Journal of Mathematics Research; Vol. 9, No. 1; February 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Absolute Valued Algebras with Strongly One Sided Unit

Alassane Diouf[1]

[1] Département de Mathématiques et Informatiques, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal

Correspondence: Alassane Diouf, Département de Mathématiques et Informatiques, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal. E-mail: dioufalassane@hotmail.fr

Received: October 30, 2016 Accepted: December 5, 2016 Online Published: December 30, 2016

doi:10.5539/jmr.v9n1p32 URL: http://dx.doi.org/10.5539/jmr.v9n1p32

Abstract

We classify the absolute valued algebras with strongly left unit of dimension ≤ 4. Also we prove that every 8-dimensional absolute valued algebra with strongly left unit contain a 4-dimensional subalgebra, next we determine the form of theirs algebras by the duplication process.

Keywords: absolute valued algebra, strongly left unit, duplication process

Mathematics Subject Classification: 17A35, 17A36

1. Introduction

The absolute valued algebras are introduced by Ostrowski 1918. It's the normed algebra A such that $\|xy\| = \|x\|\|y\|$ for all x, y in A. For an element a in an algebra A, we denote by $L_a : A \to A$ $x \mapsto ax$ and $R_a : A \to A$ $x \mapsto xa$. The algebra is called division if and only if R_a and L_a are bijective for all a in A. We denote by O the orthogonal group of linear isometries of Euclidean space $\mathbb{H}$. We recall O^+ the subgroup of proper linear isometries and O^- the subset of improper linear isometries. Let A be an absolute valued algebra with unit, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (Urbanik & Wright, 1960). The absolute valued algebras with left unit satisfying to $(x^2, x^2, x^2) = 0$, for all $x \in A$ is classified in (Diankha & all, 2013_2). These algebras are finite dimensional and isomorphic to $\mathbb{R}$, $\mathbb{C}$, ${}^\star\mathbb{C}$, $\mathbb{H}$, ${}^\star\mathbb{H}$, ${}^\star\mathbb{H}(i,1)$, $\mathbb{O}$, ${}^\star\mathbb{O}$, ${}^\star\mathbb{O}(i,1)$, $\widetilde{\mathbb{O}}$ or $\widetilde{\mathbb{O}}(i)$ and the element e satisfy to $L_e = R_e^2 = I_A$. The algebras $\mathbb{H}_i$, $\mathbb{O}_i$ (Diankha & all, 2013_1), satisfy to $L_e = R_e^2 = I_A$ and not satisfy to $(x^2, x^2, x^2) = 0$. In this paper we give a classification of the absolute valued algebras with strongly left unit of dimension ≤ 4. We proves that if A is 8-dimensional absolute valued algebra with strongly left unit, then A contain a 4-dimensional subalgebra and A is obtained by the duplication process. Otherwise A is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi,\psi)}$ with $\varphi, \psi : \mathbb{H} \to \mathbb{H}$ are linear isometries such that $\varphi(1) = 1$ and $(\varphi, \psi)^2 = (\varphi, \psi)$. The algebras $\mathbb{R}$, $\mathbb{C}$, ${}^\star\mathbb{C}$, $\mathbb{H}$, ${}^\star\mathbb{H}$, $\mathbb{H}_i$, ${}^\star\mathbb{H}(i,1)$, $\mathbb{O}$, ${}^\star\mathbb{O}$, $\mathbb{O}_i$, ${}^\star\mathbb{O}(i,1)$, $\widetilde{\mathbb{O}}$ are absolute valued algebras with strongly left unit. This list is completed by new algebras.

2. Preliminary

In this section we recall the some interest results:

Theorem 1 *The finite-dimensional absolute valued real algebras with a left unit are precisely those of the form $\mathbb{A}_\varphi$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and φ is an isometric of the euclidien espace $\mathbb{A}$ fixes 1, and $\mathbb{A}_\varphi$ denotes the absolute-valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : \mathbb{A} \to \mathbb{A}$ fixing 1, the algebras $\mathbb{A}_\varphi$ and $\mathbb{A}_\phi$ are isomorphic if and only if there exists an algebra automorphism ψ of $\mathbb{A}$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$* (Rochdi, 2003).

