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K-loop Structures Raised by the Direct Limits of Pseudo Unitary $U(p, a_n)$ and Pseudo Orthogonal $O(p, a_n)$ Groups

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Abstract

A left Bol loop satisfying the automorphic inverse property is called a K-loop or a gyrocommutative gyrogroup. K-loops have been in the centre of attraction since its first discovery by A.A. Ungar in the context of Einstein's 1905 relativistic theory. In this paper some of the infinite dimensional K-loops are built from the direct limit of finite dimensional group transversals.

Keywords: K-loops, direct limit, pseudo unitary group, pseudo orthogonal group

1. Introduction

A quasigroup is a non-empty set Q with a binary operation $\oplus : Q \times Q \to Q$ such that for all $a, b \in Q$ there exists unique $x, y \in Q$ satisfying $a \oplus x = b$ and $y \oplus a = b$. A loop $(L, \oplus)$ is a quasi-group with a two-sided neutral element $e \in L$. A K-loop $(K, \oplus)$ is a loop satisfying the left Bol loop identity (1) and the automorphic inverse property (2) for all a, b and c in K.

$$a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c. \tag{1}$$

$$(a \oplus b)^{-1} = a^{-1} \oplus b^{-1}. \tag{2}$$

The left Bol identity guarantees that each element in a loop has a two-sided inverse, so the automorphic inverse property in a K-loop makes sense. K-loops are also known as gyrocommutative gyrogroups, see (Ungar, 1997). Ungar's famaous discovery of a K-loop from Einstein's 1905 relativistic theory motivated many researchers, then many examples and theories has been studied, see (Bulut, 2015; Kerby & Wefelscheid, 1974; Kiechle, 1998; Kreuzer & Wefelscheid, 1994; Ungar, 1997, 2001).

Kreuzer and Wefelscheid (Kreuzer & Wefelscheid, 1994) undertook an axiomatic investigation and provided a method to form K-loops from the group transversals as follow:

Theorem 1.1. *Let G be a group. Let A be a subgroup of G and let K be a subset of G satisfying:*

1. *$G = KA$ is an exact decomposition, i.e., for every element $g \in G$ there are unique elements $k \in K$ and $a \in A$ such that $g = ka$.*
2. *If e is the neutral element of G, then $e \in K$.*
3. *For each $x \in K$, $xKx \subseteq K$.*
4. *For each $y \in A$, $yKy^{-1} \subseteq K$.*
5. *For each $k_1, k_2 \in K$ and $\alpha \in A$, if $k_1k_2\alpha \in K$, then there exists $\beta \in A$ such that $k_1k_2\alpha = \beta k_2k_1$.*

Then for all $a, b \in K$ there exists unique $a \oplus b \in K$ and $d_{a,b} \in A$ such that $ab = (a \oplus b)d_{a,b}$. Moreover, $(K, \oplus)$ is a K-loop.

Kiechle in (Kiechle, 1998) showed that we can form many K-loops, see Theorem 1.2, from classical groups over ordered fields by the method developed by Kreuzer and Wefelscheid in (Kerby & Wefelscheid, 1974). The underlying set of the K-loops obtained by this method is the group transversals endowed with a binary operation induced by group multiplication.

Theorem 1.2. *Let R be n-real, and $G \leq GL(n, K)$ with $G = L_G\Omega_G$, then there are $A \oplus B \in L_G$ and $d_{A,B} \in \Omega_G$ with $AB = (A \oplus B)d_{A,B}$ such that $(L_G, \oplus)$ is a K-loop.*

In above theorem R is an ordered field such that $K := R(i)$, where $i^2 = -1$. L is the set of positive definite hermitian $n \times n$

matrices over K and Ω is the unitary group as given below.

$$L = \{A \in K^{n\times n}; A = A^*, \forall v \in K^n \setminus \{0\} : v^* A v > 0\} \tag{3}$$

$$\Omega = \{U \in K^{n\times n}; UU^* = I_n\} \tag{4}$$

R is called n-real if the characteristic polynomial of every matrix in L splits over K into linear factors. We note that in this paper the classical groups are chosen over the fields $\mathbb{R}$ or $\mathbb{C}$. Therefore, we are not going to refer the term *n-real*. The real numbers $\mathbb{R}$ can be considered as a prototype of n-real field R.

Kiechle remarked in (Kiechle, 1998) that Theorem 1.2 can be generalized to the unit group of Banach algebra of bounded operators $\mathcal{H} \to \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space) by polar decomposition theorem. This generalization, see Theorem 1.3, has been studied in (Bulut, 2015) not only for $GL(\mathcal{H})$, but also some of classical complex Banach Lie subgroups of $GL(\mathcal{H})$.

Theorem 1.3. *Let $G \in \{GL(\mathcal{H}), O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$ be one of the classical complex Banach-Lie groups, and let $Pos(\mathcal{H})$ and $U(\mathcal{H})$ are collection of positive self-adjoint operators and unitary operators respectively over $\mathbb{C}$. If $P_G := G \cap Pos(\mathcal{H})$, and $U_G := G \cap U(\mathcal{H})$, then for all $A, B \in P_G$ there exist unique $A \oplus B \in P_G$ and $d_{A,B} \in U_G$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(P_G, \oplus)$ is a K-loop.*

Remark 1. *Note that Theorem 1.3 can be generalized to all classical complex Banach Lie groups.*

In this paper K-loop structures from the direct limit of some of the classical groups are studied.

