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Generalized Multi-Term Fractional Boundary Value Problems Using Integral Transform Technique

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Abstract

The integral transform technique is used to discuss the existence as well as numerical solutions for the following fractional differential equation,

$$\mathfrak{D}^q x(t) = f(t, x(t)) + p(t, x(t)), n-1 < q < n, n = [q] + 1,$$

$$x(0) = x_0 \in \mathbb{R}.$$

where $t \in \mathfrak{J} = [0, T]$ and $\mathfrak{D}^q$ denotes the fractional *Caputo* derivative of order q. $f, p : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ is continuous function.The numerical solution via sequence of successive approximations is obtained using iteration method.

Keywords: fractional derivative, fixed point theorem, Mittag -Leffler function, Laplace transform

MSC(2010) Primary: 26A33, 33E12, Secondary: 34A38, 47H10, 34B18.

1. Introduction

Researchers and mathematicians are attracted towards the fractional differential equations because many applications in a variety of Science, Engineering, Economics, Applied Mathematics and Bio-Engineering disciplines attracted the attention of many researchers and mathematicians. We will investigate the conditions applied to the generalized hybrid fixed point theorem for simplicity.As everyone is interested in getting better interpretations of their results by incorporating more and more information by utilizing different mathematical tools, one of them is the use of fractional order derivatives. On comparing fractional order differential equations (FDEs) with ordinary differential equations, then we get fractional order differential equations are more informative and have a better approach. Hybrid coupled boundary value fractional differential equations have also studied by several mathematicians. Such type of equations involving the fractional derivative of an unknown hybrid with the nonlinearity depending on it.

In this direction every interested researcher can prepare a good research paper and monographs such as (Bashir, A. & Sotiris, K. N., 2014; Bashir, A. & Sotiris, K. N., 2014; Bashir, A. & Ahmed, A., 2010; Sangita C. & Varsha, D.-G., 2014; Dhage, B. C., 2005; Bapurao, C. D. & Shyam B. D., 2015; Ahmed, E.- S. & Hind, H., 2013; Kazen, G. & Yousef, G., 2013; Yousef G., 2014; Yousef, G., 2014; Lebedev, L. P., et al., 2002; Kilbas, A. A., et al., 2006; Petras, I., 2011; Podlubny, I., 1999; Lakshmikantham, V., et al., 2009; Miller, K. S. & Ross, B., 1993) and references given. The work which is presented in this paper is inspired from masterwork of *Igor Podlubny* in (Podlubny, I., 1999) and the references (Bashir, A. & Sotiris, K. N., 2014), (Sangita C. & Varsha, D.-G., 2014), (Bapurao, C. D. & Shyam B. D., 2015) and (Bhausaheb, R. S. & Govind, P. K., 2016). The authors in (Bapurao, C. D. & Shyam, B. D. 2015) based on the iteration technique studied the approximating solution of

$$\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] + \lambda\left[\frac{x(t)}{f(t, x(t))}\right] = g(t, x(t)), t \in \mathfrak{J} = [0, T],$$

$$x(0) = x(T).$$

For the following multi-order fractional boundary value problem

$$L(D)u(t) = f(t, u(t)), t \in [0, T], T > 0,$$

$$L(D) = \lambda_n \ {}^cD^{\alpha_n} + \lambda_{n-1} \ {}^cD^{\alpha_{n-1}} + ... + \lambda_1 \ {}^cD^{\alpha_1} + \lambda_0 \ {}^cD^{\alpha_0},$$

$$\lambda_i \in \mathbb{R}, \lambda_n \neq 0, 0 \leq \alpha_0 < \alpha_1 < .... < \alpha_{n-1} < \alpha_n < 1,$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α.
As well as the authors in (Bashir, A. & Sotiris, K. N., 2014), got the existence of at least one-solution for the following coupled system equipped to the *Hadamard type* fractional derivatives.

$$D^{\alpha}u(t) = f(t, u(t), v(t)), 1 < t < e, 1 < \alpha \leq 2.$$

$$D^{\beta}v(t) = g(t, u(t), v(t)), 1 < t < e, 1 < \beta \leq 2.$$

$$u(1) = 0, u(e) = I^{\gamma}u(\sigma_1) = \tfrac{1}{\Gamma(\gamma)} \int_0^{\sigma_1} (log\tfrac{\sigma_1}{s})^{\gamma-1} \tfrac{u(s)}{s} ds.$$

$$v(1) = 0, v(e) = I^{\gamma}v(\sigma_2) = \tfrac{1}{\Gamma(\gamma)} \int_0^{\sigma_2} (log\tfrac{\sigma_2}{s})^{\gamma-1} \tfrac{v(s)}{s} ds.$$

where $\gamma > 0, 1 < \sigma_1 < e, 1 < \sigma_2 < e,\ D^{(.)}$ is *Hadamard* fractional derivative of fractional order and I^{γ} is the *Hadamard* fractional integral of order γ and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.
Inspired by papers given above,and considering the fractional coupled hybrid system:

