Journal of Mathematics Research; Vol. 8, No. 6; December 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Geometry of the 3D Pythagoras' Theorem

Luis Teia[1]

[1] Berlin, Germany

Correspondence: Dr. Luis Teia, Mehrower Allee 82, 12687 Berlin, Germany. E-mail: luistheya@gmail.com

Received: May 23, 2016 Accepted: July 19, 2016 Online Published: November 25, 2016

doi:10.5539/jmr.v8n6p78 URL: http://dx.doi.org/10.5539/jmr.v8n6p78

Abstract

This paper explains step-by-step how to construct the 3D Pythagoras' theorem by geometric manipulation of the two dimensional version. In it is shown how $x + y = z$ (1D Pythagoras' theorem) transforms into $x^2 + y^2 = z^2$ (2D Pythagoras' theorem) via two steps: a 90-degree rotation, and a perpendicular extrusion. Similarly, the 2D Pythagoras' theorem transforms into 3D using the same steps. Octahedrons emerge naturally during this transformation process. Hence, each of the two dimensional elements has a direct three dimensional equivalent. Just like squares govern the 2D, octahedrons are the basic elements that govern the geometry of the 3D Pythagoras' theorem. As a conclusion, the geometry of the 3D Pythagoras' theorem is a natural evolution of the 1D and 2D. This interdimensional evolution begs the question – Is there a bigger theorem at play that encompasses all three?

Keywords: Geometry, Pythagoras, theorem, three dimensions

1. Introduction

For the past 2,500 years, the Pythagoras' theorem, arguably the most well known theorem in the world, ranging from the times of Babylon to today, has greatly helped mankind to evolve (Friberg 1981, Roy 2003, Strogaz 2015). Its useful right angles are everywhere, whether it is a building, a table, a graph with axes, the properties of light (Einstein, 1961) or the atomic structure of a crystal (Kelly, 2005). Its numerous proofs are universally applicable (Maor, 2007), but still the Pythagoras' theorem is exclusively bound to two dimensions. Since we live in a three dimensional world, the awareness of this gap in knowledge begs the question - How does the Pythagoras' theorem looks like in three dimensions?

2. Theory

2.1 The 1D and 2D Pythagoras' Theorem

Let us start with the simple one dimensional version of the Pythagoras' theorem. The 1D Pythagoras' theorem is governed by line segments, where the geometric addition of two line segments gives a third, or $x + y = z$ (Figure 2a). This is by definition the mathematical process of summation. This can be transformed into the 2D Pythagoras' theorem via the following two steps:

> **Step 1.** Rotating line x by 90 degrees about the middle point gives a right-angled triangle (Figure 2b).
> **Step 2.** Extruding all lines perpendicularly to their length gives squares. These squares result is the two dimensional version of the Pythagoras' theorem, in which the geometric addition of two squares gives a third, or $x^2 + y^2 = z^2$ (Figure 2c).

3. Hypothesis

It is assumed that the process used to convert the 1D Pythagoras' theorem into 2D (Figure 1) also applies from 2D to 3D. It is also assumed that the 3D Pythagoras' theorem has the same geometric elements as in 2D.

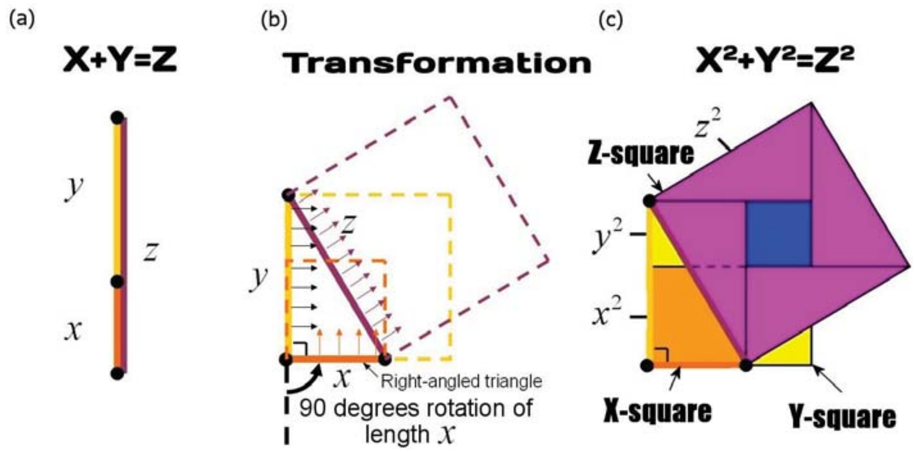

Figure 1. The geometry of (a) $x + y = z$, (b) the transformation, and (c) $x^2 + y^2 = z^2$.