2. Preliminaries

Let $(I, \leq)$ be a directed set, i.e., for any pair $i, j \in I$ with $i \leq j$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. Let $\{G_i : i \in I\}$ be the collection of groups with the collection of group homomorphisms $\left\{\gamma_{i,j} : G_i \to G_j : i \leq j\right\}$. The triple $(G_i, \gamma_{i,j}, I)$ is called a direct system if the following two axioms are satisfied.

1. $\gamma_{i,i}(x) = x$ for all $x \in G_i$,
2. $\gamma_{i,k} = \gamma_{j,k} \circ \gamma_{i,j}$ for all $i \leq j \leq k$.

The group G is called the direct limit of the direct system $(G_i, \gamma_{i,j}, I)$, if:

1. In case of existing the group homomorphisms $\alpha_i : G_i \to G$, then $\alpha_i = \alpha_j \circ \gamma_{i,j}$ for all $i \leq j$,
2. G respects the universal property, i.e., if there is another group K with the group homomorphisms $\beta_i : G_i \to K$ such that $\beta_i = \beta_j \circ \gamma_{i,j}$ for all $i \leq j$, then there exists a unique group homomorphism $\theta : G \to K$ which makes the following diagram commute.

$$\begin{tikzcd}
G_i \arrow[dd, "\gamma_{i,j}"'] \arrow[dr, "\alpha_i"] \arrow[drr, "\beta_i"] & & \\
& G \arrow[r, dashed, "\theta"] & K \\
G_j \arrow[ur, "\alpha_j"'] \arrow[urr, "\beta_j"'] & &
\end{tikzcd}$$

Figure 1. The Universal Property of the Direct Limit

The direct limit G of the direct system $(G_i, \gamma_{i,j}, I)$ is denoted by $\varinjlim G_i = G$. It can be verified that

$$\varinjlim G_i = \bigcup_{i\in I} G_i \Big/ \sim .$$

Two elements $x, y \in \bigcup_{i\in I} G_i$ where $x \in G_i$ and $y \in G_j$ are similar, $x \sim y$, if there exists $k \in I$ such that $\gamma_{i,k}(x) = \gamma_{j,k}(y)$, where $i, j \leq k$. It can be easily verified that the relation $\sim$ is an equivalence relation. Let $[x]$ be the equivalence class of x. The product of $[x]$ and $[y]$ is defined by $[x].[y] := [\gamma_{i,k}(x).\gamma_{j,k}(y)]$ for some $k \geq i, j$. It is known that the direct limit has a group structure with respect to this product.

3. Pseudo Unitary and Pseudo Orthogonal Groups

Let $M_n(\mathbb{C})$ be the set of n by n matrices with complex entries and let $GL(n,\mathbb{C})$ be the general linear group, i.e., the set of invertible matrices in $M_n(\mathbb{C})$. Restricting the entries of $GL(n,\mathbb{C})$ to real numbers gives $GL(n,\mathbb{R})$. In this paper we only focus two well known classical groups that are called pseudo-unitary and pseudo-orthogonal groups. Even they are Lie groups we only view them as algebraic groups.

Let $p, q \in \mathbb{N}$ with $0 < p \leq q$ such that $p + q = n$ and let $J_{p,q} = diag(I_p, -I_q)$. Let A^* be the conjugate transpose of A and A^T be the transpose of A. The pseudo unitary group (5) and pseudo orthogonal group (6) are given below,

$$U(p,q) = \left\{A \in GL(n,\mathbb{C}) : \ A^* J_{p,q} A = J_{p,q}\right\}. \tag{5}$$

$$O(p,q) = \left\{A \in GL(n,\mathbb{R}) : \ A^T J_{p,q} A = J_{p,q}\right\}. \tag{6}$$

Direct limit of classical groups and their unitary representations are studied by Olshanskii in (Ol'shanskii, 1978). In that paper the infinite dimensional Lie groups are viewed as the direct limit of finite dimensional Lie groups. Folowing are some of the infinite dimensional Lie groups given in (Ol'shanskii, 1978).

1. $GL(\infty,\mathbb{C}) = \bigcup_{n=1}^{\infty} GL(n,\mathbb{C})$.
2. $GL(\infty,\mathbb{R}) = \bigcup_{n=1}^{\infty} GL(n,\mathbb{R})$.
3. $U(p,\infty) = \bigcup_{n=1}^{\infty} U(p,n)$.
4. $O(p,\infty) = \bigcup_{n=1}^{\infty} O(p,n)$.

The direct limit $GL(\infty,\mathbb{C})$ is obtained by taking the infinite union of the groups $Gl(n,\mathbb{C})$. The group homomorphism $\gamma_{n,m} : GL(n,\mathbb{C}) \to GL(m,\mathbb{C})$ $(n \leq m)$ is defined by $\gamma(A) = diag(A, I_{m-n})$, where $diag(A, I_{m-n})$ is a diagonal block matrix such that I_{m-n} is the identity matrix of the size $m - n$ by $m - n$. The group homomorphism $\gamma_{n,m} : U(p,n) \to U(p,m)$ (or $\gamma_{n,m} : O(p,n) \to O(p,m)$) is defined similarly.

In this paper we define a broad class of infinite dimensional classical groups that are the proper subgroups of the groups defined in (Ol'shanskii, 1978). In our construction we use a strictly increasing sequence of positive integers (a_n) and a constant $p \in \mathbb{N}$ such that $b_n = p + a_n$. It is clear that (b_n) is also a strictly increasing sequence of positive integers. Choosing (a_n) as a strictly increasing sequence of positive integers enable us to form K-loops based on how (a_n) be chosen. Following is a list of some of the infinite dimensional classical groups that are investigated in this paper.