$$\begin{aligned} \lambda_1 \mathfrak{D}^{\alpha}_{o^+}\left(\frac{u(t)}{f(t, u(t))}\right) - \lambda_2\left(\frac{u(t)}{f(t, u(t))}\right) &= g_1(t, u(t), v(t)) + p_1(t, u(t), v(t)), \\ & n < \alpha < n+1. \\ \mu_1 \mathfrak{D}^{\beta}_{o^+}\left(\frac{v(t)}{f(t, v(t))}\right) - \mu_2\left(\frac{v(t)}{f(t, v(t))}\right) &= g_2(t, u(t), v(t)) + p_2(t, u(t), v(t)), \\ & n < \beta < n+1. \end{aligned} \tag{1}$$

$$u(0) = u(T), v(0) = v(T)$$

$$0 < \lambda_2 < \lambda_1 < \infty, 0 < \mu_2 < \mu_1 < \infty, n \in \mathbb{N}, T \in \mathbb{R}^+.$$

where $t \in \mathfrak{J} = [0, T]$ and $\mathfrak{D}^{\alpha}_{o^+}$ represents the *Caputo* derivative of order $\alpha > 0$.
To prove the existence results, we are going to use the following famous fixed point theorem
To begin the proof, we consider the following conditions be satisfied throughout this paper:
$(C_1) f, p \in C^n(\mathfrak{J} \times \mathbb{R}, \mathbb{R}^+)$ with $supf(t, u(t)) = \rho_1, supp(t, u(t)) = \rho_2,$

$$(t, u) \in \mathfrak{J} \times \mathbb{R}, \rho_1, \rho_2 \in \mathbb{R}^+.$$

$(C_2) g_1, g_2 \in C(\mathfrak{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ and $sup\{g_1(t, u, v) + p_1(t, u, v)\} = \theta_1,$

$$sup\{g_2(t, u, v) + p_2(t, u, v)\} = \theta_2,$$

for all $(t, u, v) \in \mathfrak{J} \times \mathbb{R} \times \mathbb{R}, \theta_1, \theta_2 \in \mathbb{R}^+$.
(C_3) [Positivity of L_1, L_2 defined by (14), (31)]

$$f(0, u(0)) - f(T, u(T))E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1}T^{\alpha}) > 0$$

and

$$f(0, v(0)) - f(T, v(T))E_{\beta,1}(\tfrac{\mu_2}{\mu_1}T^{\beta}) > 0$$

where $E_{.,.}(t)$ denotes the two parameter Mittag-Leffler function is defined by the remark (2.3).

$(C_4)\left[\frac{u(t)}{f(t,u(t))}\right]^{(k)}_{t=0} = M_k, \left[\frac{v(t)}{f(t,v(t))}\right]^{(k)}_{t=0} = N_l, k, l = 1, 2, 3, ..., n,$

$M_k, N_l \in \mathbb{R}^+$

2. Technical Background

In this section, we use some basic definitions and notations which are given in (Kilbas, A. A., et al., 2006), (Podlubny, I., 1999) with details and present technical preperations needed for further discussion.

Definition 2.1 The Riemann -Liouville fractional integral of order $\alpha > 0$ of a function $u(t) : (0, \infty) \to \mathbb{R}$ is defined as below

$$\mathfrak{I}^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \tag{2}$$

provided that the right hand side is pointwise defined on $(0, \infty)$

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ for a function $u(t) : (0, \infty) \to \mathbb{R}$ is given by

$$\mathfrak{D}^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^n(\tau) d\tau, \tag{3}$$

provided that the right hand side is pointwise defined on $(0, \infty)$ and $n = [\alpha] + 1$.

Definition 2.3 Laplace transform of integrable function $u(t)$ is defined by

$$\mathfrak{L}[u(t); s] = \int_0^{\infty} e^{-st} u(t) dt, s \in \mathbb{R}. \tag{4}$$

Remark 2. 1: The Laplace transform of convolution of two integrable function f, g is given by

$$\mathfrak{L}[f(t); s]\mathfrak{L}[g(t); s] = \mathfrak{L}[(f * g)(t); s] = \mathfrak{L}[\int_0^t f(t-\tau)g(\tau)d\tau; s] \tag{5}$$

Remark 2. 2: The Laplace transform of fractional Caputo derivative is as below

$$\mathfrak{L}[\mathfrak{D}^{\alpha}_{o^+}u(t); s] = s^{\alpha}\mathfrak{L}[u(t); s] - \sum_{k=0}^{n} s^{\alpha-k-1}u^{(k)}(0), n < \alpha < n+1. \tag{6}$$

Definition 2.4 The one - parameter generalization of exponential function e^z is called to be Mittag - Leffler function and defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, z \in \mathbb{C}, \alpha > 0. \tag{7}$$

Remark 2. 3 The two-parameter generalization of Mittag Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C}, \alpha > 0\beta > 0. \tag{8}$$

Lemma 2.1 The generalized Mittag - Leffler function $E_{\alpha,\beta}(z)$ defined by equation (8) has the following properties:

$(I)\frac{d^k}{dz^k}(z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})) = z^{\beta-k-1}E_{\alpha,\beta-k}(\lambda z^{\alpha}), \lambda, z \in \mathbb{C}, k = 1, 2, 3, ...$

$(II)\mathfrak{L}[z^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm\lambda z^{\alpha}); s] = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{k+1}}, Re(s) >\mid \lambda \mid^{\frac{1}{\alpha}}$