4. Development

4.1 Location of the Axes

The central square theory (Teia, 2015) is used to determine how, and where, the 2D Pythagoras' theorem is rotated when transforming into 3D. The theory states that the right side of the equation $x^2 + y^2 = z^2$ is composed geometrically of four right-angled triangles rotating around a central square $(y - x)^2$, which in turn when enclosed forms a new square about which other Pythagorean triples revolve (Figure 3a). This means that all triangles in the ternary tree of Pythagorean triples relate to each other via intermediate squares (Teia, 2016). Figure 3b shows the geometrical representation of the Pythagoras' family of triples constructed with the central square theory. For simplicity, from now on x^2, y^2, z^2 and $(y-x)^2$ are termed X-square, Y-square, Z-square and central square, respectively.

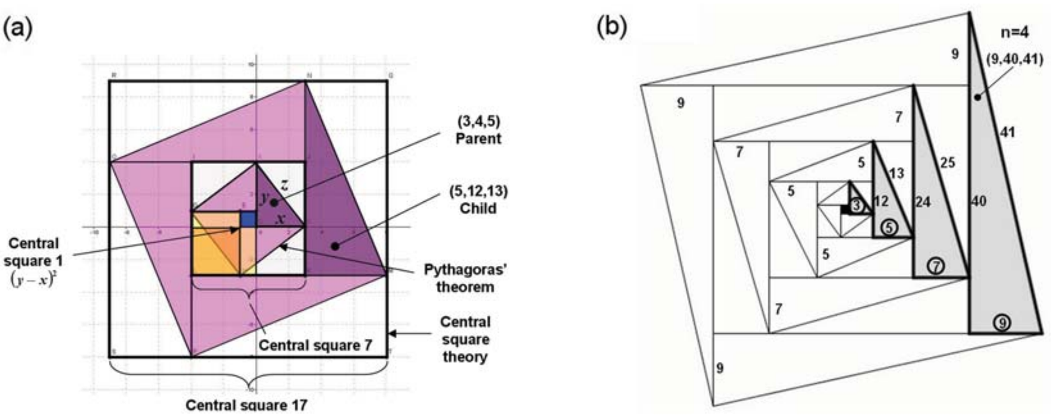

Figure 2. (a) The central square theory, and (b) the Pythagoras' family of triples (Teia, 2015).

More importantly, Figure 2b shows that the left side of the Pythagoras' theorem $x^2 + y^2 = z^2$ is governed by an axis of symmetry (redrawn and highlighted in Figure 3a). Similarly, the right side of the Pythagoras' theorem is composed of right-angled triangles revolving around an axis of rotation (redrawn and highlighted in Figure 3b). Ultimately, Figure 3c shows that central squares not only interconnect the left side of the Pythagoras' theorem (Figure 3a) with the right side (Figure 3b), but also they connect the axis of symmetry with the axis of rotation. That is, the squares are both symmetric about the axis of symmetry, as well as, revolve around the axis of rotation.

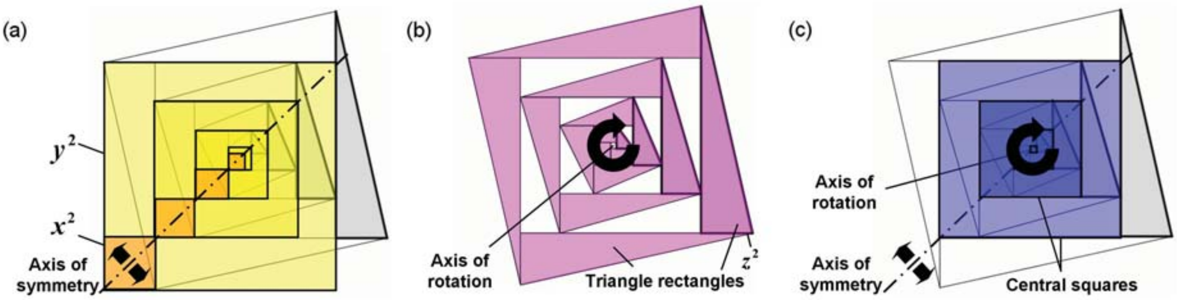

Figure 3. Definition of (a) axis of symmetry, (b) axis of rotation, and (c) their connection via central squares.