1. $GL(\infty,(a_n),\mathbb{R}) = \bigcup_{n=1}^{\infty} GL(a_n,\mathbb{R}) \Big/ \sim$
2. $GL(\infty,(a_n),\mathbb{C}) = \bigcup_{n=1}^{\infty} GL(a_n,\mathbb{C}) \Big/ \sim$
3. $U(p,(a_n),\infty) = \bigcup_{n=1}^{\infty} U(p,a_n) \Big/ \sim$
4. $O(p,(a_n),\infty) = \bigcup_{n=1}^{\infty} O(p,a_n) \Big/ \sim$

Let $A, B \in GL(\infty,(a_n),\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, then $A \in GL(a_i,\mathbb{F})$ and $B \in GL(a_j,\mathbb{F})$ for some positive integers i, j such that $i \leq j$. We say that $A \sim B$, if there exists a positive integer $k \geq i, j$ such that

$$\gamma_{i,k}(A) = \gamma_{j,k}(B) \tag{7}$$
$$diag(A, I_{a_k - a_i}) = diag(B, I_{a_k - a_j}) \tag{8}$$
$$diag(A, I_{a_j - a_i}, I_{a_k - a_j}) = diag(B, I_{a_k - a_j}) \implies B = diag(A, I_{a_j - a_i}). \tag{9}$$

This observation exposes that if $A \in GL(a_i,\mathbb{F})$ is related with $B \in GL(a_j,\mathbb{F})$ for $i \leq j$, then an appropriate choice of k is $k = j$, and replacing k with j in equation (7) yields immediately that $B = \gamma_{i,j}(A) = diag(A, I_{a_j - a_i})$. Let $[A]^+$ and $[A]^-$ be two sets defined below:

$$[A]^+ = \left\{X \in GL(a_j,\mathbb{R}) : \ X = \gamma_{i,j}(A), \ i \leq j\right\}. \tag{10}$$

$$[A]^- = \{Y \in GL(a_r,\mathbb{R}) : \ A = \gamma_{r,i}(Y), \ r \leq i\}. \tag{11}$$

It is clear that A is a common element of $[A]^+$ and $[A]^-$, hence using the transitivity of $\sim$ yields that $[A]^+ = [A]^-$. Therefore, the equivalence class of A, $[A] = [A]^+ = [A]^-$. Henceforth, we use the equivalence class of $[A]$ as defined for $[A]^+$ that is;

$$[A] = \left\{X \in GL(a_j, \mathbb{R}) : \ X = \gamma_{i,j}(A), \ i \leq j\right\} \tag{12}$$

Let $A \in GL(a_i, \mathbb{R})$ and let $B \in GL(a_j, \mathbb{R})$ such that $i \leq j$, then the product of two equivalence classes $[A]$ and $[B]$ is another equivalence class that is defined by:

$$[A][B] := [\gamma_{i,k}(A)\gamma_{j,k}(B)] \text{ for some } k \geq i, j. \tag{13}$$

It can be easily verified that this product is well defined by showing that it is independent from the representatives of each equivalence class.

We only discussed above the equivalence relation over $GL(\infty, (a_n), \mathbb{F})$, but the equivalence relation $\sim$ for $U(p, (a_n), \infty)$, and $O(p, (a_n), \infty)$ are same. If we take $a_n = n$, then our definitions for $GL(\infty, (a_n), \mathbb{F})$, $U(p, (a_n), \infty)$, and $O(p, (a_n), \infty)$ are equivalent to the infinite dimensional Lie groups given in (Ol'shanskii, 1978), but if $a_n \neq n$, then $GL(\infty, (a_n), \mathbb{R})$, $GL(\infty, (a_n), \mathbb{C})$, $U(p, (a_n), \infty)$, and $O(p, (a_n), \infty)$ are proper subgroups of $GL(\infty, \mathbb{R})$, $GL(\infty, \mathbb{C})$, $U(p, \infty)$, and $O(p, \infty)$ respectively. We also show in Theorem 4.3 that $GL(\infty, \mathbb{F}) \cong GL(\infty, (a_n), \mathbb{F})$, $U(p, \infty) \cong U(p, a_n, \infty)$, and $O(p, \infty) \cong O(p, a_n, \infty)$.

At some point we also need to define the direct limit of the set of positive definite symmetric (or hermitian) matrices. The matrix $A \in M_n(\mathbb{C})$ is called positive definite if for each nonzero column matrix $z \in \mathbb{C}^{n\times 1}$ the real part of z^*Az is positive and A is called hermitian if $A^* = A$. The matrix $A \in M_n(\mathbb{R})$ is called positive definite if z^TAz is positive for each nonzero $z \in \mathbb{R}^{n\times 1}$ and it is called symmetric if $A^T = A$.

