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The Global and Exponential Attractors for the Higher-order Kirchhoff-type Equation with Strong Linear Damping

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Abstract

In this paper, we study the longtime behavior of solution to the initial boundary value problem for a class of strongly damped Higher-order Kirchhoff type equations: $u_{tt} + (-\Delta)^m u_t + \left(\alpha + \beta \|\nabla^m u\|^2\right)^q (-\Delta)^m u + g(u) = f(x)$. At first, we do priori estimation for the equations to obtain two lemmas and prove the existence and uniqueness of the solution by the lemmas and the Galerkin method. Then, we obtain to the existence of the global attractor in $H_0^m(\Omega) \times L^2(\Omega)$ according to some of the attractor theorem. In this case, we consider that the estimation of the upper bounds of Hausdorff for the global attractors are obtained. At last, we also establish the existence of a fractal exponential attractor with the non-supercritical and critical cases.

Keywords: Nonlinear Higher-order Kirchhoff type equation, Galerkin method, The existence and uniqueness, The Global attractor, Huasdorff dimensions, The Exponential attractor

2010 Mathematics Classification: 35K10, 35K10, 35K41

1. Introduction

In this paper, we are concerned with the existence of global attractor for the following nonlinear Higher-order Kirchhoff-type equations:

$$u_{tt} + (-\Delta)^m u_t + \left(\alpha + \beta \|\nabla^m u\|^2\right)^q (-\Delta)^m u + g(u) = f(x), (x,t) \in \Omega \times [0, +\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, \ldots, m-1, x \in \partial\Omega, t \in (0, +\infty), \tag{1.3}$$

where m > 1 is an integer constant, $\alpha > 0, \beta > 0$ are constants and q is a real number. Moreover, Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and v is the unit outward normal on $\partial\Omega$. $g(u)$ is a nonlinear function specified later.

It is known that Kirchhoff (1883) first investigated the following nonlinear vibration of an elastic string for $\delta = f = 0$:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f; \quad 0 \leqslant x \leqslant L, t \geqslant 0, \tag{1.4}$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, E the Young modulus, $p0$ the initial axial tension, δ the resistance modulus, and f the external force.

When $\alpha = 0, \beta = 1$ and $q > 0$ are real number, Yunlong Gao, Yuting Sun and Guoguang Lin (2016) studied existence of weak solutions for degenerate High-order Kirchhoff equations:

$$u_{tt} + (-\Delta)^m u_t + \|\nabla^m u\|^{2q} (-\Delta)^m u + g(u) = f(x), (x,t) \in \Omega \times [0, +\infty), \tag{1.5}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.6}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, \ldots, m-1, x \in \partial\Omega, t \in (0, +\infty), \tag{1.7}$$

where m > 1 is an integer constant. Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and v is the unit outward normal on $\partial\Omega$. $g(u)$ is a nonlinear function specified later.

When $\alpha = 0, \beta = 1$ and $q > 0$ is real number and strong linear stamping $(-\Delta)^m u_t$ is replaced βu_t, Li Yan (2011) studied The Asymptotic Behavior of Solutions for a Nonlinear Higher Order Kirchhoff Type Equation:

$$u_{tt} + \left(\int_\Omega |D^m u|^2 dx\right)^q (-\Delta)^m u + \beta u_t + g(u) = 0, in \quad Q = \Omega \times (0, +\infty), \tag{1.8}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \ldots, m-1, on \quad \sum = \Gamma \times (0, +\infty), \tag{1.9}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), in \quad x \in \Omega, \tag{1.10}$$

where Ω is an open bounded set of $R^n (n \geq 1)$ with smooth boundary Γ and the unit normal vector.The function $g \in C^1$ satisfies some of conditions.

When $\left(\alpha + \beta \|\nabla^m u\|^2\right)^q$ is replaced $a + b\|\nabla^m u\|^{2q}$ and $g(u) = -|u|^p u$, Guoguang Lin, Yunlong Gao, Yuting Sun (2017) had studied local existence and blow-up of solutions:

$$u_{tt} + (-\Delta)^m u_t + \left(a + b\|D^m u\|^{2q}\right)(-\Delta)^m u = |u|^p u, (x,t) \in \Omega \times [0, +\infty), \tag{1.11}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots, m-1, x \in \partial\Omega, t \in (0, +\infty), \tag{1.12}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.13}$$

where Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and v is the unit outward normal on $\partial\Omega$. Moreover, $m > 1$ is an integer constant, and q, p, a and b are some constants such that $q \geq 1$, $p \geq 0$, $a \geq 0$, $b \geq 0$ and $a + b > 0$.

When $q = 0, m = 1, g(u) = -|u|^p u$, the equation (1.1) becomes a nonlinear wave equation:

$$u_{tt} - \Delta u - \Delta u_t = |u|^p u, \quad (x,t) \in \Omega \times [0, +\infty), \tag{1.14}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.15}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, +\infty). \tag{1.16}$$

It has been extensively studied and several results concerning existence and blowing-up have been established (Ball, J. M., 1997; KOPÁČKOVÁ, M, 1989; HARAUX, A. & ZUAZUA, E., 1988).

When $\alpha = 0, \beta = 1, m = 1, g(u) = -|u|^\alpha u$ and $q = \gamma > 0$ is real number, Kosuke Ono (1997) had studied global existence, asymptotic stability and blowing up of solutions for Some Degenerate Non-linear Wave Equations:

$$u_{tt} - \|\nabla u\|^{2\gamma} \Delta u - \Delta u_t = |u|^\alpha u, \quad (x,t) \in \Omega \times [0, +\infty), \tag{1.17}$$

$$u(0) = u_0(x), u_t(0) = u_1(x), \quad x \in \Omega, \tag{1.18}$$

$$u(x,t)|_{\partial\Omega} = 0, \quad t \in [0, +\infty), \tag{1.19}$$

where Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$.

When $\left(\alpha + \beta \|\nabla^m u\|^2\right)^q$ is replaced $-m\left(\int_\Omega |\nabla u(t,x)|^2 dx\right)$, $m = 1, g(u) = 0$ and no linear damping, Marina Ghisi and Massimo Gobbino (2009) studied spectral gap global solutions for degenerate Kirchhoff equations. Given a continuous function $m : [0, +\infty) \to [0, +\infty)$, they consider the Cauchy problem:

$$u_{tt}(t,x) + m\left(\int_\Omega |\nabla u(t,x)|^2 dx\right) \Delta u(t,x) = 0, \forall (x,t) \in \Omega \times [0, T), \tag{1.20}$$

$$u(0) = u_0, u_t(0) = u_1, \tag{1.21}$$

where $\Omega \subseteq R^n$ is an open set and ∇u and Δu denote the gradient and the Laplacian of u with respect to the space variables. They prove that for such initial data (u_0, u_1) there exist two pairs of initial data $(\bar{u}_0, \bar{u}_1), (\hat{u}_0, \hat{u}_1)$ for which the solution is global, and such that $u_0 = \bar{u}_0 + \hat{u}_0, u_1 = \bar{u}_1 + \hat{u}_1$.

When $m = 1$, $\left(\alpha + \beta \|\nabla^m u\|^2\right)^q$ and $(-\Delta)^\alpha u_t$ are replaced $M\left(\|\nabla u\|^2\right), (-\Delta) u_t$. Yang Zhijian, Ding Pengyan and Lei Li (2016) studied Longtime dynamics of the Kirchhoff equations with fractional damping and supercritical nonlinearity:

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + (-\Delta)^\alpha u_t + f(u) = g(x), x \in \Omega, t > 0, \tag{1.22}$$

$$u\,|_{\partial\Omega} = 0, u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.23}$$

where $\alpha \in (\frac{1}{2}, 1), \Omega$ is a bounded domain R_N with the smooth boundary $_{\partial\Omega}$, and the nonlinearity $f(u)$ and external force term g will be specified.The main results are focused on the relationships among the growth exponent p of the nonlinearity $f(u)$ and well-posedness. They show that (i)even if p is up to the supercritical range, that is, $1 \le p < \frac{N+4\alpha}{(N-4\alpha)^+}$, the well-posedness and the longtime behavior of the solutions of the equation are of the characters of the parabolic equation; (ii) when $\frac{N+4\alpha}{(N-4\alpha)^+} \le p < \frac{N+4}{(N-4)^+}$, the corresponding subclass G of the limit solutions exists and possesses a weak global attractor.

When $m = 1$, $\left(\alpha + \beta \|\nabla^m u\|^2\right)^q$ is replaced $\sigma(\|\Delta u\|^2)$, Yang Zhijian, I.Chueshov (Yang, Z. J. & et al., 2014; Zhijian Yang & Zhiming Liu., 2015; Igor Chueshov., 2012) studied the Global attractor and exponential attractors for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity:

$$u_{tt} - \sigma(\|\Delta u\|^2)\Delta u_t - \phi(\|\Delta u\|^2)\Delta u + f(u) = h(x) \quad in \quad \Omega \times \mathbb{R}^+, \tag{1.24}$$

$$u(x,t)|_{\partial\Omega} = 0, u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.25}$$

where Ω is a bounded domain in R^N with the smooth boundary $\partial\Omega$, $\sigma(s)$, $\phi(s)$ and $f(s)$ are nonlinear functions, and $h(x)$ is an external force term. They prove that in strictly positive stiffness factors and supercritical nonlinearity case, there exists a global finite-dimensional attractor in the natural energy space endowed with strong topology.

When $m = 1$, Xiaoming Fan (2004) consider the following non-degenerate Kirchhoff-type's Kernel sections and estimation of Hausdorff dimensions:

$$u_{tt} - \alpha\Delta u_t - \left(\beta + \gamma\left(\int_\Omega |\nabla u|^2 dx\right)^\rho\right)\Delta u + h(u_t) + f(u,t) = g(x,t), x \in \Omega, t > \tau, \tag{1.26}$$

$$u(x,t)\,|_{x\in\partial\Omega} = 0, t \ge \tau, \tag{1.27}$$

$$u(x,\tau) = u_{0\tau}(x), u_t(x,t) = u_{1\tau}(x), x \in \Omega, \tag{1.28}$$

where $\beta > 0, \rho > -1, \gamma \ge 0$. $h(u_t)$ and $g(u,t)$ are supposed in paper.

For the most of the scholars represented by Yang Zhijian have studied all kinds of low order Kirchhoff equations and only a small number of scholars have studied the blow-up and asymptotic behavior of solutions for higher-order Kirchhoff equation. So, in this context, we study the high-order Kirchhoff equation is very meaningful. In order to study the high-order nonlinear Kirchhoff equation with the damping term, we borrow some of Li Yan's (Ball, J. M., 1997) partial assumptions (2.1)-(2.3) for the nonlinear term g in the equation. In order to prove that the lemma 2.4, we have improved the results from assumptions (2.1)-(2.3) such that $0 < C_2 \le \frac{1}{2}$. Then, under all assumptions, we prove that the equation has a unique smooth solution $(u, u_t) \in L^\infty((0,+\infty); H^{2m}(\Omega) \cap H_0^m(\Omega) \times H_0^m(\Omega))$ and obtain the solution semigroup $S(t)$: $H^{2m}(\Omega) \cap H_0^m(\Omega) \times H_0^m(\Omega) \to H^{2m}(\Omega) \cap H_0^m(\Omega) \times H_0^m(\Omega)$ has global attractor $\mathcal{A}$ and the upper bounds of Hausdorff dimensions. At last, we get the exponential attractor by strong quasi-stability.

For more related results we refer the reader to (Xiaoming Fan & Shengfan Zhou., 2004; HD Nguyen., 2014; Yaojun Ye, 2013; Teman, R., 1998; S. Zhou., 1999; Ke Li., 2017; Zhang Yan & et al., 2008; Xueli Song & Yanren Hou., 2015; L. H. Fatori, et al, 2015; Lin, G. G., 2011; Teman, R., 1998; Wu, J. Z. & Lin, G. G., 2009; Robert A. Adams, et al., 2003; Z. J. Yang, 2010; Zhijian Yang & Pengyan Ding, 2016). In order to make these equations more normal, in section 2 and in section 3, some assumptions, notations and the main results are stated. Under these assumptions, we prove the existence and uniqueness of solution, then we obtain the global attractors for the problems (1.1)-(1.3). In section 4, we consider that the estimation of the upper bounds of Hausdorff for the global attractors are obtained according to (Yaojun Ye., 2013). In section 5, we obtain the fractal exponential attractor by (Yang, Z. J. & et al., 2016; Yang, Z. J. & et al., 2014; Zhijian Yang & Zhiming Liu, 2015; Igor Chueshov, 2012).

2. Preliminaries

In this section, we introduce material needed in the proof our main result. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their usual scalar products and norms. Meanwhile we define
$H_0^m(\Omega) = \left\{u \in H^m(\Omega) : \frac{\partial^i u}{\partial v^i} = 0, i = 0, 1, \ldots, m-1\right\}$ and introduce the following abbreviations: $E_0 = H_0^m(\Omega) \times L^2(\Omega), E_1 = H^{2m}(\Omega) \cap H_0^m(\Omega) \times H_0^m(\Omega), A = -\Delta, \|\cdot\|_{H^m} = \|\cdot\|_{H^m(\Omega)}, \|\cdot\|_{H_0^m} = \|\cdot\|_{H_0^m(\Omega)}, \|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for any real number $p > 1$.

