, x(t_2), y(t_2)) - h(t_2, x(t_2), y(t_2))| \\
&\quad + \left|\phi\left(t_1, \int_0^{t_1} f(t_1, s, x(s), y(s))dg(s)\right) - \phi\left(t_1, \int_0^{t_1} f(t_2, s, x(s), y(s))dg(s)\right)\right| \\
&\quad + \left|\phi\left(t_1, \int_0^{t_1} f(t_2, s, x(s), y(s))dg(s)\right) - \phi\left(t_2, \int_0^{t_1} f(t_2, s, x(s), y(s))dg(s)\right)\right| \\
&\quad + \left|\phi\left(t_2, \int_0^{t_1} f(t_2, s, x(s), y(s))dg(s)\right) - \phi\left(t_2, \int_0^{t_2} f(t_2, s, x(s), y(s))dg(s)\right)\right| \\
&\leq \varphi(|x(t_1) - x(t_2)|, |y(t_1) - y(t_2)|) + \omega(h, \varepsilon) \\
&\quad + \left|\int_0^{t_1} f(t_1, s, x(s), y(s))dg(s) - \int_0^{t_1} f(t_2, s, x(s), y(s))dg(s)\right| + \omega(\phi, \varepsilon) \\
&\quad + \left|\int_0^{t_1} f(t_2, s, x(s), y(s))dg(s) - \int_0^{t_2} f(t_2, s, x(s), y(s))dg(s)\right| \\
&\leq \varphi(\omega(x, \varepsilon), \omega(y, \varepsilon)) + \omega(h, \varepsilon) + \int_0^{t_1} \omega(f, \varepsilon)dg(s) + \omega(\phi, \varepsilon) \\
&\quad + \int_{t_2}^{t_1} |f(t_2, s, x(s), y(s))|dg(s) \\
&\leq \varphi(\omega(x, \varepsilon), \omega(y, \varepsilon)) + \omega(h, \varepsilon) + \int_0^{t_1} \omega(f, \varepsilon)dg(s) + \omega(\phi, \varepsilon) \\
&\quad + m_1(t) \int_{t_2}^{t_1} m_2(s)dg(s),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\omega(h, \varepsilon) &= \sup\{|h(t_1, x, y) - h(t_2, x, y)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \varepsilon, x, y \in [-r, r]\}, \\
\omega(\phi, \varepsilon) &= \sup\{|\phi(t_1, z) - \phi(t_2, z)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \varepsilon\}, \\
\omega(f, \varepsilon) &= \sup\{|f(t_1, s, x, y) - f(t_2, s, x, y)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \varepsilon, x, y \in [-r, r]\}, \\
\omega(x, \varepsilon) &= \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \varepsilon\}, \\
\omega(y, \varepsilon) &= \sup\{|y(t_1) - y(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \varepsilon\},
\end{aligned}$$

By $(D_1) - (D_3)$, h, f, ϕ are uniformly continuous on $[0, L]$, so

$$\omega(h, \varepsilon) \to 0, \ \omega(f, \varepsilon) \to 0, \ \omega(\phi, \varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0.$$

Moreover, since the functions $m_1(t), m_2(s)$ are continuous, we have

$$m_1(t) \int_{t_2}^{t_1} m_2(s)dg(s) \to 0 \ \text{ as } \ |t_1 - t_2| \to 0.$$

Since (x, y) is an arbitrary element of $X_1 \times X_2$ in (8), we obtain

$$\omega(F(X_1 \times X_2), \varepsilon) \leq \varphi\left(\omega(x, \varepsilon), \omega(y, \varepsilon)\right).$$

Hence,

$$\lim_{\varepsilon\to 0}\sup \omega(F(X_1\times X_2),\varepsilon) \le \varphi\left(\lim_{\varepsilon\to 0}\sup \omega(x,\varepsilon), \lim_{\varepsilon\to 0}\sup \omega(y,\varepsilon)\right). \tag{9}$$

It follows from (9) and Theorem 2.5 that

$$\mu\left(F(X_1\times X_2)\right) \le \varphi(\mu(X_1),\mu(X_2)) \tag{10}$$

According to Lemma 2.9, F has at least a coupled fixed point in the space $B_r \times B_r$. The proof is therefore complete. □

According to Theorem 3.1 and (5) the definition of the operator F, we have

Theorem 3.2. *Under the assumptions* $(D_1)-(D_4)$*, Eq. (2) has at least one solution in the space* $C[0,L]\times C[0,L]$.