Let (a_n) be a strictly increasing sequence of positive integers and $p \in \mathbb{N}$ such that $b_n = p + a_n$, and let $P(b_n, \mathbb{C})$ and $P(b_n, \mathbb{R})$ be the set of positive definite hermitian and positive definite symmetric matrices respectively. We showed in lemma 4.2 that if $n \leq m$, then $\gamma_{n,m}(P(b_n, \mathbb{F})) \subseteq P(b_m, \mathbb{F})$ for $F \in \{\mathbb{R}, \mathbb{C}\}$ and clearly $\gamma_{n,n}(A) = A$ for all $A \in P(b_n, F)$ and $\gamma_{i,k} = \gamma_{j,k} \circ \gamma_{i,j}$ for all $i \leq j \leq k$. We denote the direct limit of positive definite hermitian (or symmetric) matrices by $P(\infty, b_n, \mathbb{F}) := \varinjlim P(b_n, \mathbb{F})$, where

$$P(\infty, (b_n), \mathbb{F}) = \bigcup_{n=1}^{\infty} P(b_n, \mathbb{F}) \Big/ \sim \tag{14}$$

We say that $[A] \in P(\infty, (b_n), \mathbb{F})$ is positive definite if each matrix in $[A]$ is positive definite. Similarly, we say that $[A]$ is hermitan if $[A]^* = [A]$ and we say that $[A]$ is symmetric if $[A]^T = [A]$.

Lemma 3.1. *Let $[A] \in P(\infty, (b_n), \mathbb{F})$ and let $[P] \in GL(\infty, (b_n), \mathbb{F})$ for $F \in \{\mathbb{R}, \mathbb{C}\}$, then*

1. *$[A]$ is positive definite if and only if A is positive definite.*
2. *$[P]^* = [P^*]$ if $\mathbb{F} = \mathbb{C}$ and $[P]^T = [P^T]$ if $\mathbb{F} = \mathbb{R}$*

Proof. If $[A]$ is positive definite, then each matrix in $[A]$ is positive definite, so A is positive definite since $A \in [A]$. On the other hand, suppose that $A \in P(b_i, \mathbb{C})$ is positive definite for some $i \in \mathbb{N}$. We want to show that each matrix in $[A]$ is positive definite. Let $B \in [A]$, then $B = \gamma_{i,j}(A)$ for some $j \in \mathbb{N}$ such that $i \leq j$. Let $z = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ where $\alpha^* = [\overline{z_1}, \cdots, \overline{z_{b_i}}]$ and $\beta^* = [\overline{z_{b_i+1}}, \cdots, \overline{z_{b_j}}]$. The product $z^*Bz = \alpha^*A\alpha + \|\beta\|^2 \geq 0$ since A is positive definite. We conclude that positive definiteness of A forces that any matrix in $[A]$ is positive definite, thus $[A]$ is positive definite.

To proof of $[P]^* = [P^*]$ comes from definition of $[P]$. Let $[P] \in GL(\infty, (b_n), \mathbb{C})$, then $P \in GL(b_n, \mathbb{C})$ for some $b_n \in (b_n)$.

$$[P] = \{Y \in GL(b_m, \mathbb{C}) : diag(P, I_{b_m - b_n}) = Y, \ n \leq m\} \tag{15}$$
$$[P]^* = \{Y^* \in GL(b_m, \mathbb{C}) : diag(P^*, I_{b_m - b_n}) = Y^*, \ n \leq m\} \tag{16}$$
$$[P]^* = \{Y^* \in GL(b_m, \mathbb{C}) : diag(P^*, I_{b_m - b_n}) = Y^*, \ n \leq m\} \tag{17}$$
$$= [P^*]. \tag{18}$$

The proof of the real case is same. □

4. Main Results

In this section, let (a_n) be a strictly increasing sequence of positive integers and let $p \geq 0$ be a fixed integer such that $b_n = p + a_n$. We use $U(n)$ and $O(n)$ for usual unitary and orthogonal matrix groups respectively. For a subgroup G of $GL(b_n, \mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ we define following:

$$\Delta_G(b_n, \mathbb{C}) := P(b_n, \mathbb{C}) \cap G \tag{19}$$

$$\Delta_G(b_n, \mathbb{R}) := P(b_n, \mathbb{R}) \cap G \tag{20}$$

$$\Sigma_G(b_n, \mathbb{C}) := U(b_n) \cap G \tag{21}$$

$$\Sigma_G(b_n, \mathbb{R}) := O(b_n) \cap G \tag{22}$$

It is shown in (Kiechle, 1998) that if $G \in \{U(p, a_n), O(p, a_n)\}$, then $\Delta_G(b_n, \mathbb{C})$ and $\Delta_G(b_n, \mathbb{R})$ are K-loops with respect to a binary operation "$\oplus$" induced by group multiplication in $GL(b_n, \mathbb{C})$ and $GL(b_n, \mathbb{R})$ respectively.

Lemma 4.1. *Let (a_n) be a strictly increasing sequence of positive integers and let $p \geq 0$ be a fixed integer and define $c_{m,n} := a_m - a_n$ for each $n, m \in \mathbb{N}$ such that $n \leq m$.The map $\gamma_{n,m} : U(p, a_n) \to U(p, a_m)$ defined by $\gamma_{n,m}(A) = diag(A, I_{c_{m,n}})$ is a group homomorphism.*

Proof. Let $A \in U(p, a_n)$, then $A^* J_{p,a_n} A = A_{p,a_n}$.