According to (Li, Y., 2011), we present some assumptions and notations needed in the proof of our results. For this reason, we assume nonlinear term $g(u) \in C^1(\Omega)$ satisfies that

(H_1) Setting $G(s)=\int_0^s g(r)dr$, then

$$\lim_{|s|\to\infty}\inf\frac{G(s)}{s^2}\geq 0; \tag{2.1}$$

(H_2) If

$$\lim_{|s|\to\infty}\sup\frac{|g'(s)|}{|s|^r}=0, \tag{2.2}$$

where $0\leq r<+\infty(n=1,2), 0\leq r<2(n=3), r=0(n\geq 4)$.

(H_3) There exist constant $C_0>0$, such that

$$\lim_{|s|\to\infty}\inf\frac{sg(s)-C_0G(s)}{s^2}\geq 0. \tag{2.3}$$

(H_4) There exist constant $C_1>0$, such that

$$|g(s)|\leq C_1\left(1+|s|^p\right), \tag{2.4}$$

$$\left|g^{'}(s)\right|\leq C_1\left(1+|s|^{p-1}\right), \tag{2.5}$$

where $1\leq p\leq\frac{n+2m}{n-2m}(n>2m)$ and $1\leq p<+\infty(n\leq 2m)$.

For every $\gamma>0$, by $(H_1)-(H_3)$ and apply Poincar$\acute{e}$ inequality, there exist constants $C(\gamma)>0$, such that

$$J(u)+\gamma\|\nabla^m u\|^2+C(\gamma)\geq 0,\quad\forall u\in H^m(\Omega), \tag{2.6}$$

$$(g(u),u)-C_2J(u)+\gamma\|\nabla^m u\|^2+C(\gamma)\geq 0,\quad\forall u\in H^m(\Omega), \tag{2.7}$$

where $J(u)=\int_\Omega G(u)dx, 0<C_2\leq\frac{1}{2}$ is independent of γ.

Lemma 2.1.(Young's Inequality$^{(Lin,G.G.,2011)}$) For any $\varepsilon>0$ and $a,b\geq 0$, then

$$ab\leq\frac{\varepsilon^p}{p}a^p+\frac{1}{q\varepsilon^q}, \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1, p>1, q>1$.

Lemma 2.2.(Sobolev-Poincar$\acute{e}$ inequality$^{(YaojunYe.,2013)}$) Let s be a number with $2\leq s<+\infty, n\leq 2m$ and $2\leq s\leq\frac{2m}{n-2m}, n>2m$. Then there is a constant K depending on Ω and s such that

$$\|u\|_s\leq K\left\|(-\Delta)^{\frac{m}{2}}u\right\|, \forall u\in H_0^m(\Omega). \tag{2.9}$$

Lemma 2.3.(Gronwall's inequality$^{(Lin,G.G.,2011)}$) If $\forall t\in[t_0,+\infty), y(t)\geq 0$ and $\frac{dy}{dt}+gy\leq h$, such that

$$y(t)\leq y(t_0)e^{-g(t-t_0)}+\frac{h}{g}, t\geq t_0, \tag{2.10}$$

where $g>0, h\geq 0$ are constants.

Lemma 2.4. Assume $(H_1)-(H_3)$ hold, and $(u_0,u_1)\in H_0^m(\Omega)\times L^2(\Omega), f(x)\in L^2(\Omega)$. Then the solution (u,v) of the problem (1.1) – (1.3) satisfies $(u,v)\in L^\infty\left((0,+\infty);H_0^m(\Omega)\times L^2(\Omega)\right)$, and

$$\|\nabla^m u\|^2+\|v\|^2\leq\frac{y(0)}{\min\left\{1,\frac{\beta^q-\varepsilon_1}{2}\right\}}e^{-\varepsilon_1C_2t}+\frac{\frac{\|f\|^2}{\varepsilon_1{}^2}+C_3}{\varepsilon_1C_2\min\left\{1,\frac{\beta^q-\varepsilon_1}{2}\right\}}. \tag{2.11}$$

where $v=u_t+\varepsilon_1u$,$0<\varepsilon_1<\min\left\{\beta^q,\frac{\lambda_1{}^m}{2\lambda_1{}^m+1},\frac{\sqrt{(2+C_2)^2+16\lambda_1{}^m}-2-C_2}{4}\right\}$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $y(0)=\|u_1+\varepsilon_1u_0\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u_0\|^2\right)^{q+1}-\varepsilon_1\|\nabla^m u_0\|^2+2J(u_0)+2C(\gamma_1)+\frac{q\beta^q}{q+1}$

, $C_3 = \frac{(2\alpha)^{q+1}\varepsilon_1}{q\beta} + m_1\left(2C(\gamma_1) + \frac{q\beta^q}{q+1}\right) + \frac{4^{\frac{q+1}{q}}q}{2\beta} + 2\varepsilon_1 C(\gamma_2)$,
$m_1 = \min\left\{2\lambda_1{}^m - 2\varepsilon_1 - 2\varepsilon_1{}^2, \frac{\varepsilon_1(q+1)}{2}\right\}, \gamma_1 = \frac{\beta^q-\varepsilon_1}{4}, \gamma_2 = \frac{1}{2} - \varepsilon_1 - \frac{\varepsilon_1}{2\lambda_1{}^m}$.
Thus, there exists R_0 and $t_0 = t_0(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H_0^m\times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \le R_0{}^2, \quad (t > t_0). \tag{2.12}$$

Proof. We take the scalar product in L^2 of equation (1.1) with $v = u_t + \varepsilon_1 u$. Then

$$\left(u_{tt} + (-\Delta)^m u_t + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q(-\Delta)^m u + g(u), v\right) = (f(x), v). \tag{2.13}$$

By using Poincaré's inequality and Young's inequality, after a computation in (2.13), we have

$$\begin{aligned}(u_{tt}, v) &= \frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon_1\|v\|^2 + \varepsilon_1{}^2(u,v)\\ &\ge \frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon_1\|v\|^2 - \frac{\varepsilon_1{}^2}{2}\|u\|^2 - \frac{\varepsilon_1{}^2}{2}\|v\|^2\\ &\ge \frac{1}{2}\frac{d}{dt}\|v\|^2 - \left(\varepsilon_1 + \frac{\varepsilon_1{}^2}{2}\right)\|v\|^2 - \frac{\varepsilon_1{}^2}{2\lambda_1{}^m}\|\nabla^m u\|^2,\end{aligned} \tag{2.14}$$

$$\begin{aligned}((-\Delta)^m u_t, v) &= -\frac{\varepsilon_1}{2}\frac{d}{dt}\|\nabla^m u\|^2 + \|\nabla^m v\|^2 - \varepsilon_1{}^2\|\nabla^m u\|^2\\ &\ge -\frac{\varepsilon_1}{2}\frac{d}{dt}\|\nabla^m u\|^2 + \lambda_1{}^m\|v\|^2 - \varepsilon_1{}^2\|\nabla^m u\|^2,\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\left(\left(\alpha+\beta\|\nabla^m u\|^2\right)^q(-\Delta)^m u, v\right)\\ =& \frac{1}{2}\left(\alpha+\beta\|\nabla^m u\|^2\right)^q\frac{d}{dt}\|\nabla^m u\|^2 + \varepsilon_1\left(\alpha+\beta\|\nabla^m u\|^2\right)^q\|\nabla^m u\|^2\\ =& \frac{1}{2\beta(q+1)}\frac{d}{dt}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1} + \frac{\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1} - \frac{\alpha\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^q,\end{aligned} \tag{2.16}$$

$$(g(u), v) \quad = \tfrac{d}{dt}J(u) + \varepsilon_1(g(u), u), \tag{2.17}$$

$$(f(x), v) \le \frac{1}{2\varepsilon_1{}^2}\|f\|^2 + \frac{\varepsilon_1{}^2}{2}\|v\|^2. \tag{2.18}$$

Substituting (2.14)-(2.18) into (2.13), then

$$\begin{aligned}&\frac{d}{dt}\left[\|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2J(u)\right]\\ &+ \left(2\lambda_1{}^m - 2\varepsilon_1 - 2\varepsilon_1{}^2\right)\|v\|^2 + \frac{2\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1} - \frac{2\alpha\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^q\\ &- \left(\frac{\varepsilon_1{}^2}{\lambda_1{}^m} + 2\varepsilon_1{}^2\right)\|\nabla^m u\|^2 + 2\varepsilon_1\,(g(u), u) \le \frac{\|f\|^2}{\varepsilon_1{}^2}.\end{aligned} \tag{2.19}$$

Next, some of the items are estimated in (2.19). By Young's inequality, we have

$$\|\nabla^m u\|^2 \le \frac{1}{q+1}\|\nabla^m u\|^{2q+2} + \frac{q}{q+1}, \tag{2.20}$$

$$\|\nabla^m u\|^2 \le \frac{\beta^q}{4(q+1)}\|\nabla^m u\|^{2q+2} + \frac{q\left(\frac{4}{\beta^q}\right)^{\frac{1}{q}}}{q+1}, \tag{2.21}$$

$$\left(\alpha+\beta\|\nabla^m u\|^2\right)^q \le \frac{q}{2\alpha(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}+\frac{(2\alpha)^q}{q+1}. \tag{2.22}$$

By (2.6), (2.20) and $\varepsilon_1<\beta^q$, we get

$$\begin{aligned}
&\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\varepsilon_1\|\nabla^m u\|^2+2J(u)+2C(\gamma_1)+\frac{q\beta^q}{q+1}\\
&\ge \frac{\beta^q}{(q+1)}\|\nabla^m u\|^{2q+2}-\varepsilon_1\|\nabla^m u\|^2+2J(u)+2C(\gamma_1)+\frac{q\beta^q}{q+1}\\
&\ge (\beta^q-\varepsilon_1)\|\nabla^m u\|^2+2J(u)+2C(\gamma_1)\\
&\ge 0,
\end{aligned} \tag{2.23}$$

where $\gamma_1=\frac{\beta^q-\varepsilon_1}{4}$.

By (2.22) ,we have

$$\begin{aligned}
&\frac{\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\frac{2\alpha\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q}+\frac{(2\alpha)^{q+1}\varepsilon_1}{q\beta}\\
&\ge \frac{2\alpha\varepsilon_1}{\beta q}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q}\\
&\ge 0.
\end{aligned} \tag{2.24}$$

Inserting (2.23)-(2.24) into (2.19), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\varepsilon_1\|\nabla^m u\|^2+2J(u)+2C(\gamma_1)+\frac{q\beta^q}{q+1}\right]\\
&+\left(2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2\right)\|v\|^2+\frac{\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\left(\frac{\varepsilon_1{}^2}{\lambda_1{}^m}+2\varepsilon_1{}^2\right)\|\nabla^m u\|^2\\
&+2\varepsilon_1\left(g(u),u\right)\le\frac{\|f\|^2}{\varepsilon_1{}^2}+\frac{(2\alpha)^{q+1}\varepsilon_1}{q\beta}.
\end{aligned} \tag{2.25}$$

In (2.25), by (2.7), (2.21) and $\varepsilon_1<\frac{\lambda_1{}^m}{2\lambda_1{}^m+1}$, we have

$$\begin{aligned}
&\left(2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2\right)\|v\|^2+\frac{\varepsilon_1}{\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\left(\frac{\varepsilon_1{}^2}{\lambda_1{}^m}+2\varepsilon_1{}^2\right)\|\nabla^m u\|^2+2\varepsilon_1\left(g(u),u\right)\\
&\ge\left(2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2\right)\|v\|^2+\frac{\varepsilon_1}{2\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}\\
&\quad+\frac{\varepsilon_1\beta^q}{2}\|\nabla^m u\|^{2q+2}-\left(\frac{\varepsilon_1{}^2}{\lambda_1{}^m}+2\varepsilon_1{}^2\right)\|\nabla^m u\|^2+2\varepsilon_1\left(g(u),u\right)\\
&\ge\left(2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2\right)\|v\|^2+\frac{\varepsilon_1}{2\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}\\
&\quad+\frac{\varepsilon_1\beta^q}{2}\|\nabla^m u\|^{2q+2}-\varepsilon_1\|\nabla^m u\|^2+2\varepsilon_1C_2J(u)-2\varepsilon_1C(\gamma_2)\\
&\ge\left(2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2\right)\|v\|^2+\frac{\varepsilon_1}{2\beta}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}\\
&\quad+\left(2q\varepsilon_1+\varepsilon_1\right)\|\nabla^m u\|^2+2\varepsilon_1C_2J(u)-\frac{4^{\frac{q+1}{q}}q}{2\beta}-2\varepsilon_1C(\gamma_2)\\
&\ge m_1\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}\right]+2\varepsilon_1C_2J(u)-\frac{4^{\frac{q+1}{q}}q}{2\beta}-2\varepsilon_1C(\gamma_2)\\
&\ge m_1\left[\|v\|^2+\frac{1}{\beta(q+1)}\left(\alpha+\beta\|\nabla^m u\|^2\right)^{q+1}-\varepsilon_1\|\nabla^m u\|^2+2C(\gamma_1)+\frac{q\beta^q}{q+1}\right]\\
&\quad-m_1\left(2C(\gamma_1)+\frac{q\beta^q}{q+1}\right)+2\varepsilon_1C_2J(u)-\frac{4^{\frac{q+1}{q}}q}{2\beta}-2\varepsilon_1C(\gamma_2),
\end{aligned} \tag{2.26}$$

where $\gamma_2=\frac{1}{2}-\varepsilon_1-\frac{\varepsilon_1}{2\lambda_1{}^m}$, and $m_1=\min\left\{2\lambda_1{}^m-2\varepsilon_1-2\varepsilon_1{}^2,\frac{\varepsilon_1(q+1)}{2}\right\}$.