4.2 The 3D Pythagoras' Theorem

The symmetry and rotation axes guide the two-step transformation (similar as defined previously in section 2.1) from $x^2 + y^2 = z^2$ to $x^3 + y^3 = z^3$. For simplicity, from now on $\sqrt{3}/2x^3$, $\sqrt{3}/2y^3$, $\sqrt{3}/2z^3$ and $\sqrt{3}/2(y-x)^3$ are termed X-, Y-, Z- and central octahedron, respectively. The steps are described as follows:

Step 1. Rotating the Z-square and central square by 90 degrees with respect to the X- and Y-square about the diagonal axis (Figure 4a) forms automatically a central octahedron (Figure 4b). Both the central square (2D) and central octahedron (3D) are a direct and inherent consequence of the process of transformation from 1D to 2D, and 2D to 3D, respectively. Hence, the geometric element that governs the 3D Pythagoras' theorem is an octahedron.

Step 2. Extruding all squares perpendicularly to their surfaces gives octahedrons. Finally, careful partition of the octahedrons gives the 3D Pythagoras' theorem shown in Figure 4c, which is equivalent to the 2D Pythagoras' theorem in Figure 4a.

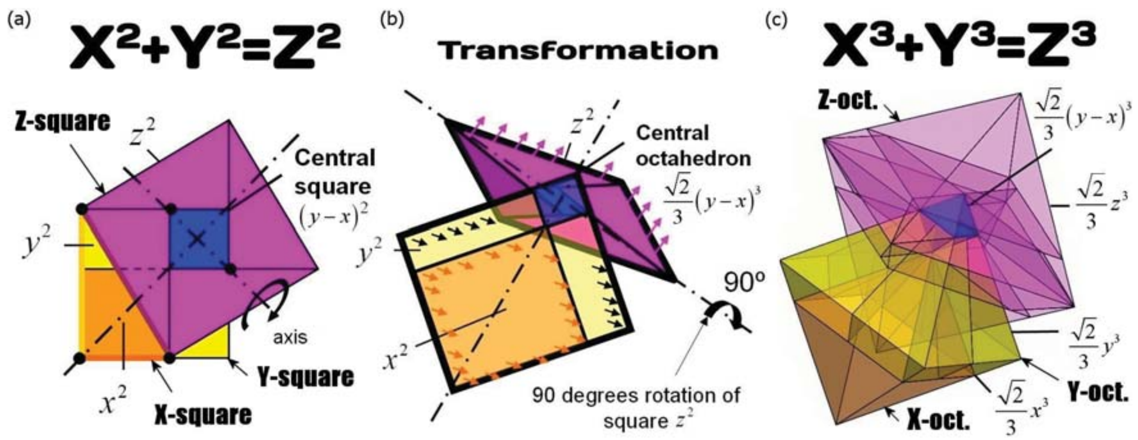

Figure 4. The geometry of (a) $x^2 + y^2 = z^2$, (b) the transformation, and (c) $x^3 + y^3 = z^3$.

The remainder of the paper focuses on explaining in more detail these two steps. First, the transformation of the X- and Y-squares into octahedrons is explained (section 4.3). Then, the formation of the 3D geometric equivalent to the symmetry rectangles is described (section 4.4). Similarly, the transformation of the Z-square into an octahedron is explained (section 4.5), and subsequently the formation of the 3D geometric equivalent to the revolving triangle is explained (section 4.6). Finally, the article concludes with the assembly of all 3D elements into the geometry of the 3D Pythagoras' theorem (shown above in Figure 4c). The details of the mathematical work involved are explained in Teia's book (2015).

4.3 Transformation of X- and Y-squares into Octahedrons

The axis of symmetry shapes the geometry of the X- and Y-squares into its 3D equivalent. Let us start by looking at the left side of the Pythagoras' theorem. The left hand side of equation $x^2 + y^2 = z^2$ is geometrically composed of the X-square, and the central square, both superimposed on to the Y-square (Figure 5a). The area outside X-square and central square, but still inside the Y-square, forms the symmetry rectangles. These have a combined area $\sum A_s = 2x(y-x)$.