We first show that $\gamma_{n,m}(A) = diag(A, I_{c_{m,n}}) := B \in U(p, a_m)$.

$$B^* J_{p,a_m} B = diag(A^*, I_{c_{m,n}}) diag(I_p, -I_{a_m}) diag(A, I_{c_{m,n}}) \tag{23}$$

$$= diag(A^*, I_{c_{m,n}}) diag(I_p, -I_{a_n}, -I_{c_{m,n}}) diag(A, I_{c_{m,n}}) \tag{24}$$

$$= diag(A^*, I_{c_{m,n}}) diag(J_{p,a_n}, -I_{c_{m,n}}) diag(A, I_{c_{m,n}}) \tag{25}$$

$$= diag(A^* J_{p,a_n} A, I_{c_{m,n}}(-I_{c_{m,n}}) I_{c_{m,n}}) \tag{26}$$

$$= diag(J_{p,a_n}, -I_{c_{m,n}}) \tag{27}$$

$$= diag(I_p, -I_{a_n}, -I_{a_m - a_n}) \tag{28}$$

$$= diag(I_p, -I_{a_m}) = J_{p,a_m} \tag{29}$$

On the other hand, $\gamma_{n,m}$ preserve the group product as follow:

$$\gamma_{n,m}(AB) = diag(AB, I_{c_{m,n}}) \tag{30}$$

$$= diag(A, I_{c_{m,n}}) diag(B, I_{c_{m,n}}) \tag{31}$$

$$= \gamma_{n,m}(A) \gamma_{n,m}(B) \tag{32}$$

Therefore, $\gamma_{n,m}$ is group homomorphism. □

Note that restricting $U(p, a_n)$ to $\mathbb{R}$ in Lemma 4.1 gives that $\gamma_{n,m} : O(p, a_n) \to O(p, a_m)$ such that $A \mapsto diag(A, I_{c_{m,n}})$ is also a group homomorphism.

Lemma 4.2. *If $F \in \{\mathbb{R}, \mathbb{C}\}$ and $\gamma_{n,m} : P(b_n, \mathbb{F}) \to M_{b_m}(\mathbb{F})$ defined by $\gamma_{n,m}(A) \mapsto diag(A, I_{c_{m,n}})$, then $\gamma_{n,m}(P(b_n, \mathbb{F})) \subseteq P(b_m, \mathbb{F})$.*

Proof. Let $x \in \mathbb{C}^{b_m \times 1}$ be a nonzero vector such that $x^* = [\overline{x_1}, \ldots, \overline{x_{b_n}}, \ldots, \overline{x_{b_m}}]$, so there exists $i \leq b_m$ such that $x_i \neq 0$. Let $v^* = [\overline{x_1}, \ldots, \overline{x_{b_n}}]$ and let $w = [\overline{x_{b_{n+1}}}, \ldots, \overline{x_{b_m}}]$, then either v or w is a nonzero vector. Observe that:

$$x^* \gamma_{n,m}(A) x = v^* A v + \sum_{i=b_{n+1}}^{b_m} \overline{x_i} x_i = v^* A v + ||w||^2 > 0. \tag{33}$$

Moreover,

$$(\gamma_{n,m}(A))^* = diag(A^*, I_{c_{m,n}}) = diag(A, I_{c_{m,n}}) = \gamma_{n,m}(A). \tag{34}$$

Therefore, $\gamma_{n,m}(P(b_n, \mathbb{C})) \subseteq P(b_m, \mathbb{C})$. The case $F = \mathbb{R}$ is similar. □

Considering lemma 4.1 and 4.2 together yields the following corollary.

Corollary 4.2.1. *Let $G = U(p, a_n)$, then $\gamma_{n,m}(\Delta_G(b_n, \mathbb{C}))$ is contained in $\Delta_G(b_m, \mathbb{C})$. If $G = O(p, a_n)$, then $\gamma_{n,m}(\Delta_G(b_n, \mathbb{R}))$ is contained in $\Delta_G(b_m, \mathbb{R})$.*

Theorem 4.3. *Let F be either the set of real numbers or complex numbers,then*

1. $GL(\infty, \mathbb{F}) \cong GL(\infty, (a_n), \mathbb{F})$.
2. $U(p, \infty) \cong U(p, (a_n), \infty)$.
3. $O(p, \infty) \cong O(p, (a_n), \infty)$

Proof. We only give the proof of first argument since the same map is used to verify (2) and (3). Suppose that $[A] \in GL(\infty, \mathbb{F})$, then there exists $i \in \mathbb{N}$ such that $A \in GL(i, \mathbb{F})$. Define $\Phi([A]) := [\gamma_{i,a_{i^*}}(A)]$ such that $a_{i^*} \in (a_n)$ is the smallest integer satisfying $a_{i^*} \geq i$. This map is well-defined. To see that suppose $[A] = [B]$ for some $A \in GL(i, \mathbb{F})$ and $B \in GL(j, \mathbb{F})$ such that $i \leq j$. The equality of $[A] = [B]$ implies that $B = \gamma_{i,j}(A)$, so

$$\begin{aligned}
\Phi([B]) &= \Phi([\gamma_{i,j}(A)]) && (35)\\
&= [\gamma_{j,a_{j^*}}(\gamma_{i,j}(A))] && (36)\\
&= [\gamma_{i,a_{j^*}}(A)] && (37)\\
&= [\gamma_{i,a_{i^*}}(A)] && (38)\\
&= \Phi([A]) && (39)
\end{aligned}$$

The map Φ preserves the product of equivalence classes.