Lemma 2.2 Assume that $h_1(t), h_2(t) \in C(\mathfrak{J}, \mathbb{R})$ and condition (C_1) be satisfied. Then $u(t)$ is the solution of fractional boundary value problem

$$\begin{aligned} \lambda_1\mathfrak{D}^{\alpha}_{o^+}\left(\frac{u(t)}{f(t,u(t))}\right) - \lambda_2\left(\frac{u(t)}{f(t,u(t))}\right) &= h_1(t) + h_2(t), \\ n < \alpha < n+1, t &\in \mathfrak{J} \\ u(0) = u(T), n \in \mathbb{N}, 0 < \lambda_2 &< \lambda_1 < \infty \end{aligned} \tag{9}$$

if and only if $u(t)$ be a solution of the Volterra type integral equation

$$u(t) = H_{\lambda1,\lambda2}(t, u(t)) + \int_0^T G_{\lambda1,\lambda2}(t, s)\{h_1(s) + h_2(s)\}\, ds \tag{10}$$

where

$$G_{\lambda_1,\lambda_2}(t, s) = \frac{f(t, u(t))}{\lambda_1}\begin{cases} G_{1(\lambda_1,\lambda_2)}(t, s); 0 \le s \le t \le T, \\ G_{2(\lambda_1,\lambda_2)}(t, s); 0 \le t \le s \le T, \end{cases} \tag{11}$$

such that

$$G_{1(\lambda_1,\lambda_2)}(t, s) = (t - s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t - s)^{\alpha}\right) + L_1E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}t^{\alpha})(T - s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(T - s)^{\alpha}\right) \tag{12}$$

and

$$G_{2(\lambda1,\lambda2)}(t, s) = L_1E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}t^{\alpha})(T - s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(T - s)^{\alpha}\right) \tag{13}$$

such that

$$L_1 = \frac{f(T, u(T))}{f(0, u(0)) - f(T, u(T))E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^{\alpha})} \tag{14}$$

also

$$\begin{aligned} H_{\lambda_1,\lambda_2}(t, u(t)) = f(t, u(t))\{\sum_{k=1}^{n}\left[\frac{u(t)}{f(t, u(t))}\right]_{t=0}^{(k)}\{t^kE_{\alpha,k+1}(\frac{\lambda_2}{\lambda_1}t^{\alpha}) \\ +L_1T^kE_{\alpha,1}(\frac{\lambda_2}{\lambda_1}t^{\alpha})E_{\alpha,k+1}(\frac{\lambda_2}{\lambda_1}T^{\alpha})\}\}. \end{aligned} \tag{15}$$

Proof: Considering lemma (2.2) we have

$$\lambda_1\mathfrak{D}^{\alpha}_{0^+}\left(\frac{u(t)}{f(t,u(t))}\right) - \lambda_2\left(\frac{u(t)}{f(t,u(t))}\right) = h_1(t) + h_2(t)$$

by using equation (6),Laplace transform of this multi-term fractional differential equation becomes

$$\lambda_1 s^{\alpha}\mathfrak{L}\left[\frac{u(t)}{f(t,u(t))}; s\right] - \lambda_1\sum_{k=0}^{n} s^{\alpha-k-1}\left[\frac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} - \lambda_2\mathfrak{L}\left[\frac{u(t)}{f(t,u(t))}; s\right]$$

$$= \mathfrak{L}[h_1(t); s] + \mathfrak{L}[h_2(t); s].$$

so we have

$$\{\lambda_1 s^{\alpha} - \lambda_2\}\mathfrak{L}\left[\frac{u(t)}{f(t,u(t))}; s\right] - \lambda_1\sum_{k=0}^{n} s^{\alpha-k-1}\left[\frac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} = \mathfrak{L}[h_1(t); s] + \mathfrak{L}[h_2(t); s]$$

Similarlly,

$$\begin{aligned} \mathfrak{L}\left[\frac{u(t)}{f(t, u(t))}; s\right] = \frac{\mathfrak{L}[h_1(t); s]}{\lambda_1 s^{\alpha} - \lambda_2} + \frac{\mathfrak{L}[h_2(t); s]}{\lambda_1 s^{\alpha} - \lambda_2} \\ +\lambda_1\sum_{k=1}^{n}\left[\frac{u(t)}{f(t, u(t))}\right]_{t=0}^{(k)}\frac{s^{\alpha-k-1}}{\lambda_1 s^{\alpha} - \lambda_2} \\ +\lambda_1\frac{u(0)}{f(0, u(0))}\frac{s^{\alpha-1}}{\lambda_1 s^{\alpha} - \lambda_2} \end{aligned} \tag{16}$$

Using Lemma (2.3) we conclude that

$$\begin{aligned} u(t) = f(t, u(t))\{\frac{1}{\lambda_1}\mathfrak{L}^{-1}\left[\mathfrak{L}[h_1(t); s]\frac{1}{s^{\alpha} - \frac{\lambda_2}{\lambda_1}}; t\right] \\ +\frac{1}{\lambda_1}\mathfrak{L}^{-1}\left[\mathfrak{L}[h_2(t); s]\frac{1}{s^{\alpha} - \frac{\lambda_2}{\lambda_1}}; t\right] \end{aligned}$$

$$+\sum_{k=1}^{n}\left[\frac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} t^k E_{\alpha,k+1}(\tfrac{\lambda_2}{\lambda_1}t^\alpha)$$

$$+\frac{u(0)}{f(0,u(0))}E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1}t^\alpha)\}.$$