Since $0 < \varepsilon_1 < \frac{\sqrt{(2+C_2)^2+16{\lambda_1}^m}-2-C_2}{4}$ and $0 < C_2 \leq \frac{1}{2}$, such that $m_1 \geq \varepsilon_1 C_2$.

Therefore, inserting (2.26) into (2.25), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2J(u) + 2C(\gamma_1) + \frac{q\beta^q}{q+1}\right] \\
&+ m_1\left[\|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2C(\gamma_1) + \frac{q\beta^q}{q+1}\right] + 2\varepsilon_1 C_2 J(u) \\
\leq\ &\frac{d}{dt}\left[\|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2J(u) + 2C(\gamma_1) + \frac{q\beta^q}{q+1}\right] \\
&+ \varepsilon_1 C_2\left[\|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2J(u) + 2C(\gamma_1) + \frac{q\beta^q}{q+1}\right] \\
\leq\ &\frac{\|f\|^2}{{\varepsilon_1}^2} + C_3,
\end{aligned} \tag{2.27}$$

with $C_3 \equiv \frac{(2\alpha)^{q+1}\varepsilon_1}{q\beta} + m_1\left(2C(\gamma_1) + \frac{q\beta^q}{q+1}\right) + \frac{4^{\frac{q+1}{q}}q}{2\beta} + 2\varepsilon_1 C(\gamma_2)$.

We set $y(t) = \|v\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u\|^2 + 2J(u) + 2C(\gamma_1) + \frac{q\beta^q}{q+1}$. Then, (2.27) is simplified as

$$\frac{d}{dt}y(t) + \varepsilon_1 C_2 y(t) \leq \frac{\|f\|^2}{{\varepsilon_1}^2} + C_3, \tag{2.28}$$

From conclusion (2.23), we know $y(t) \geq 0$. So, by Gronwall's inequality and (2.23), we obtain

$$\|v\|^2 + \frac{\beta^q - \varepsilon_1}{2}\|\nabla^m u\|^2 \leq y(t) \leq y(0)e^{-\varepsilon_1 C_2 t} + \frac{\frac{\|f\|^2}{{\varepsilon_1}^2} + C_3}{\varepsilon_1 C_2}, \tag{2.29}$$

where $y(0) = \|u_1 + \varepsilon_1 u_0\|^2 + \frac{1}{\beta(q+1)}\left(\alpha + \beta\|\nabla^m u_0\|^2\right)^{q+1} - \varepsilon_1\|\nabla^m u_0\|^2 + 2J(u_0) + 2C(\gamma_1) + \frac{q\beta^q}{q+1}$.

Therefore, we get

$$\|(u,v)\|^2_{H_0^m\times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \leq \frac{y(0)}{\min\left\{1, \frac{\beta^q-\varepsilon_1}{2}\right\}}e^{-\varepsilon_1 C_2 t} + \frac{\frac{\|f\|^2}{{\varepsilon_1}^2} + C_3}{\varepsilon_1 C_2 \min\left\{1, \frac{\beta^q-\varepsilon_1}{2}\right\}}. \tag{2.30}$$

Then,

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H_0^m\times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \leq \frac{\frac{\|f\|^2}{{\varepsilon_1}^2} + C_3}{\varepsilon_1 C_2 \min\left\{1, \frac{\beta^q-\varepsilon_1}{2}\right\}}. \tag{2.31}$$

So, there exist R_0 and $t_0 = t_0(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H_0^m\times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \leq {R_0}^2, \quad (t > t_0). \tag{2.32}$$

Lemma 2.5. In addition to the assumptions of Lemma 2.4., $(H_1)-(H_4)$ hold.If (H_5) : $f(x) \in H_0^m(\Omega)$, and $(u_0, u_1) \in H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)$. Then the solution (u, v) of the problems (1.1)-(1.3) satisfies $(u, v) \in L^\infty\left((0, +\infty); H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)\right)$, and

$$\|\nabla^m v\|^2 + \|\Delta^m u\|^2 \leq \frac{z(0)}{\min\{1, \mu-\varepsilon_2\}}e^{-m_2 t} + \frac{C_5 + \frac{1}{{\varepsilon_2}^2}\|\nabla^m f\|^2}{\min\{1, \mu-\varepsilon_2\}\, m_2}, \tag{2.33}$$

where $(-\Delta)^m v = (-\Delta)^m u_t + \varepsilon_2(-\Delta)^m u$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $z(0) = \|\nabla^m u_1 + \varepsilon\nabla^m u_0\|^2 + (\mu-\varepsilon_2)\,\|\Delta^m u_0\|^2, m_2 = \min\left\{{\lambda_1}^m - 2\varepsilon_2 - 2{\varepsilon_2}^2, \frac{-\frac{{\varepsilon_2}^2}{{\lambda_1}^m} - 2{\varepsilon_2}^2 + 2\alpha^q\varepsilon_2}{\mu-\varepsilon_2}\right\}$.

Thus, there exists R_1 and $t_1 = t_1(\Omega) > 0$, such that

$$\|(u,v)\|_{H^{2m}\bigcap H_0^m\times H_0^m} = \|\Delta^m u\|^2 + \|\nabla^m v\|^2 \leq {R_1}^2, \quad (t > t_1). \tag{2.34}$$

Proof. Taking L^2-inner product by $(-\Delta)^m v = (-\Delta)^m u_t + \varepsilon_2(-\Delta)^m u$ in (1.1), we have

$$\Big(u_{tt} + (-\Delta)^m u_t + \big(\alpha + \beta \|\nabla^m u\|^2\big)^q (-\Delta)^m u + g(u), (-\Delta)^m v\Big) = (f(x), (-\Delta)^m v). \tag{2.35}$$

After a computation in (2.35) one by one, as follow

$$\begin{aligned}(u_{tt}, (-\Delta)^m v) &= \frac{1}{2}\frac{d}{dt}\|\nabla^m v\|^2 - \varepsilon_2\|\nabla^m v\|^2 + {\varepsilon_2}^2(\nabla^m u, \nabla^m v)\\ &\geq \frac{1}{2}\frac{d}{dt}\|\nabla^m v\|^2 - \varepsilon_2\|\nabla^m v\|^2 - \frac{{\varepsilon_2}^2}{2{\lambda_1}^m}\|\Delta^m u\|^2 - \frac{{\varepsilon_2}^2}{2}\|\nabla^m v\|^2.\end{aligned} \tag{2.36}$$

$$((-\Delta)^m u_t, (-\Delta)^m v) = \|\Delta^m v\|^2 - \frac{\varepsilon_2}{2}\frac{d}{dt}\|\Delta^m u\|^2 - {\varepsilon_2}^2\|\Delta^m u\|^2. \tag{2.37}$$

By Lemma 2.4., we know it exists C_4, such that

$$\alpha^q \leq \big(\alpha + \beta \|\nabla^m u\|^2\big)^q \leq C_4. \tag{2.38}$$

We set

$$\mu = \begin{cases} \alpha^q, & \frac{d}{dt}\|\Delta^m u\|^2 \geq 0;\\ C_4, & \frac{d}{dt}\|\Delta^m u\|^2 \leq 0.\end{cases} \tag{2.39}$$

By (2.38) and (2.39), we get

$$\begin{aligned}&\Big(\big(\alpha + \beta \|\nabla^m u\|^2\big)^q (-\Delta)^m u, (-\Delta)^m v\Big)\\ =& \frac{\big(\alpha + \beta \|\nabla^m u\|^2\big)^q}{2}\frac{d}{dt}\|\Delta^m u\|^2 + \varepsilon_2\big(\alpha + \beta \|\nabla^m u\|^2\big)^q\|\Delta^m u\|^2\\ \geq& \frac{\mu}{2}\frac{d}{dt}\|\Delta^m u\|^2 + \varepsilon_2\alpha^q\|\Delta^m u\|^2.\end{aligned} \tag{2.40}$$

By Young's inequality, we get

$$(g(u), (-\Delta)^m v) \geq -\|g(u)\|\,\|\triangle^m v\| \geq -\frac{\|g(u)\|^2}{2} - \frac{\|\triangle^m v\|^2}{2}. \tag{2.41}$$

Next to estimate $\|g(u)\|^2$ in (2.41). By (H_4):$|g(s)| \leq C_1\,(1 + |s|^p)$ and Young's inequality, we have

$$\begin{aligned}\|g(u)\|^2 &\leq \int_\Omega C_1^2(1 + |u|^p)^2 dx\\ &\leq \int_\Omega \Big(C_1^2 + 2C_1^2|u|^p + C_1^2|u|^{2p}\Big)dx\\ &\leq \int_\Omega \Big(2C_1^2 + 2C_1^2|u|^{2p}\Big)dx\\ &\leq 2C_1^2\,|\Omega| + 2C_1^2\,\|u\|^{2p}_{L^{2p}(\Omega)}.\end{aligned} \tag{2.42}$$

By $1 \leq p \leq \frac{n+2m}{n-2m}, n > 2m (1 \leq p < +\infty, n \leq 2m)$. So, there exists $K > 0$, such that $\|u\|_{L^{2p}(\Omega)} \leq K\,\|\nabla^m u\|$. $\|\nabla^m u\|$ bounded by Lemma 2.4. Then, (2.42) turns into

$$\|g(u)\|^2 \leq C_5\,(p, C_1, K, |\Omega|)\,. \tag{2.43}$$

Collecting with (2.43), from (2.41) we have

$$(g(u), (-\Delta)^m v) \geq -\frac{C_5}{2} - \frac{\|\triangle^m v\|^2}{2}. \tag{2.44}$$

By $f(x) \in H_0^m(\Omega)$ and Young's inequality, we obtain

$$(f(x), (-\Delta)^m v) = (\nabla^m f(x), \nabla^m v) \leq \frac{1}{2\varepsilon_2^2}\|\nabla^m f\|^2 + \frac{\varepsilon_2^2}{2}\,\|\nabla^m v\|^2\,. \tag{2.45}$$

Integrating (2.36)-(2.40),(2.44)-(2.45), from (2.35) entails

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla^m v\|^2+(\mu-\varepsilon_2)\|\Delta^m u\|^2\right]+\|\Delta^m v\|^2\\&-2(\varepsilon_2+\varepsilon_2{}^2)\|\nabla^m v\|^2+\left(-\frac{\varepsilon_2{}^2}{\lambda_1{}^m}-2\varepsilon_2{}^2+2\alpha^q\varepsilon_2\right)\|\Delta^m u\|^2\\&\le C_5+\frac{1}{\varepsilon_2{}^2}\|\nabla^m f\|^2\end{aligned}\tag{2.46}$$

By Poincar*é* inequality, such that $\lambda_1^m\|\nabla^m v\|^2\le\|\Delta^m v\|^2$. So, (2.46) turns into

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla^m v\|^2+(\mu-\varepsilon_2)\|\Delta^m u\|^2\right]+(\lambda_1{}^m-2\varepsilon_2-2\varepsilon_2{}^2)\|\nabla^m v\|^2\\&+\left(-\frac{\varepsilon_2{}^2}{\lambda_1{}^m}-2\varepsilon_2{}^2+2\alpha^q\varepsilon_2\right)\|\Delta^m u\|^2\le C_5+\frac{1}{\varepsilon_2{}^2}\|\nabla^m f\|^2.\end{aligned}\tag{2.47}$$

Taking $m_2=\min\left\{\lambda_1{}^m-2\varepsilon_2-2\varepsilon_2{}^2,\frac{-\frac{\varepsilon_2{}^2}{\lambda_1{}^m}-2\varepsilon_2{}^2+2\alpha^q\varepsilon_2}{\mu-\varepsilon_2}\right\}$, then

$$\frac{d}{dt}z(t)+m_2z(t)\le\frac{1}{\varepsilon_2^2}\|\nabla^m f\|^2+C_5,\tag{2.48}$$

where $z(t)=\|\nabla^m v\|^2+(\mu-\varepsilon_2)\,\|\Delta^m u\|^2$.

By Gronwall's inequality, we have

$$z(t)\le z(0)e^{-m_2t}+\frac{\frac{1}{\varepsilon^2}\|\nabla^m f\|^2+C_5}{m_2},\tag{2.49}$$

where $z(0)=\|\nabla^m u_1+\varepsilon_2\nabla^m u_0\|^2+(\mu-\varepsilon_2)\,\|\Delta^m u_0\|^2$.