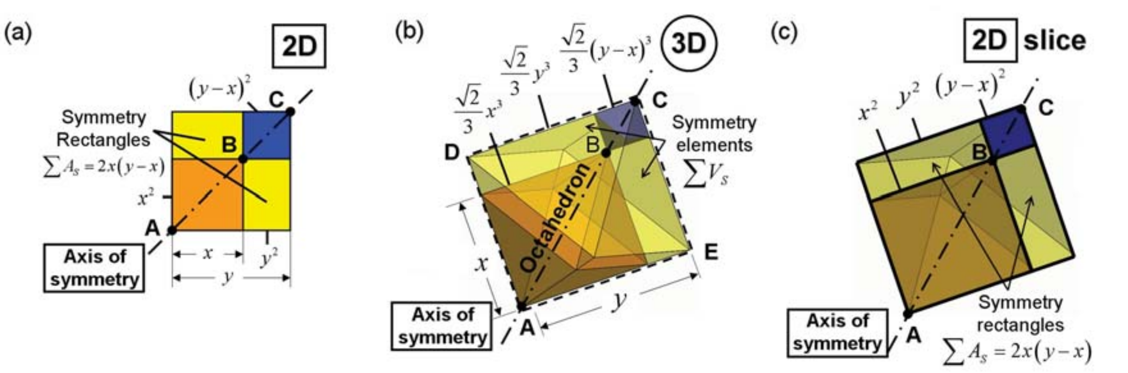

Figure 5. The geometrical transformation to 3D of x^2 and y^2.

The general rule to geometrically connect these elements is – they all share common vertices:

Point A connects the lower left corner of the X- and Y-square.
Point B connects the upper right corner of the X-square with lower left corner of the central square.
Point C connects the upper right corner of the Y-square with upper right corner of the central square.

It is assumed that when passing to three dimensions, the common points between elements (A, B and C in Figure 5a) remain the same. That is, the extreme corners of the three elements need to coincide with the axis of symmetry. Following the three-point ABC rule, the squares in Figure 5a are extruded around the axis of symmetry and form the octahedrons in Figure 5b. Taking a sectional view ADCE gives the two dimensional slice in Figure 5c, that is the same as Figure 5a. As one can imagine, the elements change size as point B translates along the axis of symmetry. In other words, as x changes towards or away from y, point B slides along the axis of symmetry between points A and C. The left hand side of equation $x^3 + y^3 = z^3$ is therefore geometrically composed of X- and central octahedron overlapping onto the Y-octahedron. The volume outside the X- and central octahedron, but still inside Y-octahedron, forms the symmetry elements that possess a combined volume of $\sum V_s$. This is similar to the symmetry rectangles with a combined area $\sum A_s$, but in three dimensions.

4.4 The Symmetry Element

Just as the symmetry rectangles are symmetric about the diagonal axis, the three-dimensional symmetry element is also symmetric about the vertical axis. The symmetry element is found by splitting the Y-octahedron as follows. Splitting along a vertical plane gives half of the Y-octahedron (Figure 6a). The same element needs to revolve around the axis, hence splitting it again about the perpendicular plane results in a segment that is the same in all quadrants (Figure 6b). The element is further split along diagonal plane resulting in a segment whose shape is an eigth of the y-octahedron (Figure 6c). Since the symmetry rectangle does not overlap with other squares, neither does the symmetry element. Plotting a plane along the surface of the X-octahedron and cutting (Figure 6d) and another along the surface of the central octahedron and cutting again, results in the final shape of the revolving element (Figure 6f). As seen, 8 symmetry elements accommodate between the X-octahedron and the central octahedron, giving the end volume of the Y-octahedron. This is equivalent to the 2D case where two symmetry rectangles accommodate between the X-square and the central square, giving the area of the Y-square.

4.5 Transformation of the Z-square into an Octahedron

The axis of symmetry shapes the geometry of the Z-square into its 3D equivalent. The right hand side of equation $x^2 + y^2 = z^2$ is geometrically composed of four congruent right-angled triangles rotated around a central square, and hence also around an axis of rotation (Figure 7a). There are specific rules that define how the elements revolve around the central square. These are inferred from Figure 7a as:

Rule 1. The extended sides of the central square are in line with the vertices of the Z-square.
Rule 2. The elements revolving around the axis are identical, and have as shape right-angled triangles.