Now by Laplace transform of convolution, we have

$$\begin{aligned} u(t) = f(t,u(t))\{&\frac{1}{\lambda_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)h_1(s)ds \\ &+\frac{1}{\lambda_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)h_2(s)ds \\ +\sum_{k=1}^{n}\left[\frac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} &t^k E_{\alpha,k+1}(\frac{\lambda_2}{\lambda_1}t^\alpha)+\frac{u(0)}{f(0,u(0))}E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}t^\alpha)\}. \end{aligned} \tag{17}$$

Now applying the boundary condition $u(0) = u(T)$, we can observe the final result of $u(t)$ as follows

$$u(t) = f(t,u(t))\{\tfrac{1}{\lambda_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\tfrac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)h_1(s)ds$$

$$+\tfrac{1}{\lambda_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\tfrac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)h_2(s)ds$$

$$+\sum_{k=1}^{n}\left[\tfrac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} t^k E_{\alpha,k+1}(\tfrac{\lambda_2}{\lambda_1}t^\alpha)$$

$$+L_1\{\tfrac{1}{\lambda_1}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\tfrac{\lambda_2}{\lambda_1}(T-s)^\alpha\right)h_1(s)ds$$

$$+\tfrac{1}{\lambda_1}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\tfrac{\lambda_2}{\lambda_1}(T-s)^\alpha\right)h_2(s)ds$$

$$+\sum_{k=1}^{n}\left[\tfrac{u(t)}{f(t,u(t))}\right]_{t=0}^{(k)} T^k E_{\alpha,k+1}(\tfrac{\lambda_2}{\lambda_1}T^\alpha)\}E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1}t^\alpha)\}$$

$$= H_{\lambda_1,\lambda_2}(t,u(t)) + \int_0^T G_{\lambda_1,\lambda_2}(t,s)\{h_1(s)+h_2(s)\}ds.$$

Lemma 2.3 The Green's function $G_{\lambda_1,\lambda_2}(t,s)$ and $H_{\lambda_1,\lambda_2}(t,u(t))$ defined by equations (11) - (13) and (15) respectively, have the following properties:

$(B_1)G_{\lambda_1,\lambda_2}(t,s) \in C(\mathfrak{J}\times\mathfrak{J},\mathbb{R}^+), H_{\lambda_1,\lambda_2}(t) \in C(\mathfrak{J},\mathbb{R}^+)$.

$(B_2)G_{\lambda_1,\lambda_2}(t,s) \le \frac{\rho T^{\alpha-1}}{\lambda_1}\left\{E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)[1+L_1E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)]\right\}$,
where $t,s\in\mathfrak{J}$ and $\rho = sup\{\rho_1,\rho_2\}$ for $(t,u)\in\mathfrak{J}\times\mathbb{R}$.

$$\rho_1 = supf(t,u(t)) \text{ and } \rho_2 = sup\,p(t,u(t))$$

$(B_3)H_{\lambda_1,\lambda_2}(t,u(t)) \le \rho\sum_{k=1}^{n} T^k M_k E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)\left[1+L_1E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)\right]$,

where $t\in\mathfrak{J}$ and $\rho = sup\{\rho_1,\rho_2\}$ for $(t,u)\in\mathfrak{J}\times\mathbb{R}$.

Proof: By using conditions $(C_3),(C_4)$ and $E_{\alpha,\alpha}(z) < E_{\alpha,1}(z)$ for $\alpha\in(n,n+1)$, $n\in\mathbb{N}$ and $z\in\mathfrak{J}$ and simple calculations, desired proof will be completed. Hence we omit it.
Now by introducing an additional principle condition as follows
(C_5) There exist positive constants $L_{1,u}, L_{2,u}$ with$L_{1,u}\le L_{2,u}$ such that

$$\begin{aligned} \mid f(t,u_1(t)) - f(t,u_2(t)) \mid &\le \frac{L_{1,u}\mid u_1(t)-u_2(t)\mid}{4min\{1+\xi_1,1+\eta_1\}(L_{2,u}+\mid u_1(t)-u_2(t)\mid)} \\ and \mid p(t,u_1(t)) - p(t,u_2(t)) \mid &\le \frac{L_{1,u}\mid u_1(t)-u_2(t)\mid}{4min\{1+\xi_1,1+\eta_1\}(L_{2,u}+\mid u_1(t)-u_2(t)\mid)} \end{aligned} \tag{18}$$

where $t \in \mathfrak{J}$ and

$$\xi_1 = \sum_{k=1}^{n} T^k M_k E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)\left[1 + L_1 E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)\right]$$

$$\eta_1 = \frac{\theta_1 T^{\alpha-1}}{\lambda_1}\left\{E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)[1 + L_1 E_{\alpha,1}(\frac{\lambda_2}{\lambda_1}T^\alpha)]\right\}. \tag{19}$$