$$\begin{aligned}
\Phi([A][B]) &= \Phi([\gamma_{i,j}(A)\gamma_{j,j}(B)]) && (40)\\
&= [\gamma_{j,a_{j^*}}(\gamma_{i,j}(A)\gamma_{j,j}(B))] && (41)\\
&= [\gamma_{j,a_{j^*}}(\gamma_{i,j}(A))\gamma_{j,a_{j^*}}((\gamma_{j,j}(B))] && (42)\\
&= [\gamma_{i,a_{j^*}}(A)\gamma_{j,a_{j^*}}(B)] && (43)\\
&= [\gamma_{a_{i^*},a_{j^*}}(\gamma_{i,a_{i^*}}(A))\gamma_{a_{j^*},a_{j^*}}(\gamma_{j,a_{j^*}}(B))] && (44)\\
&= [\gamma_{i,a_{i^*}}(A)][\gamma_{j,a_{j^*}}(B)] && (45)\\
&= \Phi([A])\Phi([B]) && (46)
\end{aligned}$$

Hence, the map Φ preserves the product of equivalence classes from $GL(\infty, \mathbb{F})$ to $GL(\infty, (a_n), \mathbb{F})$. Moreover, the map Φ is surjective. If $[A] \in GL(\infty, (a_n), \mathbb{F})$, then there exists $a_j \in (a_n)$ such that $A \in GL(a_j, \mathbb{F})$. Let $a_j = i$ for some $i \in \mathbb{N}$, but then

$$\begin{aligned}
\Phi([A]) &= [\gamma_{i,a_{i^*}}(A)] && (47)\\
&= [\gamma_{a_j,a_j}(A)] && (48)\\
&= [A] && (49)
\end{aligned}$$

Note that a_{i^*} is the smallest member of the (a_n) which is greater than or equal to i, but then $a_{i^*} = a_j$. Therefore, the equivalence classes given in equation (47) and (48) are same. Finally, suppose that $\Phi([A]) = \Phi([B])$ where $A \in GL(i, \mathbb{F})$ and $B \in GL(j, \mathbb{F})$ for some integers i, j such that $i \leq j$. The definition of Φ immediately implies that $[\gamma_{i,a_{i^*}}(A)] = [\gamma_{j,a_{j^*}}(B)]$, and clearly $\gamma_{i,a_{i^*}}(A) \in [\gamma_{j,a_{j^*}}(B)]$, thus

$$\begin{aligned}
\gamma_{a_{i^*},a_{j^*}}(\gamma_{i,a_{i^*}}(A)) &= \gamma_{j,a_{j^*}}(B) && (50)\\
\gamma_{i,a_{j^*}}(A) = \gamma_{j,a_{j^*}}(B) &\implies i = j \implies A = B \implies [A] = [B]. && (51)
\end{aligned}$$

We conclude that ϕ is a bijective group homomorphism. Therefore, $GL(\infty, \mathbb{F}) \cong GL(\infty, (a_n), \mathbb{F})$. □

Proposition 4.3.1. *Let $G_n = U(p, a_n)$ and $H_n = O(p, a_n)$, then for each $n \in \mathbb{N}$,*

$$G_n = \Delta_{G_n}(b_n, \mathbb{C})\Sigma_{G_n}(b_n, \mathbb{C}) \quad (52)$$

$$H_n = \Delta_{H_n}(b_n, \mathbb{R})\Sigma_{H_n}(b_n, \mathbb{R}) \quad (53)$$

Proof. It is well-known by polar decomposition theorem that each invertible n by n matrix in $GL(n, \mathbb{C})$ has a unique decomposition such that the first component in $P(n, \mathbb{C})$ and the second component is in $U(n)$. Therefore, for any $M \in U(p, a_n)$, $M = PQ$ where $P \in P(b_n, \mathbb{C})$ and $Q \in U(b_n)$. What we need to see is the decomposition of M actually stays in $U(p, a_n)$. To see this we use the fact that $M \in U(p, a_n)$,i.e., $M^* J_{p,a_n} M = J_{p,a_n}$, so $M = J^{-1}_{p,a_n}(M^*)^{-1} J_{p,a_n}$. Replacing the M with PQ gives that $M = J^{-1}_{p,a_n}((PQ)^*)^{-1} J_{p,a_n}$ and,

$$\begin{aligned} M &= J_{p,a_n}{}^{-1}((PQ)^*)^{-1} J_{p,a_n} && (54)\\ &= J^{-1}_{p,a_n}(P^*)^{-1}(Q^*)^{-1} J_{p,a_n} && (55)\\ &= J^{-1}_{p,a_n}(P^*)^{-1} J_{p,a_n} J^{-1}_{p,a_n}(Q^*)^{-1} J_{p,a_n} && (56)\\ &= (J^{-1}_{p,a_n}(P^*)^{-1} J_{p,a_n})(J^{-1}_{p,a_n}(Q^*)^{-1} J_{p,a_n}) && (57)\end{aligned}$$

We set $S_1 := J^{-1}_{p,a_n}(P^*)^{-1} J_{p,a_n}$ and $S_2 = J^{-1}_{p,a_n}(Q^*)^{-1} J_{p,a_n}$. Notice that $S_1{}^* = S_1$ since $J_{p,a_n} = J^{-1}_{p,a_n} = J^*_{p,a_n}$, $P^* = P$, and $Q^* = Q^{-1}$. Recall also that the inverse of a positive definite matrix is also positive definite, so let $v \in \mathbb{C}^{b_n \times 1}$ be any nonzero vector. Then

$$v^* S_1 v = (J_{p,a_n} v)^* P^{-1}(J_{p,a_n} v) > 0. \tag{58}$$

Therefore, $S_1 \in P(b_n, \mathbb{C})$. On the other hand,

$$S_2 = J^{-1}_{p,a_n}(Q^*)^{-1} J_{p,a_n} = J_{p,a_n} Q J_{p,a_n} \in U(b_n) \tag{59}$$