Therefore, we have

$$\|\nabla^m v\|^2+\|\Delta^m u\|^2\le\frac{z(0)}{\min\{1,\mu-\varepsilon_2\}}e^{-m_2t}+\frac{C_5+\frac{1}{\varepsilon_2{}^2}\|\nabla^m f\|^2}{\min\{1,\mu-\varepsilon_2\}\,m_2}.\tag{2.50}$$

Then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H^{2m}\cap H_0^m\times H_0^m}=\|\Delta^m u\|^2+\|\nabla^m v\|^2\le\frac{C_5+\frac{1}{\varepsilon_2{}^2}\|\nabla^m f\|^2}{\min\{1,\mu-\varepsilon_2\}\,m_2}.\tag{2.51}$$

So,there exists R_1 and $t_1=t_1(\Omega)>0$, such that

$$\|(u,v)\|_{H^{2m}\cap H_0^m\times H_0^m}=\|\Delta^m u\|^2+\|\nabla^m v\|^2\le R_1{}^2,\quad (t>t_1).\tag{2.52}$$

3. Global Attractor

3.1 The Existence and Uniqueness of Solution

Theorem 3.1. Assume $(H_1)-(H_4)$ hold, and $(u_0,u_1)\in H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega), f(x)\in H_0^m(\Omega), v=u_t+\varepsilon_1u$. So Equation (1.1) exists a unique smooth solution

$$(u(x,t),v(x,t))\in L^\infty\left((0,+\infty);H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)\right).\tag{3.1}$$

Proof. By the Galerkin method, Lemma 2.4. and Lemma 2.5. , we can easily obtain the existence of Solutions. Next, we prove the uniqueness of Solutions in detail.

Assume u,v are two solutions of the problems (1.1)-(1.3), let $w=u-v$, then $w(x,0)=w_0(x)=0, w_t(x,0)=w_1(x)=0$ and the two equations subtract and obtain

$$w_{tt}+(-\Delta)^m w_t+\left(\alpha+\beta\,\|\nabla^m u\|^2\right)^q(-\Delta)^m u-\left(\alpha+\beta\,\|\nabla^m v\|^2\right)^q(-\Delta)^m v+g(u)-g(v)=0.\tag{3.2}$$

By multiplying (3.2) by w_t, we get

$$\Big(w_{tt} + (-\Delta)^m w_t + \big(\alpha + \beta \|\nabla^m u\|^2\big)^q (-\Delta)^m u - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q (-\Delta)^m v + g(u) - g(v), w_t\Big) = 0. \tag{3.3}$$

$$(w_{tt}, w_t) = \frac{1}{2}\frac{d}{dt}\|w_t\|^2, \tag{3.4}$$

$$((-\Delta)^m w_t, w_t) = \|\nabla^m w_t\|^2, \tag{3.5}$$

$$\begin{aligned}
&\Big(\big(\alpha + \beta \|\nabla^m u\|^2\big)^q (-\Delta)^m u - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q (-\Delta)^m v, w_t\Big)\\
&= \big(\alpha + \beta \|\nabla^m u\|^2\big)^q ((-\Delta)^m w, w_t) + \Big[\big(\alpha + \beta \|\nabla^m u\|^2\big)^q - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q\Big]((-\Delta)^m v, w_t)\\
&= \frac{1}{2}\frac{d}{dt}\Big[\|\nabla^m w\|^2\big(\alpha + \beta \|\nabla^m u\|^2\big)^q\Big] - \frac{1}{2}\|\nabla^m w\|^2 \frac{d}{dt}\big(\alpha + \beta \|\nabla^m u\|^2\big)^q\\
&\quad + \Big[\big(\alpha + \beta \|\nabla^m u\|^2\big)^q - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q\Big]((-\Delta)^m v, w_t)\\
&= \frac{1}{2}\frac{d}{dt}\Big[\|\nabla^m w\|^2\big(\alpha + \beta \|\nabla^m u\|^2\big)^q\Big] - q\beta\big(\alpha + \beta \|\nabla^m u\|^2\big)^{q-1} \|\nabla^m u\| \|\nabla^m u_t\| \|\nabla^m w\|^2\\
&\quad + \Big[\big(\alpha + \beta \|\nabla^m u\|^2\big)^q - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q\Big]((-\Delta)^m v, w_t).
\end{aligned} \tag{3.6}$$

Exploiting (3.4)-(3.6), we receive

$$\begin{aligned}
&\frac{d}{dt}\Big[\|w_t\|^2 + \big(\alpha + \beta \|\nabla^m u\|^2\big)^q \|\nabla^m w\|^2\Big] + 2\|\nabla^m w_t\|^2\\
&= 2q\beta\big(\alpha + \beta \|\nabla^m u\|^2\big)^{q-1} \|\nabla^m u\| \|\nabla^m u_t\| \|\nabla^m w\|^2\\
&\quad - 2\Big[\big(\alpha + \beta \|\nabla^m u\|^2\big)^q - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q\Big]((-\Delta)^m v, w_t) - 2(g(u) - g(v)) = 0.
\end{aligned} \tag{3.7}$$

In (3.7), according to Lemma 2.4. and Lemma 2.5., such that

$$\begin{aligned}
&2q\beta\big(\alpha + \beta \|\nabla^m u\|^2\big)^{q-1} \|\nabla^m u\| \|\nabla^m u_t\| \|\nabla^m w\|^2\\
&\quad - 2\Big[\big(\alpha + \beta \|\nabla^m u\|^2\big)^q - \big(\alpha + \beta \|\nabla^m v\|^2\big)^q\Big]((-\Delta)^m v, w_t)\\
&\le C_6\|\nabla^m w\|^2 + 4q\beta\xi\big(\alpha + \beta\xi^2\big)^{q-1} \|\Delta^m v\| \|w_t\| \|\nabla^m w\|\\
&\le C_6\|\nabla^m w\|^2 + C_7 \|w_t\| \|\nabla^m w\|\\
&\le \Big(C_6 + \frac{C_7}{2}\Big)\Big(\|w_t\|^2 + \|\nabla^m w\|^2\Big).
\end{aligned} \tag{3.8}$$

where $\min\limits_{t\in[0,+\infty)} \{\|\nabla^m u\|, \|\nabla^m v\|\} < \xi < \max\limits_{t\in[0,+\infty)} \{\|\nabla^m u\|, \|\nabla^m v\|\}$, $C_6 > 0$, and $C_7 > 0$ are constants.

By (H_4), Lemma 2.2., Lemma 2.4. and Lemma 2.5., we obtain

$$\begin{aligned}
&|-2(g(u) - g(v), w_t)|\\
&\le 2\|g(u) - g(v)\| \|w_t\|\\
&\le 2\left\|\int_0^1 \frac{d}{ds} g(su + (1-s)v) ds\right\| \|w_t\|\\
&\le C_1 \left\|\int_0^1 \Big[1 + |su + (1-s)v|^{p-1}\Big] ds \times w\right\| \|w_t\|\\
&\le C_1 \left\|\Big[1 + (|u| + |v|)^{p-1}\Big] |w|\right\| \|w_t\|\\
&\le C_1 \Big[1 + \Big(\|\Delta^m u\|^{p+1} + \|\Delta^m v\|^{p+1}\Big)\Big] \|w_t\| \|w\|\\
&\le C_8 \Big(\|w_t\|^2 + \|\nabla^m w\|^2\Big)
\end{aligned} \tag{3.9}$$

where $C_8 > 0$ is constant. From the above, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|w_t\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2\right] \\
&\le \left(C_6 + \frac{C_7}{2} + C_8\right)\left(\|w_t\|^2 + \|\nabla^m w\|^2\right) \\
&\le \left(C_6 + \frac{C_7}{2} + C_8\right)\|w_t\|^2 + \frac{C_6 + \frac{C_7}{2} + C_8}{\left(\alpha + \beta\|\nabla^m u\|^2\right)^q}\left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2 \\
&\le \max\left\{C_6 + \frac{C_7}{2} + C_8, \frac{C_6 + \frac{C_7}{2} + C_8}{\alpha^{2q}}\right\}\left[\|w_t\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2\right].
\end{aligned} \tag{3.10}$$

By using Gronwall's inequality for (3.10), we obtain

$$\begin{aligned}
&\|w_t\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2 \\
&\le \left[\|w_t(0)\|^2 + \left(\alpha + \beta\|\nabla^m u_0\|^2\right)^q \|\nabla^m w(0)\|^2\right] e^{C_9 t} = 0,
\end{aligned} \tag{3.11}$$

where $C_9 = \max\left\{C_6 + \frac{C_7}{2} + C_8, \frac{C_6 + \frac{C_7}{2} + C_8}{\alpha^{2q}}\right\} > 0$.

Hence , we can get $\|w_t\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2 = 0$. That shows that

$$\|w_t\|^2 = 0, \quad \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m w\|^2. \tag{3.12}$$

That is

$$w(x,t) = 0. \tag{3.13}$$

Therefore,

$$u = v. \tag{3.14}$$

So, we get the uniqueness of the solution.

3.2 The Existence of Global Attractor

Theorem 3.2.(Lin,G.G.,2011) Let E be a Banach space, and $\{S(t)\}(t \ge 0)$ are the semigroup operator on E_0. $S(t) : E_0 \to E_0, S(t+\tau) = S(t)S(\tau)(\forall t, \tau \ge 0), S(0) = I$, where I is a unit operator.Set $S(t)$ satisfy the follow conditions:

1) $S(t)$ is uniformly bounded, namely $\forall R > 0, \|u\|_E \le R$, it exists a constant $C(R)$, so that

$$\| S(t)u \|_E \le C(R) \quad (t \in [0, +\infty)); \tag{3.15}$$

2) It exists a bounded absorbing set $B_0 \subset E$, namely, $\forall B \subset E$, it exists a constant t_0, so that

$$S(t)B \subset B_0 \quad (t \ge t_0); \tag{3.16}$$

where B_0 and B are bounded sets.

3) When $t > 0, S(t)$ is a completely continuous operator. Therefore, the semigroup operator S(t) exists a compact global attractor $\mathcal{A}$.

Theorem 3.3. Under the assume of Lemma 2.4., Lemma 2.5. and Theorem 3.1., equations have global attractor

$$\mathcal{A} = \omega(B_0) = \bigcap_{\tau \ge 0}\overline{\bigcup_{t \ge \tau} S(t)B_0}, \tag{3.17}$$

where $B_0 = \{(u,v) \in H^{2m}(\Omega) \bigcap H_0^m(\Omega) \times H_0^m(\Omega) : \|(u,v)\|_{H^{2m}\bigcap H_0^m \times H_0^m} = \|u\|_{H^{2m}\bigcap H_0^m} + \|v\|_{H_0^m} \le R_0 + R_1\}$, B_0 is the bounded absorbing set of $H^{2m} \times H_0^m$ and satisfies

1) $S(t)\mathcal{A} = \mathcal{A}, t > 0$;

2) $\lim_{t\to\infty} dist(S(t)B, \mathcal{A}) = 0$, here $B \subset H^{2m} \bigcap H_0^m \times H_0^m$ and it is a bounded set,

$$dist(S(t)B,\mathcal{A}) = \sup_{x\in B}(\inf_{y\in\mathcal{A}}\|S(t)x-y\|_{H^{2m}\bigcap H_0^m\times H_0^m}) \longrightarrow 0, t\longrightarrow\infty. \tag{3.18}$$

Proof. Under the conditions of Theorem 3.1., it exists the solution semigroup S(t),$S(t): H^{2m}\bigcap H_0^m\times H_0^m \to H^{2m}\bigcap H_0^m\times H_0^m$, here $E_1 = H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)$.

(1) From Lemma 2.4. to Lemma 2.5., we can get that $\forall B\subset H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)$ is a bounded set that includes in the ball $\{\|(u,v)\|_{H^{2m}\bigcap H_0^m\times H_0^m}\le R\}$,

$$\begin{aligned}\|S(t)(u_0,v_0)\|^2_{H^{2m}\bigcap H_0^m\times H_0^m} &= \|u\|^2_{H^{2m}\bigcap H_0^m} + \|v\|^2_{H_0^m}\\ &\le \|u_0\|^2_{H^{2m}\bigcap H_0^m} + \|v_0\|^2_{H_0^m} + C\\ &\le R_1^2 + C, (t\ge 0, (u_0,v_0)\in B).\end{aligned} \tag{3.19}$$

This shows that $S(t)(t\ge 0)$ is uniformly bounded in $H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)$.

(2) Furthermore, for any $(u_0,v_0)\in H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega)$, when $t\ge max\{t_0,t_1\}$, we have

$$\|S(t)(u_0,v_0)\|^2_{H^{2m}\bigcap H_0^m\times H_0^m} = \|u\|^2_{H^{2m}\bigcap H_0^m} + \|v\|^2_{H_0^m} \le R_0^2 + R_1^2. \tag{3.20}$$

So we get B_0 is the bounded absorbing set.

(3) Since $E_1 := H^{2m}(\Omega)\bigcap H_0^m(\Omega)\times H_0^m(\Omega) \hookrightarrow E_0 := H_0^m(\Omega)\times L^2(\Omega)$ is compact embedded, which means that the bounded set in E_1 is the compact set in E_0, so the semigroup operator S(t) exists a compact global attractor $\mathcal{A}$.

