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Some Bounds for the Norms of Circulant Matrices with the k-Jacobsthal and k-Jacobsthal Lucas Numbers

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Abstract

In this paper we investigate upper and lower bounds of the norms of the circulant matrices whose elements are $k-$Jacobsthal numbers and $k-$Jacobsthal Lucas numbers.

Keywords: $k-$Jacobsthal number, $k-$Jacobsthal Lucas number, circulant matrix, norm.

1. Introduction and Preliminaries

There are so many studies on special integer sequences because of meeting in science and nature, art (see Horadam, 1996; Koshy, 2001; Sloane, 2006). There have been several papers on the norms of very special matrices in the last years [7-16]. For example Solak (2005) has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as nxn circulant matrices, where $a_{ij} = F_{(mod(j-i,n))}$ and $b_{ij} = L_{(mod(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms. Fibonacci and Lucas sequences are defined by the recurrence relations $F_{n+1} = F_n + 2F_{n-1}$, ($F_0 = 0, F_1 = 1$), $L_{n+1} = L_n + 2L_{n-1}$, ($L_0 = 0, L_1 = 1$) respectively for $n \geq 1$. Shen and Cen [10] have given upper and lower bounds for the spectral norms of r- circulant matrices $A = C_r(F_0^{(k,-1)}, F_1^{(k,-1)}, ..., F_{n-1}^{(k,-1)})$ and $B = C_r(L_0^{(k,-1)}, L_1^{(k,-1)}, ..., L_{n-1}^{(k,-1)})$. In addition, they also have obtained some bounds for the spectral norms of Hadamard and Kronecker products of these matrices. Authors (Akbulak and Bozkurt, 2008) have studied the norms of Hankel matrices with Fibonacci and Lucas sequences. The authors (Yazlık and Taşkara, 2013) presented upper and lower bounds for the spectral norm of an r- circulant matrix whose entries are the generalized $k-$ Horadam numbers. The authors (Uslu, et al., 2011) have given the relation among $k-$ Fibonacci, $k-$Lucas and generalized $k-$ Fibonacci numbers and the spectral norms of the matrices involving these numbers.

Jacobsthal $\{j_n\}_{n\in\mathbb{N}}$, and Jacobsthal Lucas $\{c_n\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$j = j_{n-1} + 2j_{n-2}, \quad j_0 = 0, \ j_1 = 1, \ n \geq 2,$$

$$c_n = c_{n-1} + 2c_{n-2}, \quad c_0 = 2, \ c_1 = 1, \ n \geq 2,$$

Similarly $k-$Jacobsthal $\{j_{k,n}\}_{n\in\mathbb{N}}$, and $k-$Jacobsthal Lucas $\{c_{k,n}\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, \quad j_{k,0} = 0, \ j_{k,1} = 1, \ n \geq 2, \tag{1}$$

$$c_{k,n} = kc_{k,n-1} + 2c_{k,n-2}, \quad c_{k,0} = 2, \ c_{k,1} = 1, \ n \geq 2, \tag{2}$$

respectively. The first k-Jacobsthal numbers for $0 \leq n \leq 5$ are $0, 1, k, k^2 + 2, k^3 + 4k, k^4 + 6k^2 + 4$. The first k-Jacobsthal Lucas numbers for $0 \leq n \leq 5$ are $2, k, k^2 + 4, k^3 + 6k, k^4 + 8k^2 + 8$.

Recurrences (1) and (2) involve the characteristic equation

$$x^2 - kx - 2 = 0$$

with roots

$$\alpha = \frac{k + \sqrt{k^2+8}}{2}, \quad \beta = \frac{k - \sqrt{k^2+8}}{2}.$$

Their Binet's formulas are defined by

$$j_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad c_{k,n} = \alpha^n + \beta^n. \tag{3}$$

In this paper we give lower and upper bounds for the spectral norms of the circulant matrices with $k-$Jacobsthal $\{j_{k,n}\}_{n\in\mathbb{N}}$, and the $k-$Jacobsthal Lucas $\{c_{k,n}\}_{n\in\mathbb{N}}$numbers are denoted by $A = C(j_{k,0}, j_{k,1}, ..., j_{k,n-1})$ and $B = C(c_{k,0}, c_{k,1}, ..., c_{k,n-1})$. An (nxn) matrix C is called a circulant matrix if it is of the form for each $i; j = 1; ...; n$ and $k = 0; 1; 2; ...; n$ all the elements $(i; j)$ such that $j - i = k(mod\ n)$. Obviously, a circulant matrix is determined by its first row (or column). It can be denoted by the followig matrix:

$$A = \begin{bmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-1} \\ j_{n-1} & j_0 & j_1 & \cdots & j_{n-2} \\ j_{n-2} & j_{n-1} & j_0 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_1 & j_2 & j_3 & \cdots & j_0 \end{bmatrix}$$

For any $A = [a_{ij}] \in M_{m,n}(C)$. The Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \tag{4}$$

and the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i(A^H A)} \tag{5}$$

where A^H is the conjugate transpose of matrix A. $\lambda_i(A^H A)$ is eigenvalue of $A^H A$.