Lemma 1 *Let A be an absolute valued algebra with strongly left unit. The following equalities hold for all $x \in A$.*

1. $[(xe)x]e = x(xe)$
2. $[x(xe)]e = (xe)x$
3. $[xe, x] = < e, x > [e, x - xe]$
   If, moreover, x is orthogonal to e, then
4. $[xe, x] = 0$
5. $(xe)x^2 = 2 < e, x^2 > x - \|x\|^2 xe$
6. $(xe)^2 = 2 < e, x^2 > e - x^2$
7. $x^2 x = -\|x\|^2 xe$ (Chandid & Rochdi, 2008).

The group G_2 acts transitively on the sphere $S(Im(\mathbb{O})) := S^6$, that is the mapping $G_2 \to S^6$ $\Phi \mapsto \Phi(i)$ is surjective (Postnikov, 1985).

Let $\mathbb{A}$ be one of the unital absolute valued algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of dimension m. Consider the caley dickson product $\odot$ in $\mathbb{A} \times \mathbb{A}$, we define on the space $\mathbb{A} \times \mathbb{A}$ the product

$$(x, y) \star (x', y') = (f_1(x), f(x)) \odot (g_1(x'), g(y')).$$

With f_1, g_1, f, g be linear isometries of $\mathbb{A}$ and $f_1(1) = g_1(1) = 1$. We obtain a $2m$-dimensional absolute valued real algebra $\mathbb{A} \times \mathbb{A}_{(f_1,f),(g_1,g)}$. The process is called duplication process. Note that the algebra is left unit if $g_1 = g = I_{\mathbb{A}}$ and this case we not the algebra by $\mathbb{A} \times \mathbb{A}_{(f_1,f)}$. We have the following result (Calderon & all, 2011):

Theorem 2 *Let A be an 8-dimensional absolute valued algebra, then the following are equivalent:*

1. *A contains a 4-dimensional subalgebra.*
2. *A is obtained by the duplication process.*
3. *$Aut(A)$ contains a reflexion.*

Lemma 2 *Let* $\mathcal{I}^+ = \{f \in O^+ : f \text{ involutive}\}$, $\mathcal{I}^- = \{f \in O^- : f \text{ involutive}\}$, $\mathcal{I}_1^+ = \{f \in \mathcal{I}^+ : f(1) = 1\}$ *and* $\mathcal{I}_1^- = \{f \in \mathcal{I}^- : f(1) = 1\}$. *We have:*

1. $O^+ = \{T_{a,b} : a, b \in S(\mathbb{H})\}$
2. $O^- = \{T_{a,b} \circ \sigma_{\mathbb{H}} : a, b \in S(\mathbb{H})\} := O^+ \circ \sigma_{\mathbb{H}}$
3. $O_1^+ = \{T_{a,\bar{a}} : a \in S(\mathbb{H})\}$
4. $O_1^- = \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\} := O_1^+ \circ \sigma_{\mathbb{H}}$
5. $\mathcal{I}^+ = \{\pm I_{\mathbb{H}}\} \cup \{T_{a,b} : a, b \in S(Im(\mathbb{H}))\}$
6. $\mathcal{I}^- = \{\pm T_{a,\bar{a}} : a \in S(\mathbb{H})\}$
7. $\mathcal{I}_1^+ = \{I_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} : a \in S(Im(\mathbb{H}))\}$
8. $\mathcal{I}_1^- = \{\sigma_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S(Im(\mathbb{H}))\} := \mathcal{I}_1^+ \circ \sigma_{\mathbb{H}}$ *(Diankha & all, 2013$_2$).*

Corollary 1 *Let A be an absolute valued algebra with left unit satisfying to* $(x^p, x^q, x^r) = 0$ *with* $\{p, q, r\} \in \{1, 2\}$. *Then A contains a strongly left unit.*

Proof. Lemma 1 (Diankha & all, 2013$_2$) and proof of Proposition 4.8 (Chandid & Rochdi, 2008)..

The converse of Corollary 1 is false, an effect the algebra $A := \mathbb{O}_i$ is an absolute valued algebra with strongly left unit and A not satisfy to $(x^2, x^2, x^2) = 0$.

3. Absolute Valued Algebras with Strongly Left Unit

Definition 1 *An element $e \in A$ is called strongly left unit, if it's left unit and square root of right unit* ($L_e = R_e^2 = I_A$).

Theorem 3 *Let A be an absolute valued algebra with strongly left unit. Then A is finite dimensional. Moreover if* $dim(A) \leq 4$, *then A is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $^\star\mathbb{C}$, $\mathbb{H}$, $^\star\mathbb{H}$, $\mathbb{H}(i, 1)$ *or* $^\star\mathbb{H}(i, 1)$.