4. The Estimates of the Upper Bounds of Hausdorff Dimensions for the Global Attractor

4.1 Differentiability of the Semigroup

In order to estimate dimensions, we suppose: (H_5) for every $M>0$, there exist $k=k(M)$, such that:

$$\left\|g'(u_1)-g'(u_2)\right\|_{L(H_0^m(\Omega),L^2(\Omega))} \le k\|\nabla^m u_1 - \nabla^m u_2\|^{\delta_1}, \tag{4.1}$$

for any $u_1,u_2\in H_0^m(\Omega), \|\nabla^m u_1\|\le M, \|\nabla^m u_2\|\le M, \delta_1>0$.

We define $A=-\Delta, E_0 = H_0^m(\Omega)\times L^2(\Omega)$. The inner product and the norm in E_0 space are defined as follows:

$\forall\varphi_i = (u_i,v_i)\in E_0, (i=1,2)$, we have

$$(\varphi_1,\varphi_2)_{E_0} = \left(A^{\frac{m}{2}}u_1, A^{\frac{m}{2}}u_2\right) + (v_1,v_2), \tag{4.2}$$

$$\|\varphi_1\|^2_{E_0} = (\varphi_1,\varphi_1)_{E_0} = \left\|A^{\frac{m}{2}}u_1\right\|^2 + \|v_1\|^2. \tag{4.3}$$

Setting $\forall\varphi = (u,v)^T\in E_0, v = u_t+\varepsilon u, 0<\varepsilon<\min\left\{1, \frac{\lambda_1{}^m}{2}, \frac{-\frac{5}{2}-\lambda_1{}^m+\sqrt{\left(\frac{5}{2}+\lambda_1{}^m\right)^2+4\lambda_1{}^m}}{2}\right\}$, the equation (1.1) is equivalent to

$$\varphi_t + H(\varphi) = F(\varphi), \tag{4.4}$$

where

$$H(\varphi) = \begin{bmatrix} \varepsilon u - v \\ -\varepsilon v + A^m v + \varepsilon^2 u + (1-\varepsilon)A^m u] \end{bmatrix}, \tag{4.5}$$

$$F(\varphi) = \begin{bmatrix} 0 \\ \left[1-\left(\alpha+\beta\|\nabla^m u\|^2\right)^q\right]A^m u - g(u) + f(x) \end{bmatrix}. \tag{4.6}$$

Lemma 4.1.1. For any $\varphi = (u,v)^T\in E_0$, we have

$$(H(\varphi),\varphi)_{E_0} \ge \frac{\varepsilon}{4}\|\varphi\|^2_{E_0} + \frac{1}{2}\left\|A^{\frac{m}{2}}v\right\|^2. \tag{4.7}$$

Proof. By (4.2)-(4.6), we get

$$\begin{aligned}&(H(\varphi),\varphi)_{E_0}\\&=\left(\varepsilon A^{\frac{m}{2}}u-A^{\frac{m}{2}}v,A^{\frac{m}{2}}u\right)+\left(-\varepsilon v+A^m v+\varepsilon^2 u+(1-\varepsilon)A^m u,v\right)\\&=\varepsilon\left\|A^{\frac{m}{2}}u\right\|^2-\varepsilon\|v\|^2+\left\|A^{\frac{m}{2}}v\right\|^2+\varepsilon^2(u,v)-\varepsilon\left(A^{\frac{m}{2}}u,A^{\frac{m}{2}}v\right).\end{aligned}\tag{4.8}$$

By using hölder inequality, Young's inequality and Poincaré inequality, we deal with the terms in (4.8) by as follows:

$$\varepsilon^2(u,v)\geq-\frac{\varepsilon^2}{2}\|u\|^2-\frac{\varepsilon^2}{2}\|v\|^2\geq-\frac{\varepsilon^2}{2{\lambda_1}^m}\left\|A^{\frac{m}{2}}u\right\|^2-\frac{\varepsilon^2}{2}\|v\|^2,\tag{4.9}$$

$$-\varepsilon\left(A^{\frac{m}{2}}u,A^{\frac{m}{2}}v\right)\geq-\frac{\varepsilon}{2}\left\|A^{\frac{m}{2}}u\right\|^2-\frac{\varepsilon}{2}\left\|A^{\frac{m}{2}}v\right\|^2.\tag{4.10}$$

By $0<\varepsilon<\min\left\{1,\frac{{\lambda_1}^m}{2},\frac{-\frac{5}{2}-{\lambda_1}^m+\sqrt{\left(\frac{5}{2}+{\lambda_1}^m\right)^2+4{\lambda_1}^m}}{2}\right\}$ and substituting (4.9)-(4.10) into (4.8), we obtain

$$\begin{aligned}&(H(\varphi),\varphi)_{E_0}\\&\geq\left(\frac{\varepsilon}{2}-\frac{\varepsilon^2}{2{\lambda_1}^m}\right)\left\|A^{\frac{m}{2}}u\right\|^2+\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\left\|A^{\frac{m}{2}}v\right\|^2+\left(-\frac{\varepsilon^2}{2}-\varepsilon\right)\|v\|^2+\frac{1}{2}\left\|A^{\frac{m}{2}}v\right\|^2\\&\geq\frac{\varepsilon}{4}\left\|A^{\frac{m}{2}}u\right\|^2+\left(\frac{{\lambda_1}^m}{2}-\frac{\varepsilon{\lambda_1}^m}{2}-\varepsilon-\frac{\varepsilon^2}{2}\right)\|v\|^2+\frac{1}{2}\left\|A^{\frac{m}{2}}v\right\|^2\\&\geq\frac{\varepsilon}{4}\left(\left\|A^{\frac{m}{2}}u\right\|^2+\|v\|^2\right)+\frac{1}{2}\left\|A^{\frac{m}{2}}v\right\|^2\\&=\frac{\varepsilon}{4}\|\varphi\|_{E_0}^2+\frac{1}{2}\left\|A^{\frac{m}{2}}v\right\|^2.\end{aligned}\tag{4.11}$$

The proof of Lemma 4.1.1 is completed.

The linearized equations of (1.1)-(1.3), the above equations as follows:

$$\begin{aligned}&U_{tt}+A^mU_t+\left(\alpha+\beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^qA^mU\\&+2q\beta\left(\alpha+\beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^{q-1}\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^mu+g'(u)U=0,\end{aligned}\tag{4.12}$$

$$U(x,t)|_{x\in\partial\Omega}=0,t>0,\tag{4.13}$$

$$U(x,0)=\xi,U_t(x,0)=\zeta,\tag{4.14}$$

where $(\xi,\zeta)\in E_0,(u,u_t)=S(t)(u_0,u_1)$ is the solution of (1.1)-(1.3) with $(u_0,u_1)\in\mathcal{A}$.

Given $(u_0,u_1)\in\mathcal{A}$ and $S(t):E_0\to E_0$, the solution $S(t)(u_0,u_1)\in E_0$, by stand methods we can show that for any $(\xi,\zeta)\in E_0$, the linear initial boundary value problem (4.12)-(4.14) possess a unique solution $(U(t),U_t(t))\in L^\infty((0,+\infty);E_0)$.

Lemma 4.1.2. For any $t>0,R>0$, the mapping $S(t):E_0\to E_0$ is Fréchet differentiable on. Its differential at $\varphi=(u_0,u_1)^T$ is the linear operator on $F:(\xi,\zeta)^T\to(U(t),V(t))^T$, where $U(t)$ is the solution of (4.12)-(4.14).

Proof. Let $\varphi_0=(u_0,u_1)^T\in E_0,\tilde{\varphi}_0=(u_0+\xi,u_1\zeta)^T\in E_0$ with $\|\varphi_0\|_{E_0}\leq R,\|\tilde{\varphi}_0\|_{E_0}\leq R$, we denote $(u,u_t)^T=S(t)\varphi_0,(\tilde{u},\tilde{u}_t)^T=S(t)\tilde{\varphi}_0$. We can get the Lipchitz property of $S(t)$ on the bounded sets of E_0, that is

$$\|S(t)\varphi_0-S(t)\tilde{\varphi}_0\|_{E_0}^2\leq e^{C_{10}t}\|(\xi,\zeta)\|_{E_0}^2.\tag{4.15}$$

Let $\theta=\tilde{u}-u-U$ is the solution of problem

$$\theta_{tt}+A^m\theta_t+\left(\alpha+\beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^qA^m\theta=h,\tag{4.16}$$

$$\theta(0) = \theta_t(0) = 0, \tag{4.17}$$

with

$$\begin{aligned} h = &\left[\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^q - \left(\alpha + \beta\left\|A^{\frac{m}{2}} \tilde{u}\right\|^2\right)^q\right] A^m \tilde{u} \\ &+ 2q\beta\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^{q-1} \left(A^{\frac{m}{2}} U, A^{\frac{m}{2}} u\right) A^m u + g(u) - g(\tilde{u}) + g'(u)U. \end{aligned} \tag{4.18}$$

Taking the scalar product of each side of (4.16) with θ_t.Because of

$$\begin{aligned} &\left(\left[\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^q - \left(\alpha + \beta\left\|A^{\frac{m}{2}} \tilde{u}\right\|^2\right)^q\right] A^m \tilde{u} + 2q\beta\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^{q-1} \left(A^{\frac{m}{2}} U, A^{\frac{m}{2}} u\right) A^m u, \theta_t\right) \\ = &\left(\left[\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^q - \left(\alpha + \beta\left\|A^{\frac{m}{2}} \tilde{u}\right\|^2\right)^q\right] A^m \tilde{u}, \theta_t\right) \\ &+ \left(2q\beta\left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^{q-1} \left(A^{\frac{m}{2}} (u - \tilde{u} - \theta), A^{\frac{m}{2}} u\right) A^m u, \theta_t\right) \\ \le &\, C_{11}(R_0) \left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\| \left\|A^{\frac{m}{2}} \theta_t\right\| + C_{12}(R_0) \left\|A^{\frac{m}{2}} \theta\right\| \left\|A^{\frac{m}{2}} \theta_t\right\|. \end{aligned} \tag{4.19}$$

By (H_5), we have

$$\begin{aligned} &(g(u) - g(\tilde{u}) + g'(u)U, \theta_t) \\ = &\,(g(u) - g(\tilde{u}) - g'(u)(u - \tilde{u}) - g'(u)\theta, \theta_t) \\ \le &\, C_{12}(R_0)\left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\|^{1+\delta_1} \|\theta_t\| + C_{13}(R_0) \left\|A^{\frac{m}{2}} \theta\right\| \left\|A^{\frac{m}{2}} \theta_t\right\| \end{aligned} \tag{4.20}$$

By (4.19)-(4.20) and Young's inequality, we have

$$\begin{aligned} &\frac{d}{dt}\left[\|\theta_t\|^2 + \left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^q \left\|A^{\frac{m}{2}} \theta\right\|^2\right] \\ \le &\, C_{14}\left[\|\theta_t\|^2 + \left(\alpha + \beta\left\|A^{\frac{m}{2}} u\right\|^2\right)^q \left\|A^{\frac{m}{2}} \theta\right\|^2\right] + C_{15}\left(\left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\|^2 + \left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\|^{2+2\delta_1}\right). \end{aligned} \tag{4.21}$$

By the Gronwall's inequality and (4.15), we get

$$\begin{aligned} &\|\theta_t\|^2 + \left\|A^{\frac{m}{2}} \theta\right\|^2 \\ \le &\, C_{16} e^{C_{17} t} \int_0^t \left(\left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\|^2 + \left\|A^{\frac{m}{2}} u - A^{\frac{m}{2}} \tilde{u}\right\|^{2+2\delta_1}\right) d\tau \\ \le &\, C_{18} e^{C_{19} t}\left[\left(\left\|A^{\frac{m}{2}} \xi\right\|^2 + \|\zeta\|^2\right) + \left(\left\|A^{\frac{m}{2}} \xi\right\|^2 + \|\zeta\|^2\right)^{1+\delta_1}\right], \end{aligned} \tag{4.22}$$

where $C_{16}, C_{17}, C_{18}, C_{19} > 0$.

From (4.22), we obtain

$$\begin{aligned} &\frac{\|\tilde{\varphi}(t) - \varphi(t) - U(t)\|_{E_0}^2}{\left\|(\xi, \zeta)^T\right\|_{E_0}^2} \\ \le &\, C_{18} e^{C_{19} t}\left[\left(\left\|A^{\frac{m}{2}} \xi\right\|^2 + \|\zeta\|^2\right) + \left(\left\|A^{\frac{m}{2}} \xi\right\|^2 + \|\zeta\|^2\right)^{1+\delta_1}\right] \to 0, \end{aligned} \tag{4.23}$$

as $(\xi, \zeta)^T \to 0$ in E_0. The proof is competed.