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F. \tag{8}$$

2. The Sum Formulas of the Square of Jacobsthal and Jacobsthal Lucas Numbers

Proposition 1. *The summation of the squares of k−Jacobsthal numbers is obtained as:*

$$\sum_{i=0}^{n-1} j_{k,i}^2 = \frac{1}{k^2+8} \left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} + 2(-1)^n j_n \right). \tag{9}$$

Proof. By using Binet forms we have

$$\begin{aligned} \sum_{i=0}^{n-1} j_{k,i}^2 &= \sum_{i=0}^{n-1} \left(\frac{\alpha^i - \beta^i}{\alpha - \beta} \right)^2 = \frac{1}{k^2+8} \sum_{i=0}^{n-1} \left(\alpha^{2i} + \beta^{2i} - 2(-2)^i \right) \\ &= \frac{1}{k^2+8} \left(\frac{\alpha^{2n} - 1}{\alpha^2 - 1} + \frac{\beta^{2n} - 1}{\beta^2 - 1} + 2\frac{(-2)^n - 1}{3} \right) \\ &= \frac{1}{k^2+8} \left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} + 2(-1)^n j_n \right). \end{aligned}$$

□

Proposition 2. *The summation of the squares of k−Jacobsthal Lucas numbers is obtained as:*

$$\sum_{i=0}^{n-1} c_{k,i}^2 = \frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} - 2(-1)^n j_n. \tag{10}$$

Proof. By using Binet forms we have

$$\begin{aligned}\sum_{i=0}^{n-1} c_{k,i}^2 &= \sum_{i=0}^{n-1}\left(\alpha^i+\beta^i\right)^2 = \sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}+2(-2)^i\right)\\ &= \frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(-2)^n-1}{3} = \frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_n.\end{aligned}$$

□

3. Lower and Upper Bounds of Circulant Matrices Involving k-Jacobsthal and k-Jacobsthal Lucas Numbers

Theorem 1. *Let $A = C(j_{k,0}, j_{k,1}, ..., j_{k,n-1})$ be circulant matrix with $k-$Jacobsthal numbers, then we obtain*

$$\sqrt{\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)} \le \|A\|_2$$

$$\begin{aligned}\|A\|_2 \le\ & \frac{1}{k^2+8}\sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)}\ *\\ & \sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}+k^2+8\right)}\end{aligned}$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} j_{k,0} & j_{k,1} & j_{k,2} & \cdots & j_{k,n-1}\\ j_{k,n-1} & j_{k,0} & j_{k,1} & \cdots & j_{k,n-2}\\ j_{k,n-2} & j_{k,n-1} & j_{k,0} & \cdots & j_{k,n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ j_{k,1} & j_{k,2} & j_{k,3} & \cdots & j_{k,0}\end{bmatrix}$$

From (5), (8) and (10) we get

$$(\|A\|_F)^2 = n\sum_{k=0}^{n-1} j_k^2 = \frac{n}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)$$

$$\frac{1}{\sqrt{n}}\|A\|_F = \sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)\frac{1}{k^2+8}}$$

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2$$

$$\sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)\frac{1}{k^2+8}} \le \|A\|_2$$

On the other hand, let $A = BoC$ where the matrices B, C are defined as

$$\begin{aligned} B &= \left(b_{ij}\right) = \begin{cases} b_{ij} = j_{(mod_{(j-i,n)})}, & i \ge j\\ b_{ij} = 1, & i < j\end{cases}\\ C &= \left(c_{ij}\right) = \begin{cases} c_{ij} = j_{(mod_{(j-i,n)})}, & i < j\\ c_{ij} = 1, & i \ge j\end{cases}.\end{aligned}$$

It is denoted by matrix form as

$$B = \begin{bmatrix} j_{k,0} & 1 & 1 & \cdots & 1\\ j_{k,n-1} & j_{k,0} & 1 & \cdots & 1\\ j_{k,n-2} & j_{k,n-1} & j_{k,0} & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ j_{k,1} & j_{k,2} & j_{k,3} & \cdots & j_{k,0}\end{bmatrix}, C = \begin{bmatrix} 1 & j_{k,1} & j_{k,2} & \cdots & j_{k,n-1}\\ 1 & 1 & j_{k,1} & \cdots & j_{k,n-2}\\ 1 & 1 & 1 & \cdots & j_{k,n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}.$$

Then using the definitions of maximum row and column length norm, we get

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2} = \sqrt{\sum_{i=0}^{n-1} j_{k,i}^2} \\ &= \sqrt{\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)}, \\ c_1(C) &= \max_{1\le j\le n}\sqrt{\sum_{j=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n}\left|c_{jn}\right|^2} = \sqrt{1+\sum_{i=0}^{n-1} j_{k,i}^2} \\ &= \sqrt{\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}+k^2+8\right)}. \end{aligned}$$

By using (6) we have

$$\begin{aligned} \|A\|_2 &= \|BoC\|_2 \le r_1(B)\,c_1(C) \\ &\le \frac{1}{k^2+8}\sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}\right)} \\ &\quad * \sqrt{\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}+2(-1)^n j_{k,n}+k^2+8\right)}. \end{aligned}$$