In this step we introduce the classic Banach space for coupled system (1). Assume that $E = \{u(t)|u(t) \geq 0, u \in C(\mathfrak{J}, \mathbb{R})\}$.If we equip E to the max-norm

$$||u||_E = sup\{u(t)|t \in \mathfrak{J}\},$$

then we can simply verify that $(E, ||.||_E)$ is a Banach space.
Now we set $\mathfrak{B} = E \times E = \{(u, v)|u, v \in E\}$ endowed with the norm

$$||(u, v)||_{\mathfrak{B}} = ||u||_E + ||v||_E, (u, v) \in \mathfrak{B}.$$

Obviously$(\mathfrak{B}, ||.||_{\mathfrak{B}})$ is also Banach space. Here we remark our desired norm this paper is $||.||_{\mathfrak{B}}$.
Let us define $S \subset \mathfrak{B}$ as follows

$$S = \begin{cases}(u, v) \in \mathfrak{B}|0 \leq u(t), v(t), t \in \mathfrak{J}, ||(u, v)||_{\mathfrak{B}} \leq r\\ (u, v) \in E|0 \leq u(t), v(t), t \in \mathfrak{J}, ||u||_E + ||v||_E \leq r.\end{cases} \tag{20}$$

Definition 2.5 Define the Volterra type integral equation $T_1, T_2 : E \to E$ as

$$\begin{aligned} T_1(t) &= H_{\lambda_1,\lambda_2}(t, u(t)) + f(t, u(t))\int_0^T G_{\lambda_1,\lambda_2}(t, s)\{g_1(s, u(s), v(s)) + p_1(s, u(s), v(s))\}\,ds,\\ T_2(t) &= H_{\mu_1,\mu_2}(t, u(t)) + f(t, v(t))\int_0^T G_{\mu_1,\mu_2}(t, s)\{g_2(s, u(s), v(s)) + p_2(s, u(s), v(s))\}\,ds, \end{aligned} \tag{21}$$

where $H_{.,.}(t, s)$ defined by equation (15) and

$$G_{.,.}(t, s) = \begin{cases}G_{1(.,.)}(t, s); 0 \leq s \leq t \leq T,\\ G_{2(.,.)}(t, s); 0 \leq t \leq s \leq T,\end{cases} \tag{22}$$

is defined by equations (12) and (13)

Definition 2.6 Let us define the operator $\mathfrak{T} : \mathfrak{B} \to \mathfrak{B}$ as below

$$\mathfrak{T}(u, v)(t) = (T_1, T_2)(t) \tag{23}$$

using definition (2.5) for T_1, T_2.

Definition 2.7 [*Dhage*, *B.C.*(2005] Let X be a normed vector space. A mapping $T : X \to X$ is said D - Lipschitzian,provided that there exists a continuous and nondecreasing function $\psi_T : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $x, y \in X$

$$||T_x - T_y|| \leq \psi_T(||x - y||), \psi_T(0) = 0.$$

Remark 2. 4 Every Lipschitzian mapping is D-Lipschitzian and if $\psi_T(r) < r$, then T is called nonlinear D-contraction on X with contraction function ψ_T.
Remark 2. 5 Every nonlinear D-contraction is D-Lipschitzian while the reverse may not hold neccessarily. **Definition 2.8** (*Lebedev*, *L.P.*, *etal.*, 2002) Let X be a normed space and suppose $S \subset X$. A finiteset of N balls $B(x_n, \epsilon)$ with $x_n \in X$ and $\epsilon > 0$ is said to be a finite ϵ- covering of S, provided that every element of S lies inside one of the balls $B(x_n, \epsilon)$, i.e,

$$S \subset \bigcup_{n=1}^{N} B(x_n, \epsilon),$$

The set of centers $\{x_n\}$ of a finite ϵ - covering is called a finite ϵ - net for S.

Definition 2.9 ($Lebedev, L.P., etal.$, 2002) Let X be a normed space. A set $S \subset X$ is said to be a totally bounded if and only if it has a finite ϵ - covering for every $\epsilon > 0$.

Theorem 2.4 (Hausdroff copactness criterion). ($Lebedev, L.P., etal.$, 2002) Assume that X be a normed space. A set $S \subset X$ is compact if and only if it is closed and totally bounded.

Theorem 2.5 (Dhage fixed point theorem),($Dhage, B.C.$, (2005) Assume that S be a nonempty closed convex and bounded subset of Banach algebra X. Let $A, C : X \to X$ and $B : S \to X$ be three operators with the following properties:

(I) A,C are D- Lischitzian with D-functions ϕ_A and ϕ_C respectively.

(II) B is completely continuous.

(III) $x = A_x B_y + C_x \Rightarrow x \in S$, for all $y \in S$.

(IV)$M\phi_A(r) + \phi_C(r) < r$, for $r > 0$ where $M = \|B(S)\|$.

Then the operator $AB + C$ has a fixed point in S.

3. Main Result

Theorem 3.1 Assume that the conditions $(C_1) - (C_5)$ be satisfied. Then the fractional order multi-term coupled hybrid system (1) has at least one positive solution in S defined by equation (20).