4.2 The Upper Bounds of Hausdorff Dimensions for the Global Attractor

Consider the first variation of (4.4) with initial condition:

$$\Psi_t{}' + P(\varphi)\Psi = \Gamma_1(\varphi)\Psi + \Gamma_2(\varphi)\Psi, \Psi(0) = (\xi, \zeta)^T \in E_0, t > 0, \tag{4.24}$$

where $\Psi = (U, V)^T \in E_0, V = U_t + \varepsilon U$ and $\varphi = (u, v)^T \in E_0$ is a solution of (4.3),

$$P(\varphi) = \begin{bmatrix} \varepsilon I & -I \\ \varepsilon^2 I + (1-\varepsilon)A^m & -\varepsilon I + A^m \end{bmatrix}, \tag{4.25}$$

$$\Gamma_1(\varphi) = \begin{bmatrix} 0 & 0 \\ -g'(u) & 0 \end{bmatrix}, \tag{4.26}$$

$$\Gamma_2(\varphi) = \begin{bmatrix} 0 \\ \left[1 - \left(\alpha + \beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^q\right]A^m U - 2q\beta\left(\alpha + \beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^{q-1}\left(A^{\frac{m}{2}}U, A^{\frac{m}{2}}u\right)A^m u \end{bmatrix}. \tag{4.27}$$

It is easy to show from Lemma 4.1.2 that (4.24) is a well-posed problem in E_0, the mapping $S_\varepsilon(\tau) : \{u_0, v_1 = u_1 + \varepsilon u_0\} \longrightarrow \{u(\tau), v(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$, $\psi(\tau) = \{u(\tau), v_t(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$ is Fréchet differentiable on E_0 for any $t \geq 0$, its differential at $\varphi = (u_0, u_1 + \varepsilon u_0)^T$ is the linear operator on E_0, $(\xi, \zeta)^T \to (U(t), V(t))^T$, where $(U(t), V(t))^T$ is the solution of (4.24).

Lemma 4.2.1.(Teman,R.,1998) For any orthonormal family of elements of $\left(E_0, \|\|_{E_0}\right), (\xi_j, \zeta_j)^T, j = 1, 2, \cdots, n_1$, we have

$$\sum_{j=1}^{n_1}\left\|A^{\frac{m}{2}v}\xi_j\right\|^2 \leq 2\sum_{j=1}^{n_1}{\mu_j}^{v-1}, v \in [0, 1), \tag{4.28}$$

where $\left\{\mu_j\right\}_{j=1}^{+\infty}$ is the eigenvalue of A^m.

Proof. This is a direct consequence of Lemma VI 6.3 of [17].

Theorem 4.2.2. If we take proper α,β satisfies $\frac{1+\left(\alpha+\beta{R_0}^2\right)^q + 2q\beta{R_0}^2\left(\alpha+\beta{R_0}^2\right)^{q-1}}{2} - \frac{\varepsilon}{8} \leq 0$ and $(H_1) - (H_5)$ hold, then there exists $\rho(R_0) > 0$, such that the Hausdorff dimension of global attractor $\mathcal{A}$ in E_0 satisfies

$$d_H(\mathcal{A}) \leq \min\left\{n_1 \middle| n_1 \in N, \frac{1}{n_1}\sum_{j=1}^{n_1}{\mu_j}^{\delta-1} < \frac{\varepsilon}{8\rho n_1}\right\}, \tag{4.29}$$

where R_0 is as in Lemma 2.4, and

$$\delta = \begin{cases} \frac{(n-2)(p-1)-2}{2}, \frac{n}{n-2m} \leq p < \frac{n+2m}{n-2m}, n \geq 2m, \\ 0, n < 2m \quad or \quad 0 \leq p \leq \frac{n}{n-2m}, n \geq 2m. \end{cases} \tag{4.30}$$

Proof. Let $n_1 \in N$ be fixed. Consider m_1 solutions $\Psi_1, \Psi_2, \cdots, \Psi_{n_1}$ of (4.24). At a given time τ, let $Q_{n_1}(\tau)$ denote the orthogonal projection in E_0 onto span $\{\Psi_1(s), \Psi_2(s), \cdots, \Psi_{n_1}(s)\}$. Let $y_j(s) = (\xi_j, \zeta_j)^T \in E_0, j = 1, 2, \cdots, n_1$, be an orthonormal basis of

$$Q_{n_1}(s)E_0 = span\{\Psi_1(s), \Psi_2(s), \cdots, \Psi_{n_1}(s)\}, \tag{4.31}$$

with respect to the inner product $(,)_{E_0}$ and norm $\|\|_{E_0}$.

Suppose

$$\varphi(\tau) = (u(\tau), v(\tau))^T \in \mathcal{A}, \tag{4.32}$$

then $\|\varphi(\tau)\|_{E_0} \leq M_0, \forall s \geq \tau$. By $\left\|y_j\right\|_{E_0} = 1$ and Lemma 4.1.1, we have

$$-\left(P(\varphi(s))y_j(s), y_j(s)\right)_{E_0} \leq -\frac{\varepsilon}{4} - \frac{1}{2}\left\|A^{\frac{m}{2}}\zeta_j\right\|^2. \tag{4.33}$$

$$\left(\Gamma_1(\varphi(s))y_j(s), y_j(s)\right)_{E_0} \leq \left\|A^{-\frac{m}{2}}g'(u)\xi_j(s)\right\|\left\|A^{\frac{m}{2}}\zeta_j(s)\right\|. \tag{4.34}$$

By the hypothesis (H_4), the mean value theorem and the Sobolev embedding theorem:

$$H_0^{mv}(\Omega) \subset D(A^{\frac{m}{2}v}) \subset H^{mv}(\Omega) \subset L^q(\Omega) \subset L^2(\Omega) \subset L^{q'}(\Omega) \subset H^{-mv}(\Omega), \tag{4.35}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{mv}{n}, \frac{1}{q} + \frac{1}{q'} = 1, v \in [0, 1]$.

So, by Lemma 2.4 and (4.34), for $n = 1$, $H_0^m(\Omega) \subset L^\infty(\Omega) \subset L^1(\Omega) \subset H^{-m}(\Omega) \subset \left(H_0^m(\Omega)\right)'$. There exists $C_20(R_0) > 0$, we get

$$\left\|A^{-\frac{m}{2}}(g'(u)\xi_j(s))\right\| \le C_20\left\|g'(u)\xi_j(s)\right\|_{L^1} \le C_{21}(R_0)\left\|\xi_j(s)\right\|. \tag{4.36}$$

For $1 < n \le 2m$, $H_0^m(\Omega) \subset L^q(\Omega) \subset H^{-m}(\Omega) \subset \left(H_0^m(\Omega)\right)', q > 0$, there exists $C_{11}(R_0) > 0$, such that

$$\left\|A^{-\frac{m}{2}}(g'(u)\xi_j(s))\right\| \le C_9\left\|g'(u)\xi_j(s)\right\|_{L^{\frac{3}{2}}} \le C_{11}(R_0)\left\|\xi_j(s)\right\|. \tag{4.37}$$

For $n > 2m$, by (H_4), there exists $C_{22}(R_0) > 0$, such that

$$\left\|A^{-\frac{m}{2}}(g'(u)\xi_j(s))\right\| \le \left\|g'(u)\xi_j(s)\right\|_{L^{\frac{2n}{n+2m}}} \le C_{22}(R_0)\left\|A^{\frac{m}{2}\delta}\xi_j(s)\right\|. \tag{4.38}$$

From (4.34)-(4.38), we have

$$\left(\Gamma_1(\varphi(s))y_j(s), y_j(s)\right)_{E_0} \le \frac{C_{23}}{2}\left\|A^{\frac{m}{2}\delta}\xi_j(s)\right\|\left\|A^{\frac{m}{2}}\zeta_j(s)\right\|, \tag{4.39}$$

where $C_{23} = C_{23}(R_0) = 2\max\{C_{21}(R_0), C_{22}(R_0)\}$.

By Lemma 2.4, we obtain

$$\begin{aligned}
&\left(\Gamma_2(\varphi(s))y_j(s), y_j(s)\right)\\
&= \left[1 - \left(\alpha + \beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^q\right]\left(A^{\frac{m}{2}}\xi_j, A^{\frac{m}{2}}\zeta_j\right) - 2q\beta\left(\alpha + \beta\left\|A^{\frac{m}{2}}u\right\|^2\right)^{q-1}\left(A^{\frac{m}{2}}\xi_j, A^{\frac{m}{2}}u\right)\left(A^{\frac{m}{2}}u, A^{\frac{m}{2}}\zeta_j\right)\\
&\le \left[1 + \left(\alpha + \beta R_0{}^2\right)^q + 2q\beta R_0{}^2\left(\alpha + \beta R_0{}^2\right)^{q-1}\right]\left\|A^{\frac{m}{2}}\xi_j\right\|\left\|A^{\frac{m}{2}}\zeta_j\right\|.
\end{aligned} \tag{4.40}$$

By lemma VI 6.3 of [17], Young's inequality and choose α, β satisfying $\frac{1+\left(\alpha+\beta R_0{}^2\right)^q+2q\beta R_0{}^2\left(\alpha+\beta R_0{}^2\right)^{q-1}}{2} - \frac{\varepsilon}{8} \le 0$, we obtain

$$\begin{aligned}
p_{n_1}(s) &= \sum_{j=1}^{n_1}\left((-P(\varphi(s)) + \Gamma_2(\varphi(s) + \Gamma_1(\varphi(s)))\,y_j(s), y_j(s)\right)_{E_0}\\
&\le \left[\frac{1 + \left(\alpha + \beta R_0{}^2\right)^q + 2q\beta R_0{}^2\left(\alpha + \beta R_0{}^2\right)^{q-1}}{2} - \frac{\varepsilon}{4}\right]n_1 + \frac{\rho}{8}\left\|A^{\frac{m}{2}\delta}\xi_j\right\|\\
&\le -\frac{\varepsilon}{8}n_1 + \frac{\rho}{2}\sum_{j=1}^{n_1}\mu_j{}^{\delta-1},
\end{aligned} \tag{4.41}$$

where $\rho = C_{23}{}^2(R_0)$

If $\frac{\varepsilon}{8\rho n_1} \ge \frac{1}{n_1}\sum_{j=1}^{n_1}\lambda_j{}^{\delta-1}$, then

$$q_{n_1} = \lim_{t\to\infty}\inf\sup_{\tau\in R}\sup_{\Phi\subset E_0}\sup_{\varphi(\tau)\in\mathcal{A}}\frac{1}{t}\int_\tau^{\tau+t}p_{n_1}(s)ds \le -\rho n_1\left(\frac{\varepsilon}{4\rho n_1} - \frac{1}{n_1}\sum_{j=1}^{n_1}\lambda_j{}^{\delta-1}\right) < 0. \tag{4.42}$$

So, by lemma 4 of (S. Zhou, 1999), we obtain (4.29). The proof of Theorem 4.2.2 is completed.

5. Exponential Attractor

Definition 5.1.(Lin,G.G.,2011) Let X be a complete metric space. A set $\mathcal{A}_{exp} \subset X$ is said to be an exponential attractor of the dynamical system $(S(t), X)$ if

(i) it is a compact set in X;

(ii) it has finite fractal dimension in X, i.e. $\dim_f \left\{\mathcal{A}_{exp}, X\right\} < +\infty$;

(iii) it is a forward invariant set, i.e. $S(t)\mathcal{A}_{exp} \subset \mathcal{A}_{exp}, \quad t \geq 0$;

(iv) it attractor exponentially the bounded sets in X, that is, for any bounded set $B \subset X$, there exists a positive constant k such that

$$dist_X\left(S(t)B, \mathcal{A}_{\text{exp}}\right) \leq C\left(\|B\|_X\right) e^{-kt}, \quad t \geq 0, \tag{5.1}$$

where $\|B\|_X = \sup_{\zeta\in B} \|\zeta\|_X$.

Lemma 5.2.(Interpolation theorem(*RobertA.Adams&JohnJ.F.Fournier*, 2003)) Let $1 \leq p < q < r$, so that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}, \tag{5.2}$$

for some θ satisfying $0 < \theta < 1$. If $u \in L^p(\Omega) \cap L^r(\Omega)$, then $u \in L^q(\Omega)$ and

$$\|u\|_q \leq \|u\|_p^\theta \, \|u\|_r^{1-\theta}. \tag{5.3}$$

Lemma 5.3.(*ZhijianYang&PengyanDing.*, 2016) Let $y : R^+ \to R^+$ be an absolutely continuous function satisfying

$$\frac{d}{dt}y(t) + 2\epsilon y(t) \leq h(t)y(t) + z(t), \quad t > 0, \tag{5.4}$$

where $\epsilon > 0$, $z \in L^1_{loc}(R^+)$, $\int_s^t h(\tau)d\tau \leq \epsilon(t-s) + m$ for $t \geq s \geq 0$ and some $m > 0$. Then

$$y(t) \leq e^m\left(y(0)e^{-\epsilon t} + \int_0^t |z(\tau)|\, e^{-\epsilon(t-\tau)}d\tau\right), \quad t > 0. \tag{5.5}$$

Lemma 5.4. Under assumptions of Lemma 2.4 and Lemma 2.5, there exist $t_2, R_2 \geq 0$, such that

$$\|u_{tt}\|^2 \leq {R_2}^2, \quad t \geq t_2. \tag{5.6}$$

Proof. Differentiate to (1.1) about t, we get

$$\begin{aligned}
&u_{ttt} + (-\Delta)^m u_{tt} + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q (-\Delta)^m u_t \\
&+ 2q\beta\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q-1} (\nabla^m u, \nabla^m u_t)\,(-\Delta)^m u + g'(u)u_t = 0, \\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\
&u_{tt}(x,0) = f(x) - (-\Delta)^m u_1(x) - \left(\alpha + \beta\|\nabla^m u_0\|^2\right)^q (-\Delta)^m u_0 - g(u_0), \\
&u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots, m-1, x \in \partial\Omega, t \in (0, +\infty).
\end{aligned} \tag{5.7}$$

Now we use the multiplier u_{tt} for (5.7). We readily obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_{tt}\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2\right] + 2\|\nabla^m u_{tt}\|^2 \\
&= 2q\beta\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q-1} (\nabla^m u, \nabla^m u_t)\left[\|\nabla^m u_t\|^2 - 2\left((-\Delta)^m u, u_{tt}\right)\right] - 2\left(g'(u)u_t, u_{tt}\right).
\end{aligned} \tag{5.8}$$

By Lemma 2.4 and Lemma 2.5, we get

$$2q\beta\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q-1} (\nabla^m u, \nabla^m u_t)\, \|\nabla^m u_t\|^2 < C_{24}(R_0, R_1), \quad (t \geq 0), \tag{5.9}$$

$$4q\beta\left(\alpha + \beta\|\nabla^m u\|^2\right)^{q-1} (\nabla^m u, \nabla^m u_t)\, \|(-\Delta)^m u\| < C_{25}(R_0, R_1), \quad (t \geq 0), \tag{5.10}$$

$$2\left\|g'(u)u_t\right\| < C_{26}(R_0, R_1), \quad (t \geq 0), \tag{5.11}$$

$$2\|\nabla^m u_{tt}\|^2 \geq 2{\lambda_1}^m \|u_{tt}\|^2. \tag{5.12}$$

From (5.8)-(5.12) and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_{tt}\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2\right] + 2{\lambda_1}^m \|u_{tt}\|^2 \\
&\leq C_{27} + C_{25}\|u_{tt}\|^2 \\
&\leq C_{27} + \frac{{C_{25}}^2}{4\varepsilon} + \varepsilon\|u_{tt}\|^2,
\end{aligned} \tag{5.13}$$

where $C_{24}, C_{25}, C_{26} > 0$ and $C_{27} = C_{24} + C_{26}$.