Therefore we complete the proof. □

Theorem 2. *Let the elements of the circulant matrix be Jacobsthal Lucas numbers, $A = C(c_{k,0}, c_{k,1}, ..., c_{k,n-1})$, then we obtain*

$$\sqrt{\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}} \le \|A\|_2$$

$$\|A\|_2 \le \sqrt{\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}}\,\sqrt{\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}+1}$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} c_{k,0} & c_{k,1} & c_{k,2} & \cdots & c_{k,n-1} \\ c_{k,n-1} & c_{k,0} & c_{k,1} & \cdots & c_{k,n-2} \\ c_{k,n-2} & c_{k,n-1} & c_{k,0} & \cdots & c_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k,1} & c_{k,2} & c_{k,3} & \cdots & c_{k,0} \end{bmatrix}.$$

From (6), (8) and (9) we get

$$(\|A\|_E)^2 = n\sum_{i=0}^{n-1} c_{k,i}^2 = n\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}\right)$$

$$\begin{aligned} \frac{1}{\sqrt{n}}\|A\|_E &= \sqrt{\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}}, \\ \frac{1}{\sqrt{n}}\|A\|_E &\le \|A\|_2 \\ \sqrt{\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}}-2(-1)^n j_{k,n}} &\le \|A\|_2 \end{aligned}$$

On the other hand, let $A = BoC$ where B, C are defined as

$$B = \left(b_{ij}\right) = \begin{cases} b_{ij} = c_{(mod_{(j-i,n)})}, & i \geq j \\ b_{ij} = 1, & i < j \end{cases}$$
$$C = \left(c_{ij}\right) = \begin{cases} c_{ij} = c_{(mod_{(j-i,n)})}, & i < j \\ c_{ij} = 1, & i \geq j \end{cases} .$$

By the definition of $r_1(A)$, $c_1(C)$, we have

$$\begin{aligned} r_1(B) &= \max_i \sqrt{\sum_{j=1}^{n} \left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n} \left|b_{nj}\right|^2} = \sqrt{\sum_{i=0}^{n-1} c_{k,i}^2} \\ &= \sqrt{\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} - 2(-1)^n j_{k,n}} \\ c_1(C) &= \max_j \sqrt{\sum_{j=1}^{n} \left|c_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n} \left|c_{nj}\right|^2} = \sqrt{1 + \sum_{i=0}^{n-1} c_{k,i}^2} \\ &= \sqrt{\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} - 2(-1)^n j_{k,n} + 1} \end{aligned}$$

By using (6) we have

$$\begin{aligned} \|A\|_2 &= \|BoC\|_2 \leq r_1(B)\, c_1(C) \\ &= \sqrt{\left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} 2(-1)^n j_{k,n}\right)\left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} + 2(-1)^n j_{k,n} + 1\right)} \end{aligned}$$

□

Theorem 3. *Let $A = C(j_{k,0}, j_{k,1}, ..., j_{k,n-1})$ and $B = C(c_{k,0}, c_{k,1}, ..., c_{k,n-1})$ be circulant matrix with k−Jacobsthal and the k−Jacobsthal Lucas numbers, then the Euclidean norm of the Kronecker product of these matrices is*

$$\|A \otimes B\|_E = n \sqrt{\begin{array}{c} \frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}} + 2(-1)^n j_{k,n}\right) \\ *\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}} + 2(-1)^n j_{k,n}\right) \end{array}} .$$

Proof. By using (7), (9) and (10), we obtain

$$\begin{aligned} (\|A \otimes B\|_E)^2 &= \|A\|_E^2 \|B\|_E^2 = \left(n \sum_{s=0}^{n-1} J_{k,s}^2\right)\left(n \sum_{s=0}^{n-1} C_{k,s}^2\right) \\ (\|A \otimes B\|_E)^2 &= n^2 \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} + 2(-1)^n j_{k,n}\right)\right] \\ &\quad * \left[\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}} - 2(-1)^n j_{k,n}\right] \end{aligned}$$

$$\|A \otimes B\|_E = n \sqrt{\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2} - c_{k,2n} - c_{k,2} + 2}{5 - c_{k,2}}\right)^2 - \left(2(-1)^n j_{k,n}\right)^2}.$$

So the proof is completed. □

Theorem 4. *Let $A = C(j_{k,0}, j_{k,1}, ..., j_{k,n-1})$ and $B = C(c_{k,0}, c_{k,1}, ..., c_{k,n-1})$ be circulant matrix with k−Jacobsthal and the k−Jacobsthal Lucas numbers, then the upper bound for the spectral norm of the Hadamard product of these matrices is*

$$\|AoB\|_2 \leq \frac{n}{\sqrt{k^2+8}} \sqrt{\begin{array}{c} \left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}} + 2(-1)^n j_{k,n}\right) \\ *\left(\frac{4c_{k,2n-2}-c_{k,2n}-c_{k,2}+2}{5-c_{k,2}} - 2(-1)^n j_{k,n}\right) \end{array}}$$

Proof. The proof is seen easily by using $||AoB||_2 \leq ||A||_2 \, ||B||_2$, by using (9) and (10). □

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