4. Applications

Example 4.1. Consider the following integral equation

$$\begin{aligned}(x,y)(t) &= \frac{1}{8}e^{-t} + \frac{1}{2}\ln(1+|x(t)|) + \frac{1}{2}\ln(1+|y(t)|) + \frac{t^2}{2(1+t^2)}\\ &\quad + 2\ln\left(1+\frac{\left|\int_0^t e^{-t^2}\cdot \frac{\cos x(s)\cos y(s)}{(1+|\sin x(s)|)(1+|\sin y(s)|)}dg(s)\right|}{2}\right), \quad t\in[0,1],\end{aligned} \tag{11}$$

where g is the Cantor-Lebesgue function (Dovgoshey, Martio, Ryazanov & Vuorinen, 2006).

It is obvious that Eq. (11) is a exception of Eq. (2) with

$$\begin{aligned} h(t,x,y) &= \frac{1}{8}e^{-t} + \frac{1}{2}\ln(1+|x(t)|) + \frac{1}{2}\ln(1+|y(t)|),\\ \phi(t,z) &= \frac{t^2}{2(1+t^2)} + 2\ln\left(1+\frac{|z|}{2}\right),\\ f(t,s,x,y) &= e^{-t^2}\cdot\frac{\cos x(s)\cos y(s)}{(1+|\sin x(s)|)(1+|\sin y(s)|)},\\ \varphi(t,s) &= \ln\left(1+\frac{t+s}{2}\right)\in\Phi.\end{aligned}$$

Now we show that all the conditions of Theorem 3.2 are satisfied for Eq. (11).

(i) Obviously, h and ϕ are continuous.

(ii) Clearly, the function $|h(t,0,0)| = \frac{1}{8}e^{-t}$, so $M_1 = \frac{1}{8}$.

(iii) The function $|\phi(t,0)| = \frac{t^2}{2(1+t^2)}$, so $M_2 = \frac{1}{2}$.

(iv) Suppose that $t\in[0,1]$ and $x,y,u,v,z_1,z_2\in\mathbb{R}$ with $|x|\ge|u|, |y|\ge|v|$. Then we can get

$$\begin{aligned} |h(t,x,y)-h(t,u,v)| &= \frac{1}{2}\left(\ln(1+|x|)-\ln(1+|u|)\right) + \frac{1}{2}\left(\ln(1+|y|)-\ln(1+|v|)\right)\\ &= \frac{1}{2}\ln\left(\frac{1+|x|}{1+|u|}\right) + \frac{1}{2}\ln\left(\frac{1+|y|}{1+|v|}\right)\\ &= \frac{1}{2}\ln\left(1+\frac{|x|-|u|}{1+|u|}\right) + \frac{1}{2}\ln\left(1+\frac{|y|-|v|}{1+|v|}\right)\\ &\le \frac{1}{2}\ln(1+|x-u|) + \frac{1}{2}\ln(1+|y-v|)\\ &\le \ln\left(1+\frac{|x-u|+|y-v|}{2}\right)\\ &= \varphi(|x-u|,|y-v|).\end{aligned}$$

Moreover we can get

$$|\phi(t,z_1) - \phi(t,z_2)| = 2\left|\left(\ln\left(1+\frac{|z_1|}{2}\right) - \ln\left(1+\frac{|z_2|}{2}\right)\right)\right| \le |z_1 - z_2|.$$

(v) Further, notice that the function f is continuous, and we have

$$|f(t,s,x,y)| \le e^{-t^2}$$

for $t, s \in [0,1]$ and $x, y \in \mathbb{R}$. If we put $m_1(t) = e^{-t^2}, m_2(s) = 1$, then we have $M_3 = \sup\left\{e^{-t^2}\int_0^t dg(s) : t \in [0,1]\right\} < 1$.

(vi) It is easy to check that for each number $r \ge 5$, we have the following inequality

$$M_1 + M_2 + M_3 + \varphi(r,r) < \frac{1}{8} + \frac{1}{2} + 1 + \ln(1+r) < r.$$

Consequently, all the conditions of Theorem 3.2 are satisfied and Eq. (11) has at least one solution in the space $C[0,1] \times C[0,1]$.

Remark 4.2. In Example 4.1, the Cantor-Lebesgue function $g \in C[0,1] \cap BV[0,1]$, but g is not absolutely continuous on $[0,1]$. Therefore, the methods used to deal with integral equations involving the Lebesgue (or Riemann) integral (Chang & Cho, 1996; Roshan, 2017) are no longer applicable in this case. This means our existence result Theorem 3.2 is more general.

5. Conclusions

In this research, by using the approach associated with the technique of measure of noncompactness and some generalizations of Darbo fixed points theorem, we studied the existence of solutions for a class of integral equation involving the Henstock-Kurzweil-Stieltjes integral, and we obtained the existence of at least one solution for the functional integral equation we considered.

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