We choose $\varepsilon < 2{\lambda_1}^m$ and Lemma 2.5, then

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_{tt}\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2\right] + (2{\lambda_1}^m - \varepsilon)\left[\|u_{tt}\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2\right] \\
&\leq C_{27} + \frac{{C_{25}}^2}{4\varepsilon} + (2{\lambda_1}^m - \varepsilon)\left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2 \\
&\leq C_{28}.
\end{aligned} \tag{5.14}$$

By Gronwall's inequality for (5.14), we obtain

$$\begin{aligned}
&\|u_{tt}\|^2 + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q \|\nabla^m u_t\|^2 \\
&\leq \left[\|u_{tt}(x,0)\|^2 + \left(\alpha + \beta\|\nabla^m u_0\|^2\right)^q \|\nabla^m u_1\|^2\right] e^{-(2{\lambda_1}^m - \varepsilon)t} + \frac{C_{28}}{2{\lambda_1}^m - \varepsilon}.
\end{aligned} \tag{5.15}$$

Therefore, it exists $t_2, R_2 > 0$, such that

$$\|u_{tt}\|^2 \leq {R_2}^2, \quad t \geq t_2. \tag{5.16}$$

Lemma 5.5. One of the following requirements fulfills:

Case(I): When $n > 2m$, with $1 \leq p < \frac{n+2m}{n-2m}$;

Case(II): When $2m < n \leq 6m$, $g \in C^2(R)$ is critical, such that

$$\left|g''(u)\right| \leq C\left(1 + |u|^{\frac{6m-n}{n-2m}}\right), u \in R. \tag{5.17}$$

Then, the following Lipschitz continuity holds:

$$\|z_t\|^2 + \|\nabla^m z\|^2 \leq C\left(\|z_1\|^2 + \|\nabla^m z_0\|^2\right)e^{-kt} + C\int_0^t e^{-k(t-\tau)}\|z(\tau)\|^2 d\tau, \tag{5.18}$$

where $k > 0, z = u - v$. u, v are the solutions of problem (1.1)-(1.3) corresponding to initial data (u_0, u_1) and (v_0, v_1) in $H_0^m(\Omega) \times L^2(\Omega)$.

Proof. Obviously, we have

$$\begin{aligned}
&z_{tt} + (-\Delta)^m z_t + M(t)(-\Delta)^m z + \bar{M}(t)\left(\nabla^m(u+v), \nabla^m z\right)(-\Delta)^m(u+v) + f(u) - f(v) = 0, \\
&z(0) = u_0 - v_0, z_t(0) = u_1 - v_1,
\end{aligned} \tag{5.19}$$

where

$$M(t) = \frac{1}{2}\left[\left(\alpha + \beta\|\nabla^m u\|^2\right)^q + \left(\alpha + \beta\|\nabla^m v\|^2\right)^q\right] \geq \alpha^q, \tag{5.20}$$

$$\bar{M}(t) = \frac{1}{2}\int_0^1 q\beta\left[\alpha + \beta\left(\lambda\|\nabla^m u\|^2 + (1-\lambda)\|\nabla^m v\|^2\right)\right]^{q-1} d\lambda \geq 0. \tag{5.21}$$

By multiplying (5.19) by z_t and Lemma 2.4, Lemma 2.5, we get

$$\begin{aligned}
&\frac{d}{dt}\left[\|z_t\|^2 + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right] + 2\|\nabla^m z_t\|^2 \\
&= \left[q\beta\Big(\alpha + \beta\|\nabla^m u\|^2\Big)^{q-1}(\nabla^m u, \nabla^m u_t) + q\beta\Big(\alpha + \beta\|\nabla^m v\|^2\Big)^{q-1}(\nabla^m v, \nabla^m v_t)\right]\|\nabla^m z\|^2 \\
&\quad + 2\bar{M}(t)\,(\nabla^m(u_t+v_t), \nabla^m z)\,(\nabla^m(u+v), \nabla^m z) \\
&\quad + \int_0^1 q(q-1)\beta^2\Big(\alpha + \beta\lambda\|\nabla^m u\|^2 + \beta(1-\lambda)\|\nabla^m v\|^2\Big)^{q-2} \\
&\quad \times (\lambda\,(\nabla^m u, \nabla^m u_t) + (1-\lambda)\,(\nabla^m v, \nabla^m v_t))\,d\lambda(\nabla^m(u+v), \nabla^m z)^2 - 2\,(g(u)-g(v), z_t) \\
&\le C_{29}\,(\|\nabla^m u_t\| + \|\nabla^m v_t\|)\,\|\nabla^m z\|^2 - 2\,(g(u)-g(v), z_t)\,.
\end{aligned} \tag{5.22}$$

Case(I): When $1 \le p < \frac{n+2m}{n-2m}$, there exists a $\delta : 1 >> \delta > 0$ such that $H_0^{m-\delta}(\Omega) \hookrightarrow L^{p+1}(\Omega)$. By the interpolation and Lemma 2.5, we get

$$\begin{aligned}
&-2\,(g(u)-g(v), z_t) \\
&\le 2C_1\int_\Omega\Big(|u|^{p-1} + |v|^{p-1}\Big)|z|\,|z_t|\,dx \\
&\le 2C_1\left(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1}\right)\|z\|_{p+1}\|z_t\|_{p+1} \\
&\le 2C_1\left(\|u\|_{\frac{2n}{n+2m}}^{\theta(p-1)}\|u\|^{(1-\theta)(p-1)} + \|v\|_{\frac{2n}{n+2m}}^{\theta(p-1)}\|v\|^{(1-\theta)(p-1)}\right)\|z\|_{p+1}\|z_t\|_{p+1} \\
&\le C_{30}\|z\|_{H_0^{m-\delta}}\|\nabla^m z_t\| \\
&\le C_{31}\|z\|^\delta\|\nabla^m z\|^{1-\delta}\|\nabla^m z_t\| \\
&\le \varepsilon_3\|\nabla^m z_t\|^2 + {\varepsilon_3}^2\|\nabla^m z\|^2 + C_{32}\|z\|^2,
\end{aligned} \tag{5.23}$$

where $\theta = \frac{n(p-1)}{(p+1)(n+2m)}$, $C_{30}, C_{31}, C_{32} > 0$ and $2 > \varepsilon_3 > 0$.

Inserting (5.23) into (5.22), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|z_t\|^2 + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right] + (2-\varepsilon_3)\|\nabla^m z_t\|^2 \\
&\le {\varepsilon_3}^2\|\nabla^m z\|^2 + C_{29}\,(\|\nabla^m u_t\| + \|\nabla^m v_t\|)\,\|\nabla^m z\|^2 + C_{32}\|z\|^2.
\end{aligned} \tag{5.24}$$

We take the scalar product in L^2 of equation (5.19) with z. Then

$$\begin{aligned}
&\frac{d}{dt}\left[(z_t, z) + \frac{1}{2}\|\nabla^m z\|^2\right] + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2 \\
&= \|z_t\|^2 - (g(u)-g(v), z)\,.
\end{aligned} \tag{5.25}$$

In (5.25), by Lemma 2.5 we have

$$\begin{aligned}
-(g(u)-g(v), z) &\le C_1\left(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1}\right)\|z\|_{p+1}^2 \\
&\le C_{33}\,\|z\|_{p+1}^2 \\
&\le C_{34}\,\|z\|_{H_0^{m-\delta}}^2 \\
&\le \varepsilon_3\|\nabla^m z\|^2 + C_{35}\|z\|^2,
\end{aligned} \tag{5.26}$$

with $C_{33}, C_{34}, C_{35} > 0$.

Inserting (5.25) into (5.26), we get

$$\begin{aligned}
&\frac{d}{dt}\left[(z_t, z) + \frac{1}{2}\|\nabla^m z\|^2\right] + \|z\|^2 + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2 \\
&\le \|z_t\|^2 + \varepsilon_3\|\nabla^m z\|^2 + C_{36}\|z\|^2,
\end{aligned} \tag{5.27}$$

with $C_{36} = C_{35} + 1$.

Setting

$$P(t) = \|z_t\|^2 + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2 + \varepsilon_3\left((z_t, z) + \frac{1}{2}\|\nabla^m z\|^2\right), \tag{5.28}$$

$$\begin{aligned} Q(t) =&(2-2\varepsilon_3)\|\nabla^m z_t\|^2 + \varepsilon_3\|z\|^2 - 2\varepsilon_3{}^2\|\nabla^m z\|^2 \\ &+ \varepsilon_3 M(t)\|\nabla^m z\|^2 + \varepsilon_3\bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2. \end{aligned} \tag{5.29}$$

Obviously, there exist $a_2 \geq a_1 > 0$, $k > 0$ and $\varepsilon_3 > 0$ suitably small, such that

$$a_1\left[\|z_t\|^2 + \|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right] \leq P(t), \tag{5.30}$$

$$P(t) \leq a_2\left[\|z_t\|^2 + \|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right], \tag{5.31}$$

$$Q(t) \geq kP(t). \tag{5.32}$$

By (5.24)+$\varepsilon_3\times$(5.27) and (5.28)-(5.32), we get

$$\begin{aligned} &\frac{d}{dt}P(t) + kP(t) \\ \leq &\frac{d}{dt}P(t) + Q(t) \\ \leq &C_{29}\left(\|\nabla^m u_t\| + \|\nabla^m v_t\|\right)\|\nabla^m z\|^2 + \left(C_{32} + \varepsilon_3 C_{36}\right)\|z\|^2 \\ \leq &C_{37}\left(\|\nabla^m u_t\| + \|\nabla^m v_t\|\right)\|\nabla^m z\|^2 + C_{37}\|z\|^2, \end{aligned} \tag{5.33}$$

where $C_{37} = \max\{C_{29}, C_{32} + \varepsilon_3 C_{36}\}$.