Proof. The algebra A is left unit, hence A is left division (Rodriguez, 2004). Morover the assertion $R_e^2 = I_A$ imply that A is right division, then A is finite dimensional. Also A is of the form $\mathbb{A}_\varphi$, with φ a linear isometric fixed 1 and $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (Theorem 1). If $dim(A) \leq 2$, it's clear that A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $^\star\mathbb{C}$.
Assume now $dim(A) \geq 4$, then the assertion $R_e^2 = I_A$ imply:

$$\begin{aligned} x &= (x \odot 1) \odot 1 \\ &= \varphi^2(x). \end{aligned}$$

Then φ is an involutive linear isometric $\varphi^2 = I_{\mathbb{A}}$.
If $dim(A) = 4$, we have:
$\varphi \in \mathcal{I}_1^+ \cup \mathcal{I}_1^- = \{I_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} : a \in S(Im(\mathbb{H}))\} \cup \{\sigma_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S(Im(\mathbb{H}))\}$ (Lemma 2).

- If $\varphi = I_{\mathbb{H}}$, then A is isomorphic to $\mathbb{H}$.
- If $\varphi = \sigma_{\mathbb{H}}$, then A is isomorphic to ${}^{\star}\mathbb{H}$.
- If $\varphi = T_{a,\overline{a}}$: $a \in S(Im(\mathbb{H}))$, there exist $v \in S(\mathbb{H})$ such that $va\overline{v} = i$ and let the automorphism $\Phi = T_{v,\overline{v}}$ of $\mathbb{H}$ with $\Phi^{-1} = T_{\overline{v},v}$, we have $\Phi \circ T_{a,\overline{a}} \circ \Phi^{-1} = T_{i,\overline{i}}$. Then A is isomorphic to $\mathbb{H}_{T_{i,\overline{i}}}$ (Theorem 1) and the map $\Phi : \mathbb{H}(i,1) \to \mathbb{H}_{T_{i,\overline{i}}}$ $x \mapsto xi$ is an isomorphism algebras.
- If $\varphi = T_{a,\overline{a}} \circ \sigma_{\mathbb{H}}$: $a \in S(Im(\mathbb{H}))$, there exist $u \in S(\mathbb{H})$ such that $ua\overline{u} = i$ and let the automorphism $\Phi = T_{u,\overline{u}}$ of $\mathbb{H}$, we have:

$$\begin{aligned}
\Phi \circ T_{a,\overline{a}} \circ \sigma_{\mathbb{H}} \circ \Phi^{-1} &= T_{u,\overline{u}} \circ T_{a,\overline{a}} \circ \sigma_{\mathbb{H}} \circ T_{\overline{u},u} \\
&= T_{u,\overline{u}} \circ T_{a,\overline{a}} \circ T_{\overline{u},u} \circ \sigma_{\mathbb{H}} \\
&= T_{ua\overline{u},u\overline{a}\overline{u}} \circ \sigma_{\mathbb{H}} \\
&= T_{i,\overline{i}} \circ \sigma_{\mathbb{H}}.
\end{aligned}$$

Then A is isomorphic to $\mathbb{H}_{T_{i,\overline{i}} \circ \sigma_{\mathbb{H}}}$ (Theorem 1) and the map $\Phi :{}^{\star} \mathbb{H}(i,1) \to \mathbb{H}_{T_{i,\overline{i}} \circ \sigma_{\mathbb{H}}}$ $x \mapsto \overline{i}x$ is an isomorphism of algebras.∎

If $dim(A) = 4$, the last result can be obtained so by using the identity $R_e^2 = I_{\mathbb{H}}$ and the principal isotopes of $\mathbb{H}$: $\mathbb{H}(a,1)$, ${}^{\star}\mathbb{H}(a,1)$, where $a \in S(\mathbb{H})$. For the first isotope $e = \overline{a}$, and for the second isotope $e = a$.