The product $J_{p,a_n} Q J_{p,a_n} \in U(b_n)$ since $U(b_n)$ is a group and J_{p,a_n} and Q are both in $U(b_n)$. We found another decomposition of M that respects the polar decomposition theorem, and by the uniqueness of the polar decomposition theorem

$$P = S_1 = J^{-1}_{p,a_n}(P^*)^{-1} J_{p,a_n} \tag{60}$$

$$Q = S_2 = J^{-1}_{p,a_n}(Q^*)^{-1} J_{p,a_n} \tag{61}$$

The equations (60) and (61) imply that $P, Q \in U(p, b_n)$. Therefore, the equation (52) is valid. This proof can be also applied to the case $H_n = O(p, a_n)$ by restricting $\mathbb{C}$ to $\mathbb{R}$. □

Proposition 4.3.2. *Let $G_n = U(p, a_n)$ and $H_n = O(p, a_n)$. Then*

$$\varinjlim G_n = \varinjlim \Delta_{G_n}(b_n, \mathbb{C}) \varinjlim \Sigma_{G_n}(b_n, \mathbb{C}) \tag{62}$$

$$\varinjlim H_n = \varinjlim \Delta_{H_n}(b_n, \mathbb{R}) \varinjlim \Sigma_{H_n}(b_n, \mathbb{R}) \tag{63}$$

Proof. We only give the proof of equation (62) since the proof equation (63) is verbatim except we use equation (53) in proposition 4.3.1. Let $[A] \in \varinjlim G_n$, then there exists an $i \in \mathbb{N}$ such that $A \in G_i$. The equation (52) in proposition 4.3.1 implies that there exists unique $P \in \Delta_{G_i}(b_i, \mathbb{C})$ and $Q \in \Sigma_{G_i}(b_i, \mathbb{C})$ such that $A = PQ$, but then $[A] = [PQ] = [\gamma_{i,i}(P)\gamma_{i,i}(Q)] = [P][Q]$ and this gives that $\varinjlim G_n \subseteq \varinjlim \Delta_{G_n}(b_n, \mathbb{C}) \varinjlim \Sigma_{G_n}(b_n, \mathbb{C})$.

To see the other inclusion we use the infinite union definition for the direct limit. The sets $S_k := \cup_{n=1}^k [U(p, a_n) \cap P(b_n, \mathbb{C})]$ and $R_k := \cup_{n=1}^k [U(p, a_n) \cap U(b_n)]$ are both subgroups of $U(p, (a_n), \infty)$ for each $k \geq 1$, furthermore S_k and R_k form ascending chain of subgroups of $U(p, (a_n), \infty)$, so their infinite unions are also contained in $U(p, (a_n), \infty)$. Therefore, the product of $\cup_{n=1}^\infty [U(p, a_n) \cap P(b_n, \mathbb{C})]$ and $\cup_{n=1}^k [U(p, a_n) \cap U(b_n)]$ is contained in $U(p, (a_n), \infty)$ since $U(p, (a_n), \infty)$ is a group, and this gives the desired inclusion. □

Corollary 4.3.1. *Let $G_n = U(p, a_n)$ and $H_n = O(p, a_n)$. For all $[X] \in \varinjlim G_n$ and for all $[Y] \in \varinjlim H_n$ following assertions are hold.*

$$|\varinjlim \Delta_{G_n}(b_n, \mathbb{C}) \cap [X] \varinjlim \Sigma_{G_n}(b_n, \mathbb{C})| = 1 \tag{64}$$

$$|\varinjlim \Delta_{H_n}(b_n, \mathbb{R}) \cap [Y] \varinjlim \Sigma_{H_n}(b_n, \mathbb{R})| = 1 \tag{65}$$

Proof. Let $[X] \in \varinjlim G_n$ and let $[\alpha_1], [\alpha_2] \in \varinjlim \Delta_{G_n}(b_n, \mathbb{C}) \cap [X] \varinjlim \Sigma_{G_n}(b_n, \mathbb{C})$. There exists $[W_1]$ and $[W_2]$ in $\varinjlim \Sigma_{G_n}(b_n, \mathbb{C})$ such that $[\alpha_1] = [X][W_1]$ and $[\alpha_2] = [X][W_2]$. Solving $[X]$ in $[\alpha_2] = [X][W_2]$ and substituting it into $[\alpha_1] = [X][W_1]$ gives that

$$[\alpha_1] = ([\alpha_2][W_2]^{-1})[W_1] = [\alpha_2]([W_2]^{-1}[W_1]) \tag{66}$$

Note that the product $[W_2]^{-1}[W_1]$ is in $\varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ since $\varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ is a group. It can be easily shown that,

$$\varinjlim \Delta_{G_n}(b_n,\mathbb{C}) = \varinjlim G_n \cap \varinjlim P(b_n,\mathbb{C}) \tag{67}$$

hence $[\alpha_1] \in \varinjlim G_n$ so,

$$[\alpha_1] = [\alpha_1][1] \tag{68}$$

The decomposition of $[\alpha_1]$ given in equations (66) and (68) and uniqueness of its decomposition by proposition 4.3.2 implies that $[\alpha_1] = [\alpha_2]$.