Proof: We will prove this in four steps as follows

Step. 1 Developing the main problem in [Ahmed, E.- S., & Hind, H. (2013] and considering the definition (2.5) together with theorem (2.4), let us define

$$A_{1,u}(t) = f(t, u(t)),$$

$$B_{1,u}(t) = \int_0^T G_{\lambda_1,\lambda_2}(t, s)\{g_1(s, u(s), v(s)) + p_1(s, u(s), v(s))\}ds, \tag{24}$$

$$C_{1,u}(t) = H_{\lambda_1,\lambda_2} u(t),$$

and

$$A_{2,v}(t) = f(t, v(t)),$$

$$B_{2,v}(t) = \int_0^T G_{\mu_1,\mu_2}(t, s)\{g_2(s, u(s), v(s)) + p_2(s, u(s), v(s))\}ds, \tag{25}$$

$$C_{2,v}(t) = H_{\mu_1,\mu_2} v(t),$$

By conditions C_1, C_5, we have the following

$$|A_{1,u_1}(t) - A_{1,u_2}(t)| = |f(t, u_1(t)) - f(t, u_2(t))|$$

$$\leq \frac{L_1|u_1(t)-u_2(t)|}{4min\{1+\xi_1,1+\eta_1\}(L_2+|u_1(t)-u_2(t)|)}, t \in \mathfrak{J}$$

Now take supremum on t we have

$$\|A_{1,u_1} - A_{1,u_2}\|_E \leq \frac{L_{1,u}\|u_1 - u_2\|}{4min\{1 + \xi_1, 1 + \eta_1\}(L_{2,u} + \|u_1 - u_2\|)}, u_1, u_2 \in E. \tag{26}$$

Thus $A_{1,u}$ is nonlinear D- contraction on E with D- function

$$\psi_{A_1,u}(r) = \frac{L_{1,u} r}{4min\{1 + \xi_1, 1 + \eta_1\}(L_{2,u} + r)}. \tag{27}$$

In the same manner applying conditions $(C_1), (C_5)$ we conclude that $C_{1,u}$ also is nonlinear D- contraction with D- function

$$\psi_{C_{1,u}}(r) = \frac{L_{1,u}r}{4\{1+\xi_1\}(L_{2,u}+r)} \tag{28}$$

Recall definition of T_1 by equation(21) and gathering results represented by the equations (26)-(28), also considering the remark (2.5), we conclude that not only $A_{1,u}$,$C_{1,u}$ are nonlinear D-contraction with corresponding D-function $\psi_{A_{1,u}}$ and $\psi_{C_{1,u}}$, respectively but also in the same way $A_{2,v}, C_{2,v}$ are D-contraction with D-function $\psi_{A_{2,v}}$ and $\psi_{C_{2,v}}$, such that

$$\begin{aligned}\psi_{A_{2,v}}(r) &= \frac{L_{1,v}r}{4min\{1+\xi_2, 1+\eta_2\}(L_{2,v}+r)},\\ \psi_{C_{2,v}}(r) &= \frac{L_{1,v}r}{4\{1+\xi_2,\}(L_{2,v}+r)}\end{aligned} \tag{29}$$

such that

$$\begin{aligned}\xi_2 &= \sum_{k=1}^{n} T^k N_k E_{\beta,1}(\frac{\mu_2}{\mu_1}T^\beta)\left[1 + L_2 E_{\beta,1}(\frac{\mu_2}{\mu_1}T^\beta)\right]\\ \eta_2 &= \frac{\theta_2 T^{\beta-1}}{\mu_1}\left\{E_{\beta,1}(\frac{\mu_2}{\mu_1}T^\beta)[1 + L_2 E_{\beta,1}(\frac{\mu_2}{\mu_1}T^\beta)]\right\},\end{aligned} \tag{30}$$

where

$$L_2 = \frac{f(T, v(T))}{f(0, v(0)) - f(T, v(t))E_{\beta,1}(\frac{\mu_2}{\mu_1}T^\beta)} \tag{31}$$

Thus if we consider the operator $\mathfrak{T}$ defined by (23) as

$$\mathfrak{T}(u,v)(t) = \begin{pmatrix}C_{1,u}\\ C_{2,v}\end{pmatrix}(t) + \begin{pmatrix}A_{1,u} & 0\\ 0 & A_{2,v}\end{pmatrix}(t)\begin{pmatrix}B_{1,u}\\ B_{2,v}\end{pmatrix}(t) = C_{u,v}(t) + A_{u,v}(t)B_{u,v}(t), \tag{32}$$

setting

$$\|A_{u,v}\|_{\mathfrak{B}} = \sum_{i=1}^{2}\sum_{j=1}^{2} sup\{A_{u,v}(i,j)(t) | t \in \mathfrak{J}\}. \tag{33}$$

we conclude that $\|A_{u,v}\|_{\mathfrak{B}} = \|A_{1,u}\|_E + \|A_{2,v}\|_E$. Therefore both $A_{u,v}, C_{u,v}$ are nonlinear D-contractions with corresponding D- functions

$$\psi_{A_u,v}(t) = \psi_{A_1,u} + \psi_{A_2,v}, \psi_{C_u,v}(t) = \psi_{C_1,u} + \psi_{C_2,v} \tag{34}$$

respectively. This is end of step.1.

Step. 2 To prove $B_{u,v}$ defined by the equations(24),(25) and (32) is completely continuous. To show this we will use the Hausdroff compactness criterion in theorem (2.4) as follows: Obviously $S \subset \mathfrak{B}$ is a nonempty closed convex and bounded in $\mathfrak{B}$. Let us define

$$\begin{aligned}S_u &= \{u \in E \mid \|u(t)\| \le \frac{r}{2}, t \in \mathfrak{J}\},\\ S_v &= \{v \in E \mid \|v(t)\| \le \frac{r}{2}, t \in \mathfrak{J}\}.\end{aligned} \tag{35}$$

Clearly S_u, S_v are closed. So both of them are Banach spaces with the norm of E. Also u(t), v(t) are equicontinuous on $\mathfrak{J}$. So by means of Arzela - Ascoli theorem we conclude that S_u, S_v are compact. Hence Theorem (2.4) ensures that S_u, S_v are totally bounded.