For all alternative algebra A, Artin's theorem (Schafer, 1996) shows that for any $x, y \in A$, the set $\{x, y, \overline{x}, \overline{y}\}$ is contained in an associative subalgebra of A. We note by $T(x) = x + \overline{x}$ the tace of $x \in A$ and we have $x^2 - T(x)x + \|x\|^2 e = 0$ for all $x \in A$. As A is real alternative quadratic algebra, we have $A = \mathbb{R}e \oplus Im(A)$ (Frobenius decomposition) and their exist a unique linear form $\lambda : A \to \mathbb{R}$ such that $\lambda(1) = 1$, $ker(\lambda) = Im(A)$ and $< x, y >= \lambda(x\overline{y}) = \lambda(\overline{x}y)$ for all $x, y \in A$ $(*)$ (Koecher & Remmert, 1991). Otherwise for all $x, y \in Im(A)$ we have $xy + yx = -2 < x, y > e$ $(\star)$ and the identity $xyx = 2\lambda(xy)x - \|x\|^2\overline{y}$ for all $x, y \in Im(A)$ is called the triple product identity (**TPI**).

In 8-dimensional, by the duplication process we recover theirs algebras.

Theorem 4 *Let A be an 8-dimensional absolute valued algebra with strongly left unit. Then A contains a four-dimensional subalgeba.*

Proof. We have $\mathbb{O} = \mathbb{R} \oplus Im(\mathbb{O})$ and their exist a unique linear form $\lambda : \mathbb{O} \to \mathbb{R}$ such that $\lambda(1) = 1$ and $ker(\lambda) = Im(\mathbb{O})$. Let $u \in 1^{\perp}$, we have $0 =< 1, u >=< \Phi^n(1), \Phi^n(u) >=< 1, \Phi^n(u) >$. Then we have $\varphi^n(1^{\perp}) \subseteq 1^{\perp}$, for all $n \in \mathbb{N}$. The algebra A is of the form $\mathbb{O}_{\Phi}$ with $\Phi(1) = 1$ and $\Phi^2 = I_{\mathbb{O}}$ (Theorem 1 and Theorem 3). Otherwise we have $i \odot i = \Phi(i)i$ and $i \odot 1 = \Phi(i)$. Using the equality $(\star)$ we have $i\Phi(i) + \Phi(i)i = -2 < i, \Phi(i) > 1$. Also using Lemma 1 (7), we have $\Phi[\Phi(i)i] = \Phi(i)i$. Using the **TPI** we have

$$\Phi(i)i\Phi(i) = 2\lambda[\Phi(i)i]\Phi(i) + i = -2 < i, \Phi(i) > \Phi(i) + i. \quad (*)$$

and

$$i\Phi(i)i = 2\lambda[i\Phi(i)]i + \Phi(i) = -2 < i, \Phi(i) > i + \Phi(i). \quad (*)$$

Hence we have the products,

$$\Phi(i) \odot 1 = \Phi^2(i) = i.$$

$$\Phi(i) \odot i = \Phi^2(i)i = -1.$$

$$\Phi(i) \odot \Phi(i) = \Phi^2(i)\Phi(i) = i\Phi(i) = -2 < i, \Phi(i) > 1 - \Phi(i)i. \quad (\star)$$

$$\Phi(i) \odot \Phi(i)i = \Phi^2(i)\Phi(i)i = i\Phi(i)i = -2 < i, \Phi(i) > i + \Phi(i). \quad \textbf{(TPI)}$$

$$\Phi(i)i \odot 1 = \Phi[\Phi(i)i] = \Phi(i)i.$$

$$\Phi(i)i \odot i = \Phi[\Phi(i)i]i = \Phi(i)i^2 = -\Phi(i).$$

$$\Phi(i)i \odot \Phi(i) = \Phi[\Phi(i)i]\Phi(i) = \Phi(i)i\Phi(i) = -2 < i, \Phi(i) > \Phi(i) + i. \quad \textbf{(TPI)}$$

$$\Phi(i)i \odot \Phi(i)i = \Phi[\Phi(i)i]\Phi(i)i = (\Phi(i)i)^2 = T[\Phi(i)i]\Phi(i)i - 1.$$

$\odot$	1	i	$\Phi(i)$	$\Phi(i)i$
1	1	i	$\Phi(i)$	$\Phi(i)i$
i	$\Phi(i)$	$\Phi(i)i$	-1	$-i$
$\Phi(i)$	i	-1	$-2 < i, \Phi(i) > 1 - \Phi(i)i$	$-2 < i, \Phi(i) > i + \Phi(i)$
$\Phi(i)i$	$\Phi(i)i$	$-\Phi(i)$	$-2 < i, \Phi(i) > \Phi(i) + i$	$T[\Phi(i)i]\Phi(i)i - 1$