Therefore, $\varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ is the *left transversal* of $\varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ in $\varinjlim G_n$. The proof of the real case is similar. □

Theorem 4.4. *If* $[A]$ *and* $[B]$ *are two elements of* $\varinjlim \Delta_{G_n}(b_n,\mathbb{C})$*, then there exists a unique* $[A]\oplus[B] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ *and a unique* $d_{[A],[B]} \in \varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ *such that* $[A][B] = ([A]\oplus[B])d_{[A],[B]}$*. On the other hand,* $\varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ *is a K-loop with respect to* $\oplus$*.*

Proof. Let $[A]$ and $[B]$ are two elements of $\varinjlim \Delta_{G_n}(b_n,\mathbb{C})$, then $[A]$ and $[B]$ are also in $\varinjlim G_n$, so $[A][B] \in \varinjlim G_n$. The proposition 4.3.2 forces that $[A][B] = [\alpha][\beta]$ for some unique $[\alpha] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ and $[\beta] \in \varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$. If we define $[\alpha] := [A]\oplus[B]$ and $[\beta] := d_{[A],[B]}$, then

$$[A][B] = ([A]\oplus[B])d_{[A],[B]} \implies [A]\oplus[B] = [A][B]d^{-1}_{[A],[B]}. \tag{69}$$

To see $(\varinjlim \Delta_{G_n}(b_n,\mathbb{C}),\oplus)$ is a K-loop we verify the axioms of Theorem 1.1. We see that the decomposition of $\varinjlim \Delta_{G_n}$ given in (62) is exact by proposition 4.3.2 and corollary 4.3.1. On the other hand, $[I_{b_n}] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$.

If $[A],[B] \in (\varinjlim \Delta_{G_n}(b_n,\mathbb{C}),\oplus)$, then $A \in \Delta_{G_i}(b_i,\mathbb{C})$ and $B \in \Delta_{G_j}(b_j,\mathbb{C})$ for some $i, j \in \mathbb{N}$ with $i \le j$. Notice that $[A][B][A] = [\gamma_{i,k}(A)\gamma_{j,k}(B)\gamma_{i,k}(A)]$ for some $k \ge i, j$, and

$$\begin{aligned}
([A][B][A])^* &= [(\gamma_{i,k}(A)\gamma_{j,k}(B)\gamma_{i,k}(A))^*] \text{ by lemma 3.1} && (70)\\
&= [(\gamma_{i,k}(A^*)\gamma_{j,k}(B^*)\gamma_{i,k}(A^*))] && (71)\\
&= [\gamma_{i,k}(A)\gamma_{j,k}(B)\gamma_{i,k}(A)] && (72)\\
&= [A][B][A] && (73)
\end{aligned}$$

so $[A][B][A]$ is hermitian. Let $z \in \mathbb{C}^{b_k\times 1}$ be a non-zero vector, then

$$\begin{aligned}
z^*(\gamma_{i,k}(A)\gamma_{j,k}(B)\gamma_{i,k}(A))z &= (\gamma_{i,k}(A)z)^*\gamma_{j,k}(B)(z\gamma_{i,k}(A)) && (74)\\
&= (r)^*\gamma_{j,k}(B)r > 0 \text{ since } \gamma_{j,k}(B) \in P(b_k,\mathbb{C}) && (75)
\end{aligned}$$

hence $[A][B][A]$ is positive definite by lemma 3.1. Note that $r = \gamma_{i,k}(A)z$. The product $[A][B][A]$ is already in $U(p,a_k)$ since $\gamma_{i,k}(A)$, $\gamma_{j,k}(B)$, and $\gamma_{i,k}(A)$ are all in $U(p,a_k)$ and $U(p,a_k)$ is a group. We conclude that for each $[A] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$, $[A]\varinjlim \Delta_{G_n}(b_n,\mathbb{C})[A] \subseteq \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$.

Let $[A] \in \varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ and let $[B] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$, then the product $[A][B][A]^{-1}$ is contained in $\varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ due to $[A]^* = [A]^{-1}$ and $[B]^* = [B]$, and $[B]$ is positive definite.

On the other hand, suppose that for $[A_1],[A_2] \in \varinjlim \Delta_{G_n}(b_n,\mathbb{C})$ and $[B] \in \varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ such that $[A_1][A_2][B]$ is an element of $\varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$, then

$$[A_1][A_2][B] = ([A_1][A_2][B])^* = [B^*][A_2{}^*][A_1{}^*] = [B^{-1}][A_2][A_1]. \tag{76}$$

In the equation (76) $[B^{-1}] \in \varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ since $\varinjlim \Sigma_{G_n}(b_n,\mathbb{C})$ is a group.

Therefore, $(\varinjlim \Delta_{G_n}(b_n,\mathbb{C}),\oplus)$ is a K-loop by Theorem 1.1. □

The following corollary is immediate from Theorem (4.4) if $\mathbb{C}$ is restricted to $\mathbb{R}$.

Corollary 4.4.1. *If* $[A]$ *and* $[B]$ *are two elements of* $\varinjlim \Delta_{H_n}(b_n,\mathbb{R})$*, then there exists a unique* $[A]\oplus[B] \in \varinjlim \Delta_{H_n}(b_n,\mathbb{R})$ *and a unique* $d_{[A],[B]} \in \varinjlim \Sigma_{H_n}(b_n,\mathbb{R})$ *such that* $[A][B] = ([A]\oplus[B])d_{[A],[B]}$*. On the other hand,* $\varinjlim \Delta_{H_n}(b_n,\mathbb{R})$ *is a K-loop with respect to* $\oplus$*.*

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