Journal of Mathematics Research; Vol. 9, No. 5; October 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Study on the Minimum and Maximum Sum of C2 Problem in IMO2014

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Received: June 23, 2017 Accepted: July 17, 2017 Online Published: August 9, 2017

doi:10.5539/jmr.v9n5p1 URL: https://doi.org/10.5539/jmr.v9n5p1

Abstract

The focus of this paper is primarily on a problem: the principle of the extreme value under some special operations. After enumerating from the maximum sum to minimum and solving these cases, I found that the use of the two mathematical models enabled the derivation of the general form of the use of the two mathematical models enabled the derivation of the general form of the maximum and the minimum sum. This program looks into the principles of minimum and maximum sum, and the various patterns that come along with it. In order to further discuss this kind of problems, we set up other different conditions, solving them with two mathematical models and principle of sequence recursive relationship, induction proof, etc. We also extend all these problems to explore the generating functions of the maximum and the minimum sum with operating number m based on the parity of the number of papers. Finally, using computer generated software, we demonstrate the various sums of a particular state, along with coming up with a general rule for all states that can predict the maximum and the minimum sum through the usage of induction.

Keywords: extreme value, minimum, maximum, operation

1. Introduction

1.1 Research Motivation

In a math's project study we met with a math problem, which was described as follows:

We have 2^m sheets of papers, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets. If the numbers on the two sheets are *a* and *b*, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m (IMO, 2014).

1.2 Research Purpose

(1) Constructing a mathematical model for the maximum and minimum sum after a particular operation, also exploring the principle of non-existing sum value (Wang et al., 2002).

(2) Exploring the principle about the maximum and minimum sum of a particular operation based on the quantity and size (Richard, 2011).

(3) Creating a general form for the maximum and minimum sum of a particular operation (Djordjevic & Srivastava, 2005).

(4) Predicting the number of steps depending upon whether the corresponding state exists given the sum value.

2. Method

2.1 Definition of the Notations

2.1.1 The Operation Mode of *a* and *b*, Denoted by $(a,b) \to (m,n)$

It is a process satisfying that there exists a specific operation by which two positive integers *m* and *n* are summed up for any two positive integers a and b. For instance, $(a,b) \to (a+b, a+b)$ indicates the operation mode of a and b by addition operation for two positive integers a and b. The sum of all the numbers after one step of the operation mode of a and b is denoted by S_1, and the notation S_k denotes the sum of all the numbers after arbitrary k steps of the operation mode.

2.1.2 The Sequence$< (i_1, j_1), (i_2, j_2), (i_3, j_3), \cdots, (i_k, j_k) >$

This sequence, consisted of the numbers i_r and j_r, where $r = 1,2, \cdots, k$, which are the resulting numbers chosen in the r^{th} step of the operation mode of a and b is called the operation sequence of k steps of the operation mode of a and b. And if there exists a liner combination of $x_1, x_2, x_3, \cdots, x_n$, say $c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$ with positive integers $c_1, c_2, c_3, \cdots, c_n$, then the notation ($c_1, c_2, c_3, \cdots, c_n$) is called the characteristic vector of the sequence $< x_1, x_2, x_3, \cdots, x_n >$.

2.1.3 Exploring Some Cases

To discuss the property of Maximizing strategy model, we demonstrate an example starting from ten positive integers of $1, 2, 3, \cdots, 10$ through 3 steps of the operation model, as follows:

First, the initial sum of all the number is $1 + 2 + 3 + 4 + \cdots + 9 + 10$, so the characteristic vector of the sequence $< 1, 2, 3, \cdots, 10 >$ is $(1,1,1, \cdots, 1)$.

Next, we perform the following operation of some cases as follows:

Case1:

$< (1, 2), 3, 4, \cdots, 9,10 > \rightarrow < (3, 3), 3, 4, \cdots, 9,10 > \rightarrow < (6, 6), 3,4, \cdots, 9,10 > \rightarrow < 12, 12, 3,4, \cdots, 9,10 >$

i.e. The operation sequence is $< (i_1, j_1), (i_2, j_2), (i_3, j_3) > = < (1,2), (3,3), (6,6) >$.

So the sum of all the numbers after 3 steps of the operation model is $S_3 = 8 \times 1 + 8 \times 2 + 3 + 4 + \cdots + 9 + 10$=76 and we might obtain this result deducing by 12+12=24=(1+2)*8.

Thus the characteristic vector of the sequence $< 1, 2, 3, \cdots, 10 >$ is $(8,8,1, \cdots, 1)$.

Case2:

$< (1, 2), 3, 4, \cdots, 9,10 > \rightarrow < 3, 3, (3, 4), \cdots, 9,10 > \rightarrow < 3, 3, 7, 7, (5, 6), \cdots, 9,10 > \rightarrow < 3, 3, 7,7,11,11, \cdots, 9,10 >$

i.e. The operation sequence is $< (i_1, j_1), (i_2, j_2), (i_3, j_3) > = < (1,2), (3,4), (5,6) >$.

So the sum of all the numbers after 3 steps of the operation mode is $S_3 = 2 \times 1 + 2 \times 2 + 2 \times 3 + 2 \times 4 + 2 \times 5 + 2 \times 6 + \cdots + 9 + 10$=76 and we might obtain this result deducing by 3+3+7+7+11+11=42=(1+2)*2+(3+4)*2+(5+6)*2.

Thus the characteristic vector of the sequence $< 1, 2, 3, \cdots, 10 >$ is $(2,2,2,2,2,2,1, \cdots, 1)$.

2.2 Creating Mathematical Models and Operations

2.2.1 The Principle about the Maximum

As can be seen from above, we could observe the process $< 1, 2, \cdots, 7, 8, (9,10) > \rightarrow < 1, 2, \cdots, 7, 8, (19,19) > \rightarrow < 1, 2, \cdots, 7, 8, (38, 38) > \rightarrow < 1, 2, \cdots, 7, 8, 76, 76 >$, then the operation sequence is $< (i_1, j_1), (i_2, j_2), (i_3, j_3) > = < (9,10), (19,19), (38,38) >$.

From the result of $S_3 = 1 + 2 + \cdots + 7 + 8 + 8 \times 9 + 8 \times 10$=188 and we might obtain this result deducing by 76+76=42=(9+10)*8.

Thus the characteristic vector of the sequence $< 1, 2, 3, \cdots, 10 >$ is $(1,1, \cdots, 1,1,8,8)$.

We discover some properties as follows:

(1) The operation sequence $< (i_1, j_1), (i_2, j_2), (i_3, j_3) > = < (9,10), (19,19), (38,38) >$ produces the maximal sum of all the numbers after 3 steps of the operation model starting from ten positive integers of $1, 2, 3, \cdots, 10$. On the other hand, its characteristic vector is $(1,1, \cdots, 1,1,8,8)$.

(2) The operation sequence $< (i_1, j_1), (i_2, j_2), (i_3, j_3) > = < (1,2), (3,3), (6,6) >$ produces the minimal sum of all the numbers after 3 steps of the operation model starting from ten positive integers of $1, 2, 3, \cdots, 10$. On the other hand, its characteristic vector is $(8,8,1, \cdots, 1)$.

Therefore, through the previous section of the inquiry we summarize and guess that when the two largest number to choose for the operation, the maximum sum of all the numbers are obtained; similarly, when the two smallest number to choose for the operation, the minimum sum of all the numbers are obtained.

2.2.2 Mathematical Models

Assuming that the number of digits is a certain value and all the numbers on the board are any positive integers. By the operation mode$(a, b) \rightarrow (a + b, a + b)$,

the maximum value of the sum S_k of all the numbers after arbitrary k steps of the operation mode, we start to explore whether there is a specific mode of operation and discuss the general rules of and the strategy with maximum value.

Maximizing Strategy Model:

The intuitive strategy to maximize the sum of all the numbers after *n* steps of the operation mode is that in every step we chose the two greatest numbers.

【Proof】Its proof process is placed in the appendix of the paper (see Appendix 1 for complete proof).

Minimizing Strategy Model:

The intuitive strategy to minimize the sum of all the numbers after some steps of the operation mode is that in every step we chose the two least numbers.