The three dimensional equivalent to Figure 7a is built by extruding the Z- and central squares perpendicularly along the axis of rotation forming two pyramids for each, which combined form two octahedrons (Figure 7a becomes Figure 7b). Taking a sectional view FGHI gives the two dimensional slice in Figure 7c, which is the same as Figure 7a. The central element is the central octahedron, and its midplane is the central square. The central octahedron was clocked

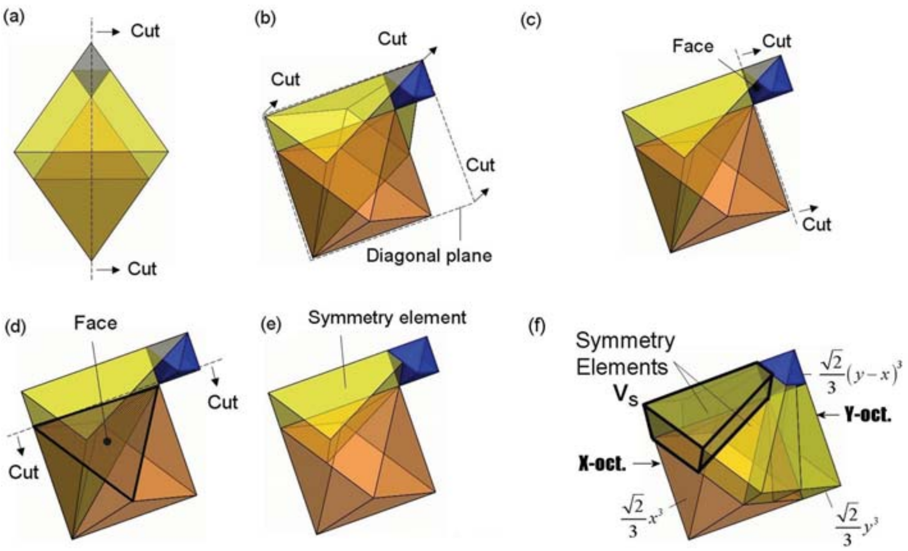

Figure 6. Formation of the symmetry element.

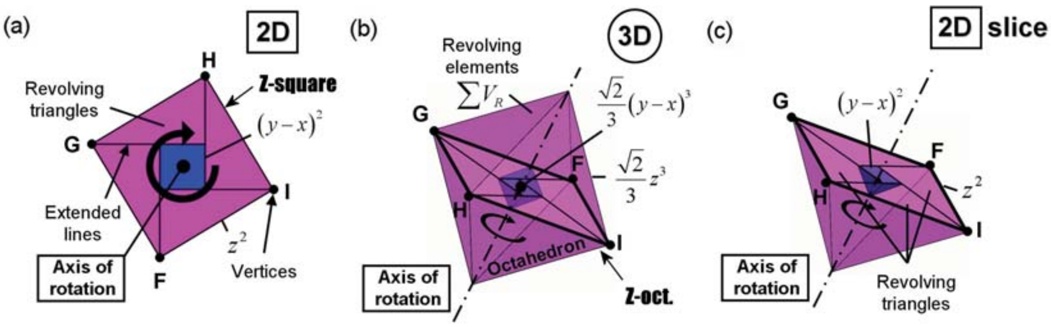

Figure 7. The geometrical transformation to 3D of z^2.

around the axis of rotation until its edges aligned with the corners FGHI of the Z-octahedron, satisfying Rule 1. The lines that fan out at midplane from the central octahedron indicate where the Z-octahedron needs to be split to form the revolving element, satisfying Rule 2. Even though the midplane section has the same ”footprint” as 2D (i.e., a right-angled triangle), the 3D element that revolves around the central octahedron is not regular. The revolving element (not yet shown) is the 3D equivalent to the revolving triangle (Figure 7a). The volume outside the central octahedron, but still inside the Z-octahedron, forms the revolving elements that possess a combined volume of $\sum V_r$.

4.6 The Revolving Element

The revolving element is found by splitting the Z-octahedron as follows. Making an horizontal cut at the top and bottom edges of the central octahedron (Figure 8a) forms two pyramids and a central section (Figure 8b). Cutting at midplane and deleting the bottom allows us to simplify using symmetry. The element needs to follow the 2D rules, that is, its base is the 2D right-angled triangle. First, make a diagonal plane along one face of the central octahedron, and along one side of the triangle (Figure 8c). Second, another diagonal cut along another adjacent face and triangle side (Figure 8d).