Then the algebra A contains a four-dimensional sub-algebra.∎

Theorem 5 *Let A be an 8-dimensional absolute valued algebra with strongly left unit. Then A is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi,\psi)}$ where (φ,ψ) are linear isometries of $\mathbb{H}$ belong to $\mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{S}_3 \cup \mathbb{S}_4$ with:*

$$\begin{aligned}
\mathbb{S}_1 &= \{I_{\mathbb{H}}\} \times \{\pm I_{\mathbb{H}},\ T_{a,b} : a, b \in S^2,\ \pm T_{c,\bar{c}} \circ \sigma_{\mathbb{H}} : c \in S^3\} \\
\mathbb{S}_2 &= \{\sigma_{\mathbb{H}}\} \times \{\pm I_{\mathbb{H}},\ T_{a,b} : a, b \in S^2,\ \pm T_{c,c} \circ \sigma_{\mathbb{H}} : c \in S^3\} \\
\mathbb{S}_3 &= \{T_{a,\bar{a}} : a \in S^2\} \times \{\pm I_{\mathbb{H}},\ T_{b,c} : b, c \in S^2,\ \pm T_{d,\bar{d}} \circ \sigma_{\mathbb{H}} : d \in S^3\} \\
\mathbb{S}_4 &= \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S^2\} \times \{\pm I_{\mathbb{H}},\ T_{b,c} : b, c \in S^2,\ \pm T_{d,\bar{d}} \circ \sigma_{\mathbb{H}} : d \in S^3\}.
\end{aligned}$$

Proof. Using the Theorem 2 and Theorem 4, the algebra A is obtained by the duplication process. It's clear that the algebra A is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi,\psi)}$, with $\varphi(1) = 1$. The linear isometric (φ,ψ) is involutive, then $\varphi^2 = \psi^2 = I_A$. We have $\varphi \in \mathcal{I}_1^+ \cup \mathcal{I}_1^-$ and $\psi \in \mathcal{I}^+ \cup \mathcal{I}^-$. Then the lemma 2 gives the result.∎

Problem 1 *In dimension 8, it will be interesting to specify these algebras by reducing the isomorphism classes.*

Acknowledgements

I thank the reviewers for the relevant remarks and suggestions.

References

Albert, A. A. (1947). Absolute valued real algebras. *Ann. Math., 48,* 495-501.

Calderon, A., Kaidi, A., Martin, C., Morales, A., Ramirez, M., & Rochdi, A. (2011). Finite - dimensional absolute - valued algebras. *israel journal of mathematics*, 184, 193 - 220.

Chandid, A. & Rochdi, A. (2008). A survey on absolute valued algebras satisfying $(x^i, x^j, x^k) = 0$. *Int. J. Algebra, 2*, 837-852.

Diankha, O., Diouf, A., & Rochdi, A. (2013_1). A brief statement on the absolute-valued algebras with one-sided Unit. *Int. J. Algebra, 7*(17), 833-838.

Diankha, O., Diouf, A., Ramirez, M. I., & Rochdi, A. (2013_2). Absolute-valued algebras with one-sided unit satisfying $(x^2, x^2, x^2) = 0$. *Int. J. Algebra*, *7*(19), 935-958.

Koecher, M., & Remmert. (1991). Numbers. *Springer Verlag.*

Postnikov, M. (1985), Leçons de Géométrie. Groupes et algèbres de Lie. *Editions Mir.*

Rochdi, A. (2003). Eight-dimensional real absolute valued algebras with left unit whose automorphism group is trivial. *IJMMS, 70*, 4447-4454.

Rodriguez, A. (1992). One-sided division absolute valued algebras. *Publ. Math., 36,*925-954.

Rodriguez, A. (2004). Absolute valued algebras, and absolute valuable Banach spaces. *Advanced courses of mathematical analysis I*, 99-155, World Sci. Publ., Hackensack, NJ.

Ramirez, M. I. (1999). On four absolute valued algebras. *Proceedings of the International Conference on Jordan Structures* (Malaga, 1997), 169-173.

Schafer, R. (1996). An Introduction to Nonassociative Algebras. *Academic Press.*

Urbanik, K., & Wright£F. B. (1960). Absolute valued algebras. *Proc. Amer. Math. Soc. 11*, 861-866.

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).