By Lemma 5.3, there exists $C > 0$, we get

$$\|z_t\|^2 + \|\nabla^m z\|^2 \leq C\left(\|z_1\|^2 + \|\nabla^m z_0\|^2\right)e^{-kt} + C\int_0^t e^{-k(t-\tau)}\|z(\tau)\|^2 d\tau. \tag{5.34}$$

Case(II): When $p = \frac{n+2m}{n-2m}$, we have

$$(g(u) - g(v), z_t) = \frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1 (g'(\lambda u + \lambda(u-v))z^2 d\lambda dx + \bar{H}(t), \tag{5.35}$$

with

$$\bar{H}(t) = -\frac{1}{2}\int_\Omega\int_0^1 g''(\lambda u + \lambda(u-v))(\lambda u_t + (1-\lambda)v_t)z^2 d\lambda dx. \tag{5.36}$$

By the growth condition of g'', we have

$$\bar{H}(t) \leq C\int_\Omega\left(1 + |u|^{\frac{6m-n}{n-2m}} + |v|^{\frac{6m-n}{n-2m}}\right)(|u_t| + |v_t|)|z|^2 dx. \tag{5.37}$$

Therefore the Hölder inequality and $H_0^m(\Omega) \hookrightarrow L^{\frac{2n}{n-2m}}(\Omega)$ imply that

$$\begin{aligned} \bar{H}(t) \leq &C\left(1 + \|u\|^{\frac{6m-n}{n-2m}}_{\frac{2n}{n-2m}} + \|v\|^{\frac{6m-n}{n-2m}}_{\frac{2n}{n-2m}}\right)\left(\|u_t\|_{\frac{2n}{n-2m}} + \|v_t\|_{\frac{2n}{n-2m}}\right)\|z\|^2_{\frac{2n}{n-2m}} \\ \leq &C_{28}\left(\|\nabla^m u_t\| + \|\nabla^m v_t\|\right)\|\nabla^m z\|^2 + \varepsilon_3\|z_t\|^2 + C_{29}\|z\|^2. \end{aligned} \tag{5.38}$$

By Lemma 2.2, we get

$$-(g(u) - g(v), z) \leq \varepsilon_3\|\nabla^m z\|^2 + C_{30}\|z\|^2. \tag{5.39}$$

So, from (5.22) and (5.27), we obtain

$$\begin{aligned}\bar{P}(t) &= \|z_t\|^2 + M(t)\|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2 \\ &+ \int_\Omega \int_0^1 g'(\lambda u + \lambda(u-v))z^2 d\lambda dx + \varepsilon_3\left((z_t, z) + \frac{1}{2}\|\nabla^m z\|^2\right),\end{aligned} \tag{5.40}$$

$$\begin{aligned}\bar{Q}(t) &= 2\|\nabla^m z_t\|^2 - 2\varepsilon_3{}^2\|z_t\|^2 + \varepsilon_3\|z\|^2 + \varepsilon_3 M(t)\|\nabla^m z\|^2 \\ &+ \varepsilon_3\bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2 - \varepsilon_3{}^2\|\nabla^m z\|^2,\end{aligned} \tag{5.41}$$

$$\frac{d}{dt}\bar{P}(t) + \bar{Q}(t) \le (C_{19} + 2C_{28})\left(\|\nabla^m u_t\| + \|\nabla^m v_t\|\right)\|\nabla^m z\|^2 + (2C_{29} + \varepsilon_3 C_{30})\|z\|^2. \tag{5.42}$$

Obviously, there exist $b_2 \ge b_1 > 0$, $k > 0$ and $\varepsilon_3 > 0$ suitably small, such that

$$b_1\left[\|z_t\|^2 + \|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right] \le \bar{P}(t), \tag{5.43}$$

$$\bar{P}(t) \le b_2\left[\|z_t\|^2 + \|\nabla^m z\|^2 + \bar{M}(t)(\nabla^m(u+v), \nabla^m z)^2\right], \tag{5.44}$$

$$\bar{Q}(t) \ge k\bar{P}(t), \tag{5.45}$$

where $\varepsilon_3 > 0$ is suitably small.

Therefore, Omit Case(I), we easily obtain (5.18).

Lemma 5.6.[28] Let X be a Banach space and M be a bounded closed set in X. Assume that the mapping $V : M \to M$ possesses the properties:

(i) V is Lipschitz on M, i.e. there exists an $L > 0$ such that

$$\|Vv_1 - Vv_2\| \le L\|v_1 - v_2\|, \forall v_1, v_2 \in M; \tag{5.46}$$

(ii) there exist compact seminorms $n_1(x)$, $n_2(x)$ on X such that

$$\|Vv_1 - Vv_2\| \le \eta\|v_1 - v_2\| + K\left(n_1(v_1 - v_2) + n_2(Vv_1 - Vv_2)\right) \tag{5.47}$$

for any $v_1, v_2 \in M$, where $0 < \eta < 1$ and $K > 0$ are constants.

Then for any $k > 0$ and $\delta \in (0, 1-\eta)$ there exists a forward invariant compact set $A_{k,\delta} \subset M$ of finite fractal dimension such that

$$dist\left(V^k M, A_{k,\delta}\right) \le q^k, k = 1, 2, \cdots, \tag{5.48}$$

where $q = \eta + \delta < 1$, and

$$\dim_f A_{k,\delta} \le \left[\ln\frac{1}{\delta+\eta}\right]^{-1} \cdot \left[\ln m_0\left(\frac{2K(1+L^2)^{1/2}}{1-\eta}\right) + k\right], \tag{5.49}$$

where $m_0(R)$ is the maximal number of pairs (x_i, y_i) in $X \times X$ possessing the properties

$$\|x_i\|^2 + \|y_i\|^2 \le R^2, n_1(x_i - x_j) + n_2(y_i - y_j) > 1, i \ne j. \tag{5.50}$$

That is, the discrete dynamical system $\left(V^k, M\right)$ possesses an exponential attractor $A_{k,\delta}$.

Theorem 5.7. Let assumption of Lemma 2.4, 2.5 and 5.5 be valid, with $1 \le p \le \frac{n+2m}{n-2m}$. Then the dynamical system $(S(t), E_0)$ has an exponential attractor $\mathcal{A}_{exp}$.

Proof. It is proved by omitting [28]. By Theorem 3.3, we known $S(t)$ has a bounded absorbing B_0 in E_1. So B_0 is closed in E_2. From Lemma 2.4, 2.5 and Lemma 5.4, B_0 is bounded in $H_0^m(\Omega) \times H_0^m(\Omega)$, and for any $\xi_u = (u_0, u_1) \in B_0, \xi_u(t) = S(t)\xi_u = (u(t), u_t(t)) \in B_0$, and

$$\|\nabla^m u\| + \|\nabla^m u_t\| + \|u_{tt}\| \le C, t \ge 0. \tag{5.51}$$

Define the operator

$$V = S(T) : B_0 \to B_0. \tag{5.52}$$

Obviously, $VB_0 \subset B_0$ and V is Lipschitz onB_0. For any $\xi_u, \xi_v \in B_0$, we infer from Lemma 5.5 that

$$\begin{aligned} \|V\xi_u - V\xi_v\|_{E_0}^2 &\le Ce^{-kT} \|\xi_u(0) - \xi_v(0)\|_{E_0}^2 + C\int_0^T e^{-k(t-\tau)}\|u(\tau) - v(\tau)\|^2 d\tau \\ &\le \eta_T^2 \|\xi_u - \xi_v\|_{E_0}^2 + C \max_{0\le\tau\le T} \|u(\tau) - v(\tau)\|^2, \end{aligned} \tag{5.53}$$

that is

$$\|V\xi_u - V\xi_v\|_{E_0} \le \eta_T \|\xi_u - \xi_v\|_{E_0} + Cn_1(\xi_u - \xi_v), \tag{5.54}$$

where $\eta_T^2 = Ce^{-kT}, n_1(\xi_u) = \max_{0\le\tau\le T} \|u(\tau)\|$. Because of $H_0^m(\Omega) \hookrightarrow L^2(\Omega)$, such that $n_1(\xi_u)$ is a compact semi-norm.

By Lemma 5.6, the discrete dynamical system $\left(V^k, B_0\right)$ has an exponential attractor $\mathbb{A}$, where $V^k = S(kT)$.

Let

$$\mathcal{A}_{\exp} = \bigcup_{0\le t\le T} S(t)\mathbb{A}. \tag{5.55}$$

By the standard method (Z. J. Yang, 2010), one easily knows that $\mathcal{A}_{\exp}$ is an exponential attractor of dynamical system $(S(t), B_0)$. So there exists a $\gamma > 0$ such that

$$dist_{E_0}\left\{S(t)B_0, \mathcal{A}_{\exp}\right\} \le Ce^{-\gamma t}, t \ge 0. \tag{5.56}$$

Similar to (Zhijian Yang & Pengyan Ding, 2016), we easily obtain conclusion of definition 5.1. So, we obtain $\mathcal{A}_{\exp}$ is an exponential attractor of $(S(t), E_0)$.

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References

Ball, J. M. (1997). Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser.*, *28*(112), 473-486.

Chueshov, I., & Lasiecka, I. (2008). Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. *Mem. Amer. Math. Soc, 195*. https://doi.org/10.1090/memo/0912

Fatori, L. H., Jorge Silva, M. A., & Ma, T. F. (2015). Long-time behavior of a class of thermoelastic plates with nonlinear strain. *J. Differential Equations, 259*, 4831-4862. https://doi.org/10.1016/j.jde.2015.06.026

Ghisi, M., & Gobbino, M. (2009). Spectral gap global solutions for degenerate Kirchhoff equations. *Nonlinear Analysis*, *71*, 4115-4124. https://doi.org/10.1016/j.na.2009.02.090

Guoguang Lin, Yunlong Gao, & Yuting Sun. (2017). On Local Existence and Blow-Up of Solutions for Nonlinear Wave Equations of Higher-Order Kirchhoff Type with Strong Dissipation. *International Journal of Modern Nonlinear Theory and Application, 6*, 11-25. https://doi.org/10.4236/ijmnta.2017.61002

HARAUX, A, & ZUAZUA, E. (1988). Decay estimates for some semilinear damped hyperbolic problems. *Arch. Rational Mech. Anal., 100*(2), 191-206. https://doi.org/10.1016/j.jde.2011.08.022

Igor Chueshov. (2012). Longtime dynamics of Kirchhoff wave models with strong nonlinear damping. *J. Differential Equations, 252*, 1229-1262. https://doi.org/10.1016/j.jde.2011.08.022

Ke, Li. (2017). A Gronwall-type lemma with parameter and its application to Kirchhoff type nonlinear wave equation. *Journal of Mathematical Analysis and Applications, 447*, 683-704. https://doi.org/10.1016/j.jmaa.2016.10.017

Kirchhoff, G., & Vorlesungen *ü*ber, Mechanik. (1883). Teubner, Leipzig.

KOP*Á*ČKOV*Á*, M. (1989). Remarks on bounded solutions of a semilinear dissipative hyperbolic equation. *Conment. Math. Univ. Carolin., 30*(4), 713-719.

Kosuke, Ono. (1997). On Global Existence, Asymptotic Stability and Blowing Up of Solutions for Some Degenerate Non-linear Wave Equations of Kirchhoff Type with a Strong Dissipation. *Mathematical Methods in the Applied Sciences, 20*, 151-177.
https://doi.org/10.1002/(SICI)1099-1476(19970125)20:2¡151::AID-MMA851¿3.0.CO;2-0

Li, Y. (2011). The Asymptotic Behavior of Solutions for a Nonlinear Higher Order Kirchhoff Type Equation. *Journal of Southwest China Normal University, 36*, 24-27.

Lin, G. G. (2011). *Nonlinear evolution equation*. Yunnan University Press.

Nguyen, H. D. (2014). A Digital Binomial Theorem. *Mathematics*, 1-5.

Robert, A. A., & Fournier, J. J. F. (2003). *Sobolev spaces.* Department of Mathematics The University of British Columbia Vancouver, Canada.

Teman, R. (1998). *Infinite Dimensional Dynamics Systems in Mechanics and Physics.* Springer, New York.

Wu, J. Z., & Lin, G. G. (2009). The global attractor of the Bossinesq equation with damping term and its dimension estimation. *Journal of Yunnan University, 31*, 335-340.

Xiaoming, Fan, & Shengfan, Zhou. (2004). Kernel sections for non-autonomous strongly damped wave equations of non-degenerate Kirchhoff-type. *Applied Mathematics and Computation, 158*, 253-266.
https://doi.org/10.1016/j.amc.2003.08.147

Xueli, Song, & Yanren, Hou. (2015). Uniform attractors for three-dimensional Navier-Stokes equations with nonlinear damping. *Journal of Mathematical Analysis and Applications, 422*, 337-351.
https://doi.org/10.1016/j.aml.2014.02.014

Yang, Z. J. (2010). Finite-dimensional attractors for the Kirchhoff models. *J. Math. Phys, 51.*

Yang, Z. J., Ding, P. Y., & Liu, Z. M. (2014). Global attractor for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity. *Applied Mathematics Letters, 33*, 12-17. https://doi.org/10.1016/j.aml.2014.02.014

Yang, Z. J., Ding, P. Y. , & Li, L. (2016). Longtime dynamics of the Kirchhoff equations with fractional damping and supercritical nonlinearity. *Journal of Mathematical Analysis Application, 442*, 485-510.
https://doi.org/10.1016/j.jmaa.2016.04.079

Yaojun, Ye. (2013). Global existence and energy decay estimate of solutions for a higher-order Kirchhoff type equation with damping and source term. *Nonlinear Analysis, Real World Applications, 14*, 2059-2067.
https://doi.org/10.1016/j.nonrwa.2013.03.001

Yunlong Gao, Yuting Sun, & Guoguang Lin. (2016). The Global Attractor and Their Hausdorff and Fractal Dimensions Estimation for the Higher-Order Nonlinear Kirchhoff-Type Equation with Strong Linear Damping. *International Journal of Modern Nonlinear Theory and Application, 5*, 185-202. https://doi.org/10.4236/ijmnta.2016.54018

Zhijian Yang, & Zhiming Liu. (2015). Exponential attractor for the Kirchhoff equations with strong nonlinear damping and supercritical nonlinearity. *Applied Mathematics Letters, 46*, 350-359.
https://doi.org/10.1016/j.jmaa.2015.10.013

Zhang Yan, Pu Zhilin, & Chen Botao. (2008) Boundedness of the Solution to the Nonlinear Kirchhoff Equation. *Journal of Southwest China Normal University, 6*, 5-8.

Zhijian Yang, & Pengyan Ding. (2016). Longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on R^N. *Journal of Mathematical Analysis and Applications, 434*, 1826-1851.
https://doi.org/10.1016/j.jmaa.2015.10.013

Zhou, S. (1999). *Global attractor for strongly damped nonlinear wave equations, 6*, 451-470.

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