【Proof】Its proof process is placed in the appendix of the paper (see Appendix 2 for complete proof).

2.3 Exploring the Principle about the Maximum and Minimum Sum

2.3.1 The General Rule of Maximum Sum

Discuss a question: Given m sheets of paper with the number 1 through *k* steps of the operation mode. What is the maximum of the sum of the numbers on all the sheets of paper? Similarly, what is the minimum of the sum of the numbers on all the sheets of paper?

【Process】

(1) When m=2 for all positive integer k

For $\mathrm{k}=1,$ the sequence $<\ (1,1)>\rightarrow<2{,}2>,$ then the sum is 4.

For $\mathrm{k}=2,$ the sequence $<(1{,}1)>\rightarrow<(2,2)>\rightarrow<4{,}4>,$ then the sum is 8.

For k= n, the sequence $<\ (1,1)>\rightarrow<(2\ ,2)>\rightarrow<(4\ ,4)>\rightarrow\cdots.\rightarrow<2^{n},2^{n}>, then\ the\ sum\ is\ 2^{n+1}$.

Assume that k=n holds, we want to show that k=n+1 also holds.

We already know that the form for k=n is $(2^{n},2^{n}\ )$, and following the rule, we find that the next step, k=n+1 step, the form becomes $((2^{n}+2^{n}\ ),(2^{n}+2^{n}\ ))$. This form would then become $(2^{n+1},2^{n+1})$ with the sum being $2^{n+1}+2^{n+1}=2^{n+2}$.

By the induction hypothesis, thus fulfills for all positive integers k.

(2) When m=3 for all positive integers k

For k=1, the sequence $<\ (1,1),1>\rightarrow<(2\ ,2),1>$.

The sum equals $2+2+1=5$, which is the maximum and the minimum.

For k=2, the sequence $<\ (1,1),1>\rightarrow<(2\ ,2),1>\rightarrow<4{,}4{,}1>$.

The sum is $4+4+1=9,$ which is the maximum.

For k=3, the sequence $<\ (1,1),1>\rightarrow<(2\ ,2),1>\rightarrow<(4{,}4){,}1>\rightarrow<8{,}8{,}1>$.

The sum is $8+8+1=17,$ which is the maximum.

We see that this trend continues on till k=n as we can see that during k=n, the trend is as follows:$<\ (1,1),1>\rightarrow<(2\ ,2),1>\rightarrow<(4{,}4){,}1>\rightarrow\cdots\rightarrow<2^{n},2^{n},1>$

The sum is $2^{n}+2^{n}+1=2^{n+1}+1.$ this is the maximum sum of k=n.

Knowing that, let us assume that when k=n, the sum is $2^{n+1}+1$. We want to show that the next step also has a maximum sum in the same trend. Therefore there are two different states with our next step, either combining a 2^{n} with 1 or $2^{n}\ with\ 2^{n}$. Obviously, the latter of the two states yields a larger sum. Therefore we see that during the k=n+1 step the maximum state is as below:$<(2^{n},2^{n}),1>\ \rightarrow<2^{n+1},2^{n+1},1>$. The sum is $2^{n+1}+2^{n+1}+1=2^{n+2}+1$. This will hold for all positive integers k since we have shown by induction that if k=n works, then k=n+1 will also work. Hence, by mathematical induction, the property that the maximum sum is $2^{n+1}+1$ for m=3 and all positive integers k.

2.3.2 Induction Proof of the Maximal Sum for All Positive m and k.

Given m sheets of paper with the number 1 through *k* steps of the operation mode, let's predict the maximum general formula for the maximum and the minimum sheets total.

【proof】

Let's first look at k=2,

When k=2 we see that the maximum has a form of (4,4,1,1,1….1) with the maximizing strategy model.

With $m-2$ ones, the sum is $(\mathrm{m}-2)\times 1+4\times 2=\mathrm{m}+6$.

When k=3 we see that the maximum has a form of (8,8,1,1,….1) with the maximizing strategy model.

With $m-2$ ones, the sum is $(\mathrm{m}-2)\times 1+8\times 2=\mathrm{m}+14$.

As we began seeing a pattern, we can predict that the maximum sum's general rule is $\mathrm{m}+2^{k+1}-2$.

Let's assume that when k=n, the inferred general rule holds.

We want to show that when k=n+1 also holds. As known, the maximum form of k=n is ($2^n, 2^n, 1, 1, 1, \ldots\ldots.1)$ There are two movements that could be made in the next step, but from theorem 3, we already know that we will get the maximum if we only add up 1 pair sheets at a time. Hence, we add the 2^n together and we can find that this equals the form $(2^{n+1}, 2^{n+1}, 1, 1, 1 \ldots..1)$ with the maximizing strategy model. Hence the sum is $2^{n+1}+2^{n+1}+m-2=2^{n+1}*2+m-2=2^{n+2}+m-2=2^{(n+1)+1}+m-2$.

Since k= n +1 works, this general rule holds for all n.

By mathematical induction, the maximal sum is $2^{n+1}+1$ for m=3 and all positive integers k.

As a result of the above discussion and argument, we organized the result into the following theorem:

Theorem 1

Given *m* sheets of paper with the number 1, the general formula of the maximum sum for a positive integer *m* after *n* steps of the operation mode with the maximizing strategy model is $\mathrm{m}+2^{k+1}-2,\ \forall m \in N, n \in$N.

2.4 Different Forms of the Minimum Sum of m, Even and Odd

2.4.1 When m is Even

1. For m=2:

The sequence $<(1,1)>\to<2,2>\to<4,4>\to<8,8>$ is obtained after 3 steps of the operation mode with the minimize strategy model. Upon observing the minimum sum, we construct a table about the minimum sum after 5 steps of the operation mode with the minimize strategy model as follow:

k	1	2	3	4	5	6	7	8
Min Sum	2*2	4*2	8*2	16*2	32*2	64*2	128*2	256*2

We see an obvious pattern of when k=n, the minimum sum will be 2^{n+1}.

By induction, k=1 the minimum sum will be $2^{1+1}=4$. Let's assume that when k=n, the minimum sum will be 2^{n+1}. Since there are only two terms after each step, these two terms will still have the same value. We can find the form for k=n. The value for each term is $\frac{2^{n+1}}{2}=2^n$. We want to show that this rule also holds for k=n+1. We already found the form for $\mathrm{k}=\mathrm{n}$, which is $<2^n, 2^n>$, and the next step would be to combine them together, forming $<2^{n+1}, 2^{n+1}>$. Thus, the sum will be $2^{n+1}+2^{n+1}=2^{n+2}$.

Hence, by induction, the sum holds for all k.

2. For m=4

The sequence $<(1,1),1,1,1,1>\to<2,2,(1,1),1,1>\to<2,2,2,2,(1,1)>\to<(2,2),2,2,2,2>\to<4,4,(2,2),2,2>\to<4,4,4,4,(2,2)>\to<4,4,4,4,4,4>$ is obtained after 6 steps of the operation mode with the minimizing strategy model. Upon observing the minimum sum, we construct a table about the minimum sum after 8 steps of the operation mode with the minimizing strategy model as follow:

k	1	2	3	4	5	6	7	8
Min Sum	4+2*1	4+2*2	4+2*4	4+2*6	4+2*10	4+2*14	4+2*22	4+2*30
sum	6	8	12	16	24	32	48	64

We infer that the pattern for m=4 is as follows:

minimum sum $\begin{cases} a_{2n+1}=3\times 2^{n+1} \\ a_{2n+2}=4\times 2^{n+1} \end{cases}, n \in N \cup \{0\}$.

Plugging in values, we see that this rule holds and we will prove by induction.

【Proof】

Observing, we see a reoccurring trend over a cycle of 2. After every 2 moves in k=4, the list would be in a state where every term is the same.

When m=4 for all non-negative integer n:

For $n = 0$, the sequence $<(1,1),1,1> \to <2,2,1,1>$ is obtained after 1 step of the operation mode with the minimizing strategy model. The sum is 6. The sequence $<(1,1),1,1> \to <2,2,(1,1)> \to <2,2,2,2>$ is obtained after 2 steps of the operation mode with the minimizing strategy model, and the sum is 8.

Assume that $n = k$ holds, i.e. the minimum sum obtained after 2k+1 steps of the operation mode with the minimizing strategy model. Then the sum is $a_{2k+1} = 3 \times 2^{k+1}$. The minimum sum is obtained after 2k+2 steps of the operation mode with the minimizing strategy model, and the sum is $a_{2k+2} = 4 \times 2^{k+1}$.

We want to show that $n = k + 1$ also holds.

Assume that this rule also works for taking 2k+2 steps of the operation mode with the minimizing strategy model, the minimum sum would be $4 \times 2^{k+1}$. Since every term is the same we know that each term equals $\frac{4\times 2^{k+1}}{4} = 2^{k+1}$. Thus the form is $< 2^{k+1}, 2^{k+1}, 2^{k+1}, 2^{k+1}, >$.

Now we want to show that for taking 2k+3 steps of the operation mode with the minimizing strategy model, the minimum sum is $3 \times 2^{k+2}$.

Observing that the form one step before was $< 2^{k+1}, 2^{k+1}, 2^{k+1}, 2^{k+1} >$, we know that the next move would be to add any two of the terms together and we get the form $< 2^{k+2}, 2^{k+2}, 2^{k+1}, 2^{k+1}, >$, with the sum $2^{k+2} + 2^{k+2} + 2^{k+1} + 2^{k+1} = 3 \times 2^{k+2}$.

We see that the same thing happens from taking 2k+4 steps of the operation mode with the minimizing strategy model, the minimum sum is $4 \times 2^{k+2}$.

Observing that the form one step before was $< 2^{k+2}, 2^{k+2}, 2^{k+1}, 2^{k+1} >$, we know that the next move would be to add any two of the terms together and we get the form $< 2^{k+2}, 2^{k+2}, 2^{k+2}, 2^{k+2} >$, with the sum $2^{k+2} + 2^{k+2} + 2^{k+2} + 2^{k+2} = 4 \times 2^{k+2}$. Those outcomes lead us to conclude that this also holds for $n = k + 1$, thus proving the general form.

By the induction hypothesis, thus fulfills for all non − negative integers n.

On the other hand, we use the symbol w of period 2, $a\ non-1\ root\ of\ equation\ x^2 = 1$, to translate the form of the minimum sum $\begin{cases} a_{2n+1} = 3 \times 2^{n+1} \\ a_{2n+2} = 4 \times 2^{n+1} \end{cases}, n \in N \cup \{0\}$ as the form of

$$a_n = [3 + \frac{w^n + w^{2n}}{2}] \times 2^{\left[\frac{n-1}{2}\right]+1}, n \in N.$$

3. For m=6:

The sequence $<(1,1),1,1,1,1> \to <2,2,(1,1),1,1> \to <2,2,2,2,(1,1)> \to <(2,2),2,2,2,2> \to <4,4,(2,2),2,2> \to <4,4,4,4,(2,2)> \to <4,4,4,4,4,4>$ is obtained after 6 steps of the operation mode with the minimizing strategy model. Upon observing the minimum sum, we construct a table about the minimum sum after 8 steps of the operation mode with the minimizing strategy model as follow:

K	1	2	3	4	5	6	7	8
Min Sum	6+2*1	6+2*2	6+2*3	6+2*5	6+2*7	6+2*9	6+2*13	6+2*17
1st difference of the times 2	0	1	2	3	5	7	9	13
2nd difference of the times 2	1	1	1	2	2	2	4	4

We infer that the pattern for m=6 is as follows:

minimum sum $\begin{cases} a_{3n+1} = 4 \times 2^{n+1} \\ a_{3n+2} = 5 \times 2^{n+1} \\ a_{3n+3} = 6 \times 2^{n+1} \end{cases}, n \in N \cup \{0\}.$

Plugging in values, we see that this rule holds and we will prove by induction.

【Proof】

We see a reoccurring trend over a cycle of 3. After every 3 moves in k=6, the list would be in a state if every term is the same.

When m=6 for all non-negative integer n:

For $n=0$, the sequence $<(1,1),1,1,1,1>\to<2,2,1,1,1,1>$ is obtained after 1 step of the operation mode with the minimizing strategy model, and the sum is 8.

The sequence $<(1,1),1,1,1,1>\to<2,2,(1,1),1,1>\to<2,2,2,2,(1,1)>$ is obtained after 2 steps of the operation mode with the minimizing strategy model, and the sum is 10. The sequence $<(1,1),1,1,1,1>\to<2,2,(1,1),1,1>$ $\to<2,2,2,2,(1,1)>\to<(2,2),2,2,2,2>$ is obtained after 3 steps of the operation mode with the minimizing strategy model, and the sum, 12.

Assume that $n=k$ holds, i.e. the minimum sum obtained after 3k+1 steps of the operation mode with the minimizing strategy model, then the sum is $a_{3k+1}=4\times 2^{k+1}$. The minimum sum is obtained after 3k+2 steps of the operation mode with the minimizing strategy model, then the sum is $a_{3k+2}=5\times 2^{k+1}$.

With the minimum sum obtained after 3k+3 steps of the operation mode with the minimizing strategy model, the sum is $a_{3k+3}=6\times 2^{k+1}$. We want to show that $n=k+1$ also holds.

Let's say this rule also works for taking 3k+3 steps of the operation mode with the minimizing strategy model, and the minimum sum would be $6\times 2^{k+1}$. Since every term is the same we know that each term equals $\frac{6\times 2^{k+1}}{6}=2^{k+1}$. Thus the form is $<2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$.

Now we want to show that for taking 3k+4 steps of the operation mode with the minimizing strategy model, the minimum sum is $8\times 2^{k+1}$.

Observing that the form one step before was $<2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$, we know that the next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$, with the sum $2^{k+2}+2^{k+2}+2^{k+1}+2^{k+1}+2^{k+1}+2^{k+1}=4\times 2^{k+2}$.

Similarly after taking 3k+5 steps of the operation mode with the minimizing strategy model, the minimum sum is $5\times 2^{k+2}$.

The form one step before was $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$. The next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+2},2^{k+2},2^{k+1},2^{k+1}>$, with the sum $2^{k+2}+2^{k+2}+2^{k+2}+2^{k+2}+2^{k+1}+2^{k+1}=5\times 2^{k+2}$. We see that taking 3k+6 steps of the operation mode with the minimizing strategy model, the minimum sum is $6\times 2^{k+2}$.Those outcomes lead us to conclude that this also holds for $n=k+1$, thus proving the general form.

By the induction hypothesis, thus fulfills for all non − negative integers n.

On the other hand, we use the symbol w of period 3, $a\ root\ of\ equation\ x^3=1$, to translate the form of the minimum sum $\begin{cases}a_{3n+1}=4\times 2^{n+1}\\ a_{3n+2}=5\times 2^{n+1}\\ a_{3n+3}=6\times 2^{n+1}\end{cases}$, $n\in N\cup\{0\}$ as the form of

$$a_n=[4+\sum_{k=1}^{2}k\times\frac{w^{n+2-k}+w^{2(n+2-k)}+w^{3(n+2-k)}}{3}]\times 2^{\left[\frac{n-1}{3}\right]+1}, n\in N.$$

4. For m=8:

The sequence $<(1,1),1,1,1,1,1,1>\to<2,2,(1,1),1,1,1,1>\to<2,2,2,2,(1,1),1,1>\to<2,2,2,2,2,2,(1,1)>\to<(2,2),2,2,2,2,2,2>\to<4,4,(2,2),2,2,2,2>\to<4,4,4,4,(2,2),2,2>\to<4,4,4,4,4,4,2,2>$ is obtained after 6 steps of the operation mode with the minimizing strategy model.

Upon observing the minimum sum, we construct a table about the minimum sum after 8 steps of the operation mode with the minimizing strategy model as follow:

K	1	2	3	4	5	6	7	8
Min Sum	8+2*1	8+2*2	8+2*3	8+2*4	8+2*6	8+2*8	8+2*10	8+2*12
Min Sum	5*2	6*2	7*2	8*2	5*4	6*4	7*4	8*4

We infer that the pattern for m=8 is as follows:

minimum sum $\begin{cases} a_{4n+1} = 5 \times 2^{n+1} \\ a_{4n+2} = 6 \times 2^{n+1} \\ a_{4n+3} = 7 \times 2^{n+1} \\ a_{4n+4} = 8 \times 2^{n+1} \end{cases}$, $n \in N \cup \{0\}$.

Plugging in values, we see that this rule holds and we will prove by induction.

【Proof】

We see a reoccurring trend over a cycle of 4. After every 4 moves in k=8, the list would be in a state, were every term is the same.

When m=8 for all non-negative integer n:

For $n = 0$, the sequence $<(1,1),1,1,1,1,1,1> \to <2,2,1,1,1,1,1,1>$ is obtained after 1 step of the operation mode with the minimizing strategy model. The sum is 10.

The sequence $<(1,1),1,1,1,1,1,1> \to <2,2,(1,1),1,1,1,1> \to <2,2,2,2,(1,1),1,1>$ is obtained after 2 steps of the operation mode with the minimizing strategy model, and the sum is 12.

The sequence $<(1,1),1,1,1,1,1,1> \to <2,2,(1,1),1,1,1,1> \to <2,2,2,2,(1,1),1,1> \to <(2,2),2,2,2,2,1,1>$ is obtained after 3 steps of the operation mode with the minimizing strategy model, then the sum is 14.

Similarly the sequence $<(1,1),1,1,1,1,1,1> \to <2,2,(1,1),1,1,1,1> \to <2,2,2,2,(1,1),1,1>$

$\to <2,2,2,2,2,2,(1,1)> \to <(2,2),2,2,2,2,2,2>$ is obtained after 4 steps of the operation mode with the minimizing strategy model, and the sum, 16.

Assume that $n = k$ holds, i.e. the minimum sum obtained after 4k+1 steps of the operation mode with the minimizing strategy model, then the sum is $a_{4k+1} = 5 \times 2^{k+1}$. With the minimum sum obtained after 4k+2 steps of the operation mode with the minimize strategy model, the sum is $a_{4k+2} = 6 \times 2^{k+1}$.

The minimum sum is obtained after 4k+3 steps of the operation mode with the minimizing strategy model, then the sum is $a_{4k+3} = 7 \times 2^{k+1}$. With the minimum sum obtained after 4k+4 steps of the operation mode with the minimizing strategy model, the sum is $a_{4k+4} = 8 \times 2^{k+1}$.

We want to show that $n = k + 1$ also holds.

Assume that it also works for taking 4k+4 steps of the operation mode with the minimizing strategy model. Then the minimum sum would be $8 \times 2^{k+1}$. Since every term is the same we know that each term equals $\frac{8 \times 2^{k+1}}{8} = 2^{k+1}$. Thus the form is $<2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$.

Now we want to show that for taking 4k+5 steps of the operation mode with the minimizing strategy model, the minimum sum is $10 \times 2^{k+1}$.

The form one step before was $<2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$. The next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$, with the sum $2^{k+2} + 2^{k+2} + 2^{k+1} + 2^{k+1} + 2^{k+1} + 2^{k+1} + 2^{k+1} + 2^{k+1} = 5 \times 2^{k+2}$. We see that from taking 4k+6 steps of the operation mode with the minimizing strategy model, the minimum sum is $6 \times 2^{k+2}$.

Observing that the form one step before was $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$, we know that the next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1}>$, with the sum $2^{k+2} + 2^{k+2} + 2^{k+2} + 2^{k+2} + 2^{k+1} + 2^{k+1} + 2^{k+1} + 2^{k+1} = 6 \times 2^{k+2}$. Similarly taking 4k+7 steps of the operation mode with the minimizing strategy model, the minimum sum is $7 \times 2^{k+2}$. After taking 4k+8 steps of the operation mode with the minimizing strategy model, the minimum sum is $8 \times 2^{k+2}$. Those outcomes lead us to conclude that this also holds for $n = k + 1$, thus proving the general form.

By the induction hypothesis, thus fulfills for all non − negative integers n.

On the other hand, we use the symbol w of period 4, $a\ root\ of\ equation\ x^4 = 1$, to translate the form of the minimum sum $\begin{cases} a_{4n+1} = 5 \times 2^{n+1} \\ a_{4n+2} = 6 \times 2^{n+1} \\ a_{4n+3} = 7 \times 2^{n+1} \\ a_{4n+4} = 8 \times 2^{n+1} \end{cases}, n \in N \cup \{0\}$ as the form of

$$a_n = [5 + \sum_{k=1}^{3} k \times \frac{w^{n+3-k} + w^{2(n+3-k)} + w^{3(n+3-k)} + w^{4(n+3-k)}}{4}] \times 2^{\left[\frac{n-1}{4}\right]+1}, n \in N.$$

5. The general formula of the minimum sum for an even integer m after n steps of the operation mode with the minimizing strategy model:

$$a_n = \{{}^{m}/_{2} + 1 + \sum_{k=1}^{{}^{m}/_{2}-1} [k \times \frac{\sum_{i=1}^{{}^{m}/_{2}} w^{i(n+{}^{m}/_{2}-1-k)}}{{}^{m}/_{2}}]\} \times 2^{\left[\frac{n-1}{{}^{m}/_{2}}\right]+1}, \forall \frac{m}{2} \in N, n \in N$$

【Discover】

For an even integer m, consider the sequence $<(1,1),1,1,1,1,\cdots,1,1>$ that has m sheets of paper with the number 1. $<(1,1),1,1,1,1,\cdots,1,1> \to <2,2,(1,1),1,1,\cdots,1,1> \to <2,2,2,2,(1,1),\cdots,1,1> \to \cdots \to <2,2,2,2,2,2,\cdots,(1,1)> \to <(2,2),2,2,2,2,\cdots,2,2>$ is obtained after $\frac{m}{2}$ steps of the operation mode with the minimizing strategy model.

Upon observing the minimum sum, we construct a table about the minimum sum after $\frac{m}{2}$ steps of the operation mode with the minimizing strategy model as follow:

K	1	2	3	…	m/2
Min Sum	m+2*1	m+2*2	m+2*3	…	m+2*(m/2)
Min Sum	m+2	m+4	m+6	…	m+m

And $<(2,2),2,2,2,2,\cdots,2,2> \to <4,4,(2,2),2,2,\cdots,2,2> \to <4,4,4,4,(2,2),\cdots,2,2> \to \cdots \to <4,4,4,4,4,4,\cdots,(2,2)> \to <4,4,4,4,4,4,\cdots,4,4>$

Upon observing the minimum sum, we construct a table about the minimum sum after again $\frac{m}{2}$ steps of the operation mode with the minimizing strategy model as follow:

K	m/2+1	m/2+2	m/2+3	…	m/2+m/2
Min Sum	2m+2*2	2m+2*4	2m+2*6	…	2m+2*m
Min Sum	2m+4	2m+8	2m+12	…	2m+2m

We infer that the pattern for an even integer m is as follows:

the minimum sum $\begin{cases} a_{\frac{m}{2}\cdot n+1} = (\frac{m}{2}+1) \times 2^{n+1} \\ a_{\frac{m}{2}\cdot n+2} = (\frac{m}{2}+2) \times 2^{n+1} \\ a_{\frac{m}{2}\cdot n+3} = (\frac{m}{2}+3) \times 2^{n+1} \\ \vdots \\ a_{\frac{m}{2}\cdot n+\frac{m}{2}} = (\frac{m}{2}+\frac{m}{2}) \times 2^{n+1} \end{cases}, \forall \frac{m}{2} \in N, n \in N \cup \{0\}.$

Plugging in values, we see that this rule holds and we will prove by induction.

【Proof】

We observe a reoccurring trend over a cycle of $\frac{m}{2}$. After every $\frac{m}{2}$ moves in k=m, the list would be in a state where every term is the same. When n=0, consider the sequence $<(1,1),1,1,1,1,\cdots,1,1>$ that has m sheets of paper with the number 1.

First, the sequence $<(1,1),1,1,1,1,\cdots,1,1> \to <2,2,1,1,1,1,\cdots,1,1>$ is obtained after 1 step of the operation mode with the minimizing strategy model, then the sum is $m+2$.

The sequence $<\ (1,1),1,1,1,1,\cdots,1,1>\to<2,2,(1,1),1,1,\cdots,1,1>\to<2,2,2,2,(1,1),\cdots,1,1>$ is obtained after 2 steps of the operation mode with the minimizing strategy model. The sum is $\mathrm{m}+4$.

The sequence $<\ (1,1),1,1,1,1,\cdots,1,1>\to<2,2,(1,1),1,1,\cdots,1,1>\to<2,2,2,2,(1,1),\cdots,1,1>\to<(2,2),2,2,2,2,\cdots,1,1>$ is obtained after 3 steps of the operation mode with the minimizing strategy model, and the sum is $\mathrm{m}+6$.

Similarly the sequence $<\ (1,1),1,1,1,1,\cdots,1,1>\to<2,2,(1,1),1,1,\cdots,1,1>\to<2,2,2,2,(1,1),\cdots,1,1>\to\cdots\to<2,2,2,2,2,2,2,\cdots(1,1)>\to<(2,2),2,2,2,2,\cdots,2,2>$ is obtained after $\frac{m}{2}$ steps of the operation mode with the minimizing strategy model. The sum is $\mathrm{m}+\mathrm{m}$.

Assume that $n=k$ holds, i.e. the minimum sum obtained after $\frac{m}{2}\cdot k+1$ steps of the operation mode with the minimizing strategy model, the sum is $a_{\frac{m}{2}\cdot k+1}=(\frac{m}{2}+1)\times 2^{k+1}$. With the minimum sum obtained after $\frac{m}{2}\cdot k+2$ steps of the operation mode in the minimizing strategy model, the sum is $a_{\frac{m}{2}\cdot k+2}=(\frac{m}{2}+2)\times 2^{k+1}$.

The minimum sum is obtained after $\frac{\mathrm{m}}{2}\cdot\mathrm{k}+3$ steps of the operation mode with the minimizing strategy model. The sum is $\mathrm{a}_{\frac{\mathrm{m}}{2}\cdot\mathrm{k}+3}=(\frac{\mathrm{m}}{2}+3)\times 2^{\mathrm{k}+1}$.

With the minimum sum obtained after $\frac{\mathrm{m}}{2}\cdot\mathrm{k}+\frac{\mathrm{m}}{2}$ steps of the operation mode in the minimizing strategy model, the sum is $\mathrm{a}_{\frac{\mathrm{m}}{2}\cdot\mathrm{k}+\frac{\mathrm{m}}{2}}=\left(\frac{\mathrm{m}}{2}+\frac{\mathrm{m}}{2}\right)\times 2^{\mathrm{k}+1}$.

We want to show that $n=k+1$ also holds.

Assume that this rule also works for taking $\frac{\mathrm{m}}{2}\cdot\mathrm{k}+\frac{\mathrm{m}}{2}$ steps of the operation mode with the minimizing strategy model, and the minimum sum would be $8\times 2^{\mathrm{k}+1}$. Now since every term is the same we know that each term equals $\frac{\mathrm{m}\times 2^{\mathrm{k}+1}}{\mathrm{m}}=2^{\mathrm{k}+1}$. Thus the form is $<2^{\mathrm{k}+1},2^{\mathrm{k}+1},2^{\mathrm{k}+1},2^{\mathrm{k}+1},2^{\mathrm{k}+1},2^{\mathrm{k}+1},\cdots,2^{\mathrm{k}+1},2^{\mathrm{k}+1}>$.

We want to show that for taking $\frac{m}{2}\cdot(k+1)+1$ steps of the operation mode with the minimize strategy model, the minimum sum is $(\frac{m}{2}+1)\times 2^{k+2}$.

Observing that the form one step before was $<2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},2^{k+1},\cdots,2^{k+1},2^{k+1}>$, we know that the next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1},\cdots,2^{k+1},2^{k+1}>$, with the sum $2^{k+2}+2^{k+2}+2^{k+1}+2^{k+1}+2^{k+1}+2^{k+1}+\cdots+2^{k+1}+2^{k+1}=(\frac{m}{2}+1)\times 2^{k+2}$.

From taking $\frac{m}{2}\cdot(k+1)+2$ steps of the operation mode with the minimizing strategy model, the minimum sum is $(\frac{m}{2}+2)\times 2^{k+2}$.

Since the form one step before was $<2^{k+2},2^{k+2},2^{k+1},2^{k+1},2^{k+1},2^{k+1},\cdots,2^{k+1},2^{k+1}>$, we know that the next move would be to add any two of the terms together and we get the form $<2^{k+2},2^{k+2},2^{k+2},2^{k+2},2^{k+1},2^{k+1},\cdots,2^{k+1},2^{k+1}>$, with the sum $2^{k+2}+2^{k+2}+2^{k+2}+2^{k+2}+2^{k+1}+2^{k+1}+\cdots+2^{k+1}+2^{k+1}=(\frac{m}{2}+2)\times 2^{k+2}$. Similarly, we see that the same thing happens from taking $\frac{m}{2}\cdot(k+1)+3$ steps of the operation mode with the minimizing strategy model, and the minimum sum is $(\frac{m}{2}+3)\times 2^{k+2}$. Similarly from taking $\frac{m}{2}\cdot(k+1)+\frac{m}{2}$ steps of the operation mode with the minimize strategy model, the minimum sum is $(\frac{m}{2}+\frac{m}{2})\times 2^{k+2}$. Those outcomes lead us to conclude that this also holds for $n=k+1$, thus proving the general form.

$\text{By the induction hypothesis, thus fulfills for all non} - \text{negative integers n.}$

On the other hand, we use the symbol w of period $\frac{m}{2}$, $in\ whch\ w\ is\ a\ root\ of\ equation\ x^{\frac{m}{2}}=1$, to translate the form of the minimum sum

$$\begin{cases} a_{\frac{m}{2}\cdot n+1}=\left(\frac{m}{2}+1\right)\times 2^{n+1} \\ a_{\frac{m}{2}\cdot n+2}=\left(\frac{m}{2}+2\right)\times 2^{n+1} \\ a_{\frac{m}{2}\cdot n+3}=\left(\frac{m}{2}+3\right)\times 2^{n+1} \\ \quad\vdots \\ a_{\frac{m}{2}\cdot n+\frac{m}{2}}=\left(\frac{m}{2}+\frac{m}{2}\right)\times 2^{n+1} \end{cases},\forall \frac{m}{2}\in N, n\in N\cup\{0\}$$

as the form of

$$a_n=\{{}^{m}/_{2}+1+\sum_{k=1}^{m/_2-1}[k\times\frac{\sum_{i=1}^{m/_2} w^{i(n+m/_2-1-k)}}{m/_2}]\}\times 2^{\left\lceil\frac{n-1}{m/_2}\right\rceil+1},\forall\frac{m}{2}\in N, n\in N.$$

As a result of the above discussion and argument, we organized the result into the following theorem:

Theorem 2

Given m sheets of paper with the number 1, the general formula of the maximum sum for an even integer *m* after *n* steps of the operation mode with the maximizing strategy model is

$$a_n=\{{}^{m}/_{2}+1+\sum_{k=1}^{m/_2-1}[k\times\frac{\sum_{i=1}^{m/_2} w^{i(n+m/_2-1-k)}}{m/_2}]\}\times 2^{\left\lceil\frac{n-1}{m/_2}\right\rceil+1},\forall\frac{m}{2}\in N, n\in N.$$

2.4.2 when m is Odd

Obviously the case *m*=1, only one number, is not necessary to discuss.

1. For m=3:

The sequence $<(1,1),1>\rightarrow<2,(2,1)>\rightarrow<(2,3),3>\rightarrow<5,5,3>$ is obtained after 3 steps of the operation mode with the minimizing strategy model.

Upon observing the minimum sum, we construct a table about the minimum sum after 8 steps of the operation mode with the minimizing strategy model as follow:

k	1	2	3	4	5	6	7	8
Min	5	8	13	21	34	55	89	144
Min	5	8	5+8	8+13	13+21	21+34	34+55	55+89

Upon observing the minimum sum, we observed that it's states included the Fibonacci number sequence $<F_n>$.

The minimum state will always be the larger number chosen with the smallest number and we see the general trend of these states.

For k=n, the sequence $<(1,1),1>\rightarrow<2,(2,1)>\rightarrow<2,3,3>\rightarrow\cdots\rightarrow(F_{n+2},F_{n+2},F_{n+1})$ is obtained after k steps of the operation mode with the minimizing strategy model.

The sum is $F_{n+4}=F_{n+2}+F_{n+2}+F_{n+1}$

We assume by induction that k=n also holds and we want to show that when k=n+1, the state is $F_{n+3}\,F_{n+3}\,F_{n+2}$, and the sum follows the trend before.

Let k=n+1 be the next step, we already know that the state is $F_{n+3}\, F_{n+3}\, F_{n+2}$, since that is adding the smallest one of the three, F_{n+1}, to the largest one, F_{n+2}, the largest one.

We conclude that the state is $F_{n+3}, F_{n+3}, F_{n+2}$ and the sum is $F_{n+3} + F_{n+3} + F_{n+2} = F_{n+5}$.That is, the minimum of the sum of the numbers on all the sheets of paper after k steps of the operation mode with the minimizing strategy model is $F_{k+3} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+3}\right], \forall k \in N.$

2. For m=5:

The sequence $<(1,1),1,1,1> \to <2,2,(1,1),1> \to <2,2,2,(2,1)> \to <(2,2),2,3,3> \to <4,4,(2,3),3> \to <4,4,5,5,3>$ is obtained after 5 steps of the operation mode with the minimize strategy model.

Upon observing the minimum sum, we construct a table about the minimum sum after 5 steps of the operation mode with the minimize strategy model as follow:

k	1	2	3	4	5	6	7	8
Min	7	9	12	16	21	28	37	49
Min	7	9	12	7+9	9+12	12+16	16+21	21+28

We observed that its recurrence relation of the sequence $< m_n >$ about the minimum sum is as follow:

$$\begin{cases} m_1 = 7, m_2 = 9, m_3 = 12 \\ m_{k+2} = m_k + m_{k-1}, k \geq 2 \end{cases}$$

The minimum state will always be the larger number chosen with the smallest number and we see the general trend of these states.

For k=n, the sequence $<(1,1),1,1,1> \to <2,2,(1,1),1> \to <2,2,2,(2,1)> \to \cdots \to < m_{n-7}, m_{n-6}, m_{n-6}, m_{n-5}, m_{n-5} >$ is obtained after k steps of the operation mode with the minimize strategy model.

The sum is $m_n = m_{n-7} + 2m_{n-6} + 2m_{n-5}$

We assume by induction that k=n also holds and we want to show that when k=n+1, the state is $m_{n-6}, m_{n-5}, m_{n-5}, m_{n-4}, m_{n-4}$ and we want to show that this sum follows the trend before.

Let *k*=*n*+1 be the next step, we already know that the state is $m_{n-6}, m_{n-5}, m_{n-5}, m_{n-4}, m_{n-4}$, since that is adding the smallest one of the three, m_{n-6}, to the largest one, m_{n-5}, the largest one.

We conclude that the state is $m_{n-6}, m_{n-5}, m_{n-5}, m_{n-4}, m_{n-4}$ and the sum is $m_{n-6} + m_{n-5} + m_{n-5} + m_{n-4} + m_{n-4} = m_{n+1}$.That is, the minimum of the sum of the numbers on all the sheets of paper after *k* steps of the operation mode with the minimize strategy model is $m_k = \left[x\left(\sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}} + \sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}}\right)^k + y\left(\sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}}w + \sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}}w^2\right)^k + z\left(\sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}}w^2 + \sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}}w\right)^k\right], \forall k \in N, w = \frac{-1+\sqrt{3}i}{2},$

where the values of x, y, z, are listed on the following appendix (see Appendix 4 for the values of x, y, z).

3. Results

3.1 Summary

3.1.1 Maximizing Strategy Model

The intuitive strategy to maximize the sum of all the numbers after *n* steps of the operation mode is that in every step we chose the two greatest numbers.

3.1.2 Minimizing Strategy Model

The intuitive strategy to minimize the sum of all the numbers after some steps of the operation mode is that in every step we chose the two least numbers.

3.1.3 Theorem 1

Given *m* sheets of paper with the number 1, the general formula of the maximum sum for a positive integer *m* after *n* steps of the operation mode with the maximizing strategy model is

$$\mathrm{m} + 2^{k+1} - 2,\ \forall m \in N, n \in \mathrm{N}.$$

3.1.4 Theorem 2

Given m sheets of paper with the number 1, the general formula of the maximum sum for an even integer *m* after *n* steps of the operation mode with the maximizing strategy model is

$$a_n = \{{}^{m}/_{2} + 1 + \sum_{k=1}^{m/_2-1}[k \times \frac{\sum_{i=1}^{m/_2} w^{i(n+m/_2-1-k)}}{m/_2}]\} \times 2^{\left[\frac{n-1}{m/_2}\right]+1}, \forall \frac{m}{2} \in N, n \in N.$$

1. If m is an even integer, then the general formula of the minimum sum is

$$a_n = \{{}^{m}/_{2} + 1 + \sum_{k=1}^{m/_2-1}[k \times \frac{\sum_{i=1}^{m/_2} w^{i(n+m/_2-1-k)}}{m/_2}]\} \times 2^{\left[\frac{n-1}{m/_2}\right]+1}, \forall \frac{m}{2} \in N, n \in N.$$

2. If m is an odd integer, then the general formula of the minimum sum is

(1) $m = 1$, the formula is trivial.

(2) $m = 3$, the formula is $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+3}\right], \forall n \in N.$

(3) $m = 5$, the formula is $\left[x\left(\sqrt[3]{\frac{1}{2}-\frac{\sqrt{69}}{18}} + \sqrt[3]{\frac{1}{2}+\frac{\sqrt{69}}{18}}\right)^k + y\left(\sqrt[3]{\frac{1}{2}-\frac{\sqrt{69}}{18}}w + \sqrt[3]{\frac{1}{2}+\frac{\sqrt{69}}{18}}w^2\right)^k + z\left(\sqrt[3]{\frac{1}{2}-\frac{\sqrt{69}}{18}}w^2 + \sqrt[3]{\frac{1}{2}+\frac{\sqrt{69}}{18}}w\right)^k\right], \forall k \in N, w = \frac{-1+\sqrt{3}i}{2}$, where the values of x, y, z, list on the following appendix.

4. Discussions

4.1 Conclusions

This study uses the even and odd relationships to prove the general formula about the maximum sum and the minimum sum (Djordjevic & Srivastava, 2005). From the pattern of the general formula, the general formula for m sheets of paper with the number 1 was predicted and proven by induction. Similarly, the general formulas for both the maximum and the minimum sums were proven. The mathematical model, in parentheses form, was found through experimenting with the number of papers in total and adding the specific numbers on each two papers (Wang et al., 2002). Using the same method, the relationship demonstrated the outcome by adding the specific numbers on each two papers among the total number of papers, and created a general form that confirms the maximum and minimum general forms of the project which in turn was generated by a computer program (see Appendix 3, Liang, 2017). The use of the two mathematical models enabled the derivation of the general form of the maximum and the minimum sum. These forms were eventually proven through induction, after finding the even and odd relationships by increasing the total amount of papers and also adding the specific numbers on each two papers. As a result, for future work, we would like to define a general formula by increasing the total amount of papers added and also assigning specific numbers for each paper.

4.2 Applications

The results of this project can be applied to the created computer software, which can show the maximum and the minimum sum for that particular scenario. Using this program allows for the inference of the number based on the number of terms and the numbers themselves. A further application is that this game can be set so that multiple people can play in which a number is first chosen for the number of papers in the game. Next, the players will choose any random number in the range of the maximum and minimum. Then the players will take turns adding the numbers together, following the rules until finally the numbers on the paper sum up to the chosen number. In this type of game, there are only two scenarios, 1 winner or everyone loses because it is not possible to add to that number. Another application for this project is using the possibility that it is not possible to add to that number to create a passcode. Since it is impossible to follow the rules of the game and add numbers to arrive at the passcode, we have created an impenetrable code, which can not be broken since there is no way one can get to the number by following the rules.

Last application is using the sum of the array to see how many integers add up to it in a multivariable linear equation with constant coefficients.

5. References

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Appendix:

1. Maximizing Strategy Model

The intuitive strategy to maximize the sum of all the numbers after *n* steps of the operation mode is that in every step we chose the two greatest numbers.

【Proof】

We apply the two methods of induction on *n* and the rearrangement inequality to prove this strategy indeed provides the greatest possible sum of numbers.

First, suppose an operation sequence $S_n =< (i_1, j_1), (i_2, j_2), (i_3, j_3), \cdots, (i_n, j_n) >$ be the maximal sum of all the numbers after *n* steps of the operation model, where for $r = 1,2, \cdots, n$ the numbers $i_r\ and\ j_r$ are the resulting numbers that are chosen in the r^{th} step of the operation mode.

Let *P*(*n*) be the statement that "the strategy to choose the two greatest numbers in every step forever produces the maximal sum S_n."

For $n = 1$, the statement *P*(1) is obvious.

Assume that $n = k, k \geq 2$, the statement is true.

i.e. Choose a sequence $S_k =< (i_1, j_1), (i_2, j_2), (i_3, j_3), \cdots, (i_k, j_k) >$ of all the numbers after *k* steps of the operation mode that produces the maximal sum, starting from $x_1, x_2, x_3, \cdots, x_m$, and let $(c_1, c_2, c_3, \cdots, c_m)$, assumed in a such order $c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_m$, be the characteristic vector of the sequence $< x_1, x_2, x_3, \cdots, x_m >$.

If the numbers are permuted by a permutation π of the indices $(1, 2, 3, \cdots, m)$, we can obtain the sum $\sum_{t=1}^{m} c_t x_{\pi(t)}$ from the sum $\sum_{t=1}^{m} c_t x_t$. By the rearrangement inequality, the greatest possible sum can be achieved when the number $x_1, x_2, x_3, \cdots, x_m$are in increasing order. So we can assume also that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_m$.

Let s be the smallest index with $c_m = c_{m-1} = c_{m-2} = \cdots = c_s$, and let the r^{th} step be the first step for which $c_{i_r} = c_m$ or $c_{j_r} = c_m$. The role of i_r and j_r is symmetrical, so we can assume $c_{i_r} = c_m$ and thus $i_r \geq s$.

We show that $c_{j_r} = c_m$ and $j_r \geq s$ hold, too.

Before the r^{th} step, on the ${i_r}^{th}$ sheet we had the number x_{i_r}. On the ${j_r}^{th}$ sheet there was a linear combination that contains the number we had the number x_{j_r} with a positive integer coefficient, and possibly some other terms. In the r^{th} sheet, the number x_{i_r} joins that linear combination. From this point, each sheet contains a linear of $x_1, x_2, x_3, \cdots, x_m$, with the coefficient of x_{j_r}being not smaller than the coefficient of x_{i_r}. This is preserved to the end of the procedure, so we have $c_{j_r} \geq c_{i_r}$. But $c_{i_r} = c_m$ is maximal among the coefficients, so we have $c_{j_r} = c_{i_r} = c_m$ and thus $j_r \geq s$.

Either from $c_{j_r} = c_{i_r} = c_m$ or from the arguments in the previous paragraph we can see that none of the ${i_r}^{th}$ and the ${j_r}^{th}$ sheets were used before step *r*. Therefore, the final linear combination of the numbers does not change if the step (i_r, j_r) is performed first: the sequence of steps $S_1 =< (i_r, j_r), (i_1, j_1), \cdots, (i_{r-1}, j_{r-1}), (i_{r+1}, j_{r+1}), \cdots, (i_m, j_m) >$ also produces the same maximal sum at the end. Therefore, we can replace S_n by S_1 and we may assume that $r = 1$ and $c_{j_1} = c_{i_1} = c_m$.

As $i_1 \neq j_1$, we can see that $s \leq m - 1$ and $c_m = c_{m-1} = c_{i_1} = c_{j_1}$. Let π be such a permutation of the indices $(1, 2, 3, \cdots, m)$ that exchanges $m, m - 1$ with i_r, j_r and does not change the remaining indices.

Let $S_2 =< (\pi(i_1), \pi(j_1)), (\pi(i_2), \pi(j_2)), \cdots, (\pi(i_m), \pi(j_m)) >$.

Since $c_{\pi(i)} = c_i$ for all incidences *i*, this sequence of steps produce the same maximal sum. Moreover, in the first step, we chose $x_{\pi(i_1)} = x_m$ and $x_{\pi(i_2)} = x_{m-1}$, the two largest numbers.

Hence, it is possible to achieve the optimal sum if we follow the maximal strategy in the first step. By the induction hypothesis, following the maximizing strategy in the remaining steps we achieve the optimal sum.

2. Minimizing Strategy Model

The intuitive strategy to minimize the sum of all the numbers after some steps of the operation mode is that in every step we chose the two least numbers.

【Proof】

We apply induction on *n* to prove this strategy indeed provides the least possible sum of numbers.

First, suppose an operation sequence $S_n =< (i_1, j_1), (i_2, j_2), (i_3, j_3), \cdots, (i_n, j_n) >$ be the minimal sum of all the numbers after *n* steps of the operation model, where for $r = 1,2, \cdots, n$ the numbers $i_r\ and\ j_r$ are the resulting numbers that are chosen in the r^{th} step of the operation mode.

Let *P* (*n*) be the statement that "the strategy to choose the two smallest numbers in every step forever produces the minimal sum S_n."

For $n = 1$, the statement *P*(1) is obvious.

Assume that $n = k, k \geq 2$, the statement is true.

i.e. Choose a sequence $S_k =< (i_1, j_1), (i_2, j_2), (i_3, j_3), \cdots, (i_k, j_k) >$ of all the numbers after *k* steps of the operation mode that produces the minimal sum, starting from $x_1, x_2, x_3, \cdots, x_m$, and let $(c_1, c_2, c_3, \cdots, c_m)$, assumed in a such order $c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_m$ be the characteristic vector of the sequence $< x_1, x_2, x_3, \cdots, x_m >$.

If the numbers are permuted by a permutation π of the indices $(1, 2, 3, \cdots, m)$, we can obtain the sum $\sum_{t=1}^{m} c_t x_{\pi(t)}$ from the sum $\sum_{t=1}^{m} c_t x_t$. By the rearrangement inequality, the greatest possible sum can be achieved when the number $x_1, x_2, x_3, \cdots, x_m$are in increasing order. So we can assume also that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_m$.

Let s be the largest index with $c_1 = c_2 = c_3 = \cdots = c_s$, and let the r^{th} step be the first step for which $c_{i_r} = c_1$ or $c_{j_r} = c_1$. The role of i_r and j_r is symmetrical, so we can assume $c_{i_r} = c_m$ and thus $i_r \leq s$.

We show that $c_{j_r} = c_1$ and $j_r \leq s$ hold, too.

Before the r^{th} step, on the ${i_r}^{th}$ sheet we had the number x_{i_r}. On the ${j_r}^{th}$ sheet there was a linear combination that contains the number we had the number x_{j_r} with a positive integer coefficient, and possibly some other terms. In the r^{th} sheet, the number x_{i_r} joins that linear combination. From this point, each sheet contains a linear of $x_1, x_2, x_3, \cdots, x_m$, with the coefficient of x_{j_r}being not smaller than the coefficient of x_{i_r}. This is preserved to the end of the procedure, so we have $c_{j_r} \geq c_{i_r}$. But $c_{i_r} = c_1$ is maximal among the coefficients, so we have $c_{j_r} = c_{i_r} = c_1$ and thus $j_r \leq s$.

Either from $c_{j_r} = c_{i_r} = c_1$ or from the arguments in the previous paragraph we can see that none of the ${i_r}^{th}$ and the ${j_r}^{th}$ sheets were used before step *r*. Therefore, the final linear combination of the numbers does not change if the step (i_r, j_r) is performed first: the sequence $S_1 =< (i_r, j_r), (i_1, j_1), \cdots, (i_{r-1}, j_{r-1}), (i_{r+1}, j_{r+1}), \cdots, (i_m, j_m) >$ also produces the same minimal sum at the end. Therefore, we can replace S_n by S_1 and we may assume that $r = 1$ and $c_{j_1} = c_{i_1} = c_1$.

As $i_1 \neq j_1$, we can see that $s \geq 2$ and $c_1 = c_2 = c_{i_1} = c_{j_1}$. Let π be such a permutation of the indices $(1, 2, 3, \cdots, m)$ that exchanges $1, 2$ with i_r, j_r and does not change the remaining indices.

Let $S_2 =< (\pi(i_1), \pi(j_1)), (\pi(i_2), \pi(j_2)), \cdots, (\pi(i_m), \pi(j_m)) >$.

Since $c_{\pi(i)} = c_i$ for all indices *i*, this sequence of steps produces the same maximal sum. Moreover, in the first step, we chose $x_{\pi(i_1)} = x_1$ and$x_{\pi(i_2)} = x_2$, the two smallest numbers.

Hence, it is possible to achieve the optimal sum if we follow the minimal strategy in the first step. By the induction hypothesis, following the minimal strategy in the remaining steps, we achieve the optimal sum. Q.E.D.

3. Computer Program

```
import java.util.Scanner;
public class addtwo{
public static void main(String[] args)throws InterruptedException {
int i,k ,m = 0,n=0;
System.out.println ("Enter the number of inputs");
Scanner sc = new Scanner(System.in);
```

```
k=sc.nextInt();
int a[] = new int [k];
System.out.println("Enter the inputs");
for(i=0;i<k;i++){
a[i] = sc.nextInt();
System.out.println("Inputs are " +a[i]);
}
while( k>0)
{if(k>0){
for(i=0;i<k;){
System.out.println("choose the first term ");
Thread.sleep(500);
m= sc.nextInt();
System.out.println("choose the first term " +a[m]);
Thread.sleep(500);
System.out.println("choose the second term " );
Thread.sleep(500);
n=sc.nextInt();
System.out.println("choose the second term " +a[n]);
Thread.sleep(500);
i=m;
a[i] =a[m]+a[n];
System.out.println("a[m]= " +a[m]);
Thread.sleep(500);
a[n]=a[m];
System.out.println("a[n]= " +a[n]);
Thread.sleep(500);
int sum = 0;
Thread.sleep(500);
for(i=0;i<k;i++){
sum = sum +a[i];
  }
System.out.println("Sums : " +sum);
}
}
}
}
}
```

4. The Values of *x*, *y*, *z* in the Theorem 2 as Follow:

(1) The value of x is

$$
\begin{aligned}
x = \frac{1}{414}\Big(& 690 + 111 \times 2^{2/3}\sqrt[3]{3\left(9-\sqrt{69}\right)} + 29\sqrt[3]{2}\left(3\left(9-\sqrt{69}\right)\right)^{2/3} + \\
& 111 \times 2^{2/3}\sqrt[3]{3\left(9+\sqrt{69}\right)} + 29\sqrt[3]{2}\left(3\left(9+\sqrt{69}\right)\right)^{2/3}\Big),
\end{aligned}
$$

(2) The value of y is

$$
\begin{aligned}
y = \frac{1}{23}\Bigg(& 48 - \frac{29}{9}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)^{2/3} - \frac{37\sqrt[3]{\frac{1}{2}\left(9-\sqrt{69}\right)}}{2\times 3^{2/3}} + \\
& \frac{29\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)\left(9-\sqrt{69}\right)}}{6\times 3^{2/3}} - \frac{37\sqrt[3]{\frac{1}{2}\left(9+\sqrt{69}\right)}}{2\times 3^{2/3}} - \\
& \frac{29\left(9+\sqrt{69}\right)^{2/3}}{6\times 2^{2/3}\sqrt[3]{3}} + \frac{29\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)\left(9+\sqrt{69}\right)}}{6\times 3^{2/3}} - \\
& \frac{29}{3}\left(\frac{2}{3}\right)^{2/3}\sqrt[3]{\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)\left(9+\sqrt{69}\right)} + \\
& \frac{29\sqrt[3]{\frac{1}{3}\left(9-\sqrt{69}\right)\left(9+\sqrt{69}\right)}}{6\times 2^{2/3}} - \frac{37}{12}\sqrt{\Big(48 - 6\sqrt[3]{2}\left(3\left(9-\sqrt{69}\right)\right)^{2/3} - 6\sqrt[3]{2}\left(3\left(9+\sqrt{69}\right)\right)^{2/3} + 4\times 3^{2/3}\sqrt[3]{2\left(9-\sqrt{69}\right)\left(9+\sqrt{69}\right)}\Big)} + \\
& \frac{29}{36}\sqrt[3]{\frac{27}{2}-\frac{3\sqrt{69}}{2}}\sqrt{\Big(48 - 6\sqrt[3]{2}\left(3\left(9-\sqrt{69}\right)\right)^{2/3} - 6\sqrt[3]{2}\left(3\left(9+\sqrt{69}\right)\right)^{2/3} + 4\times 3^{2/3}\sqrt[3]{2\left(9-\sqrt{69}\right)\left(9+\sqrt{69}\right)}\Big)} + \\
& \frac{1}{12\times 3^{2/3}}29\sqrt[3]{\frac{1}{2}\left(9+\sqrt{69}\right)}\sqrt{\Big(48 - 6\sqrt[3]{2}\left(3\left(9-\sqrt{69}\right)\right)^{2/3} - 6\sqrt[3]{2}\left(3\left(9+\sqrt{69}\right)\right)^{2/3} + 4\times 3^{2/3}\sqrt[3]{2\left(9-\sqrt{69}\right)\left(9+\sqrt{69}\right)}\Big)}\Bigg),
\end{aligned}
$$

(3) The value of z is

$$
\begin{aligned}
z = \frac{1}{23}\Bigg(& 48 - \frac{37}{3}\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} + \frac{37\sqrt[3]{\frac{1}{2}(9-\sqrt{69})}}{2\times 3^{2/3}} - \\
& \frac{29\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)(9-\sqrt{69})}}{6\times 3^{2/3}} - \frac{37\sqrt[3]{\frac{1}{2}(9+\sqrt{69})}}{2\times 3^{2/3}} - \\
& \frac{29(9+\sqrt{69})^{2/3}}{6\times 2^{2/3}\sqrt[3]{3}} - \frac{29\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)(9+\sqrt{69})}}{6\times 3^{2/3}} - \\
& \frac{29\sqrt[3]{\frac{1}{3}(9-\sqrt{69})(9+\sqrt{69})}}{6\times 2^{2/3}} + \frac{37}{12}\sqrt{\vphantom{X}}\Big(48 - 6\sqrt[3]{2}\,(3(9-\sqrt{69}))^{2/3} - \\
& \qquad 6\sqrt[3]{2}\,(3(9+\sqrt{69}))^{2/3} + 4\times 3^{2/3}\sqrt[3]{2(9-\sqrt{69})(9+\sqrt{69})}\Big) - \\
& \frac{29}{36}\sqrt[3]{\frac{27}{2}-\frac{3\sqrt{69}}{2}}\sqrt{\vphantom{X}}\Big(48 - 6\sqrt[3]{2}\,(3(9-\sqrt{69}))^{2/3} - \\
& \qquad 6\sqrt[3]{2}\,(3(9+\sqrt{69}))^{2/3} + 4\times 3^{2/3}\sqrt[3]{2(9-\sqrt{69})(9+\sqrt{69})}\Big) - \\
& \frac{1}{12\times 3^{2/3}}29\sqrt[3]{\frac{1}{2}(9+\sqrt{69})}\sqrt{\vphantom{X}}\Big(48 - 6\sqrt[3]{2}\,(3(9-\sqrt{69}))^{2/3} - \\
& \qquad 6\sqrt[3]{2}\,(3(9+\sqrt{69}))^{2/3} + 4\times 3^{2/3}\sqrt[3]{2(9-\sqrt{69})(9+\sqrt{69})}\Big)\Bigg)
\end{aligned}
$$