Thus definition (2.9) implies that there exist two finite ϵ - coverings as
$\mathfrak{U}_\epsilon(u_i), \mathfrak{U}_\epsilon(v_j), i = 1, 2, 3, ...l_1, j = 1, 2, 3....., l_2$ such that

$$\begin{aligned}S_u &\subset \bigcup_{i=1}^{l_1} \mathfrak{U}_\epsilon(u_i),\\ S_v &\subset \bigcup_{j=1}^{l_2} \mathfrak{U}_\epsilon(v_j),\end{aligned} \tag{36}$$

where

$$\mathfrak{U}_\epsilon(u_i) = \{u \in S_u \mid \|u - u_i\|_E < \epsilon\}, \\ \mathfrak{U}_\epsilon(v_j) = \{v \in S_v \mid \|v - v_j\|_E < \epsilon\}, \tag{37}$$

Define

$$Sij = \{(u, v) \in S_u \times S_v | u \in \mathfrak{U}_\epsilon(u_i), v \in \mathfrak{U}_\epsilon(v_j)\},$$

And this gives $S \subset S_u \times S_v \subset \bigcup_{i,j} S_{ij}, 1 \le i \le l_1, 1 \le j \le l_2$
In fact if we take $(u_{ij}, v_{ij}) \in S_{ij}$ then $S_u \times S_v$ can be covered by finite 4ϵ - covering
$\mathfrak{U}_{4\epsilon}(u_{ij}, v_{ij}) = \{(u, v) \in S_u \times S_v \mid \|(u, v) - (u_{ij}, v_{ij})\|_{\mathfrak{B}} < 4\epsilon\}$
In other means for every $(u, v) \in S_u \times S_v$, there exist indices i,j such that

$$u \in \mathfrak{U}_\epsilon(u_i), v \in \mathfrak{U}_\epsilon(v_j)$$

Therefore

$$|u - u_{ij}| \le |u - u_i| + |u_i - u_{ij}| < \epsilon + \epsilon = 2\epsilon, \\ |v - v_{ij}| \le |v - v_i| + |v_i - v_{ij}| < \epsilon + \epsilon = 2\epsilon. \tag{38}$$

(38) implies that $\|(u, v) - (u_{ij}, v_{ij})\|_{\mathfrak{B}} < 4\epsilon$. So our claim has been proved that S has a finite $4\epsilon-$ covering. Hence using theorem (2.4) we conclude that S is compact.
Now let us return to equation of $B_{u,v}(t)$ in (32).According to conditions $(C_1), (C_2)$ and Lemma (2.3), we know that $B_{u,v}$ is continuous on S. Thus $B_{u,v}(S)$ is relatively compact and consequently $B_{u,v}$ is completely continuous on S. This completes the step 2.
step. 3 To show $T_1 \le r_1, T_2 \le r_2$ for $u, v \in E$. Using definition (2.5) and Lemma (2.3) also conditions $(C_1) - (C_5)$, we have

$$T_1(t) = H_{\lambda_1,\lambda_2}(t, u(t)) + f(t, u(t)) \int_0^T G_{\lambda_1,\lambda_2}(t, s)\{g_1(s, u(s), v(s)) + p_1(s, u(s), v(s))\}ds \tag{39}$$

$$\le \sum_{k=1}^n \rho T^k M_k E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1} T^\alpha) \left[1 + L_1 E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1} T^\alpha)\right]$$

$$+\tfrac{\rho\theta_1 T^{\alpha-1}}{\lambda_1} \left\{E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1} T^\alpha)[1 + L_1 E_{\alpha,1}(\tfrac{\lambda_2}{\lambda_1} T^\alpha)]\right\}$$

$= r_1$
Similarly we can prove that

$$T_2(t) = H_{\mu_1,\mu_2}(t, v(t)) + f(t, v(t)) \int_0^T G_{\mu_1,\mu_2}(t, s)\{g_2(s, u(s), v(s)) + p_2(s, u(s), v(s))\}ds \tag{40}$$

$$\le \sum_{k=1}^n \rho T^k N_k E_{\beta,1}(\tfrac{\mu_2}{\mu_1} T^\beta) \left[1 + L_2 E_{\beta,1}(\tfrac{\mu_2}{\mu_1} T^\beta)\right]$$

$$+\tfrac{\rho\theta_2 T^{\beta-1}}{\mu_1} \left\{E_{\beta,1}(\tfrac{\mu_2}{\mu_1} T^\beta)[1 + L_2 E_{\beta,1}(\tfrac{\mu_2}{\mu_1} T^\beta)]\right\},$$

$= r_2$
So considering (39) and (40) we conclude that

$$\|T_1\|_E \le r_1, \|T_2\|_E \le r_2, u, v \in E. \tag{41}$$

Finally

$$\|\mathfrak{T}(u, v)\|_{\mathfrak{B}} = \|T_1\|_E + \|T_2\|_E \le r = 2max\{r_1, r_2\}, (u, v) \in \mathfrak{B}. \tag{42}$$

Hence we have been proved that $\mathfrak{T}(S) \subset S$.
In other words if we consider

$$x_1 = C_{1,x_1} + A_{1,x_1} B_{1,y_1}, \\ x_2 = C_{2,x_2} + A_{2,x_2} B_{2,y_2}, \tag{43}$$

then (42) ensures that $(x_1, x_2) \in S$ for all $(y_1, y_2) \in S$, the step 3. completed.
step. 4 First of all by means of equations (19), (30) and definition (2.5), it is clear that

$$\|B_{1,u}\|_E \leq \eta_1, \|B_{2,V}\|_E \leq \eta_2. \tag{44}$$

Hence $\|B_{u,v}\|_{\mathfrak{B}} \leq \wedge = 2max\{\eta_1, \eta_2\}$
Therefore according to obtained results in step. 1. containing (27)-(34), we deduce that $\mathfrak{T}(u, v)$ is nonlinear D -contraction with corresponding D-function

$$\phi_{Cu,v} + \wedge\phi_{Au,v}$$

This completes the step. 4.
Since all conditions of theorem (2.4) are satisfied, then operator $\mathfrak{T}$ defined by (23) or equivalently by (32) has a fixed point in S. In other means the fractional order coupled system (1) has one positive solution S.The proof is complete.

Example 3.2 Consider the fractional order coupled system

$$\begin{aligned} 5\mathfrak{D}_{0^+}^{\frac{5}{4}}\left(\frac{u(t)}{f(t,u(t))}\right) - \frac{4u(t)}{f(t,u(t))} &= g_1(t,u(t),v(t)) + p_1(t,u(t),v(t)) \\ 6\mathfrak{D}_{0^+}^{\frac{12}{7}}\left(\frac{v(t)}{f(t,v(t))}\right) - \frac{v(t)}{f(t,v(t))} &= g_1(t,u(t),v(t)) + p_1(t,u(t),v(t)) \\ u(0) = u(1), v(0) &= v(1), \end{aligned} \tag{45}$$

where

$$f(t,u(t)) = \tfrac{2-u(t)}{500(2-exp(4t+1))}$$

$$f(t,v(t)) = \tfrac{2-v(t)}{550(2-exp(4t+1))}$$

$$g_1(t,u(t),v(t)) = cosec(u) + v^2; (u,v) \in [0,5] \times [0,1]$$

$$= 2cosec(u) + v^2 - 1; (u,v) \in [0,5] \times [1,3]$$

$$= cosec(u) + v^2 + 1; (u,v) \in [0,5] \times [3,5]$$

$$p_1(t,u(t),v(t)) = u^4 + sinv; (u,v) \in [0,1] \times [0,5]$$

$$= u^4 + 2sinv - 1; (u,v) \in [1,3] \times [0,5]$$

$$= u^4 + sinv + 1; (u,v) \in [3,5] \times [0,5]$$

$$g_2(t,u(t),v(t)) = sec^2(v) + u; (u,v) \in [0,1] \times [0,5]$$

$$= sec^2(v) + 2u - 1; (u,v) \in [1,\sqrt{3}] \times [0,5]$$

$$= sec^2(v) + u + \sqrt{3} - 1; (u,v) \in [\sqrt{3},5] \times [0,5]$$

$$p_2(t,u(t),v(t)) = cos^4(u) + v; (u,v) \in [0,5] \times [0,1]$$

$$= cos^4(u) + 2v - 1; (u,v) \in [0,5] \times [1,\sqrt{3}]$$

$$= cos^4(u) + v + \sqrt{3} - 1; (u,v) \in [0,5] \times [\sqrt{3},5]$$

Calculations shows that for $t \in \mathfrak{J}$

$$| f(t,u_1) - f(t,u_2) | \leq \frac{|u_1-u_2|}{500(2-e)}$$

$$| f(t,v_1) - f(t,v_2) | \leq \frac{|v_1-v_2|}{550(2-e)}$$

$$\theta_1 = 8, \theta_2 = 5 + \sqrt{3}$$

$$M_1 = 5, N_1 = 4,$$

$$f(0,u(0)) = 4, f(0,v(0)) = 3.70$$

$$f(1,u(1)) = f(1,v(1)) = 2$$

Hence

$$| f(t,u_1) - f(t,u_2) | \leq \frac{|u_1(t)-u_2(t)|}{4min\{1+\xi_1, 1+\eta_1\}(2+|u_1(t)-u_2(t)|)}$$

$$| f(t,v_1) - f(t,v_2) | \leq \frac{|v_1(t)-v_2(t)|}{4min\{1+\xi_2, 1+\eta_2\}(2+|v_1(t)-v_2(t)|)}$$

Hence conditions$(C_1) - (C_5)$ are satisfied, therefore according to theorem (3.1), fractional coupled system (45) has one positive solution in S.

4. Conclusion

The integral transform technique applied to the given fractional differential equation and solution obtained using these techniques with the help of approximations.

5. Further Scope

The fractional differential equation solved by various techniques, here the integral transform technique used which will be very helpful for such type of non-integer order fractional differential equations and gives the approximate result for the fractional differential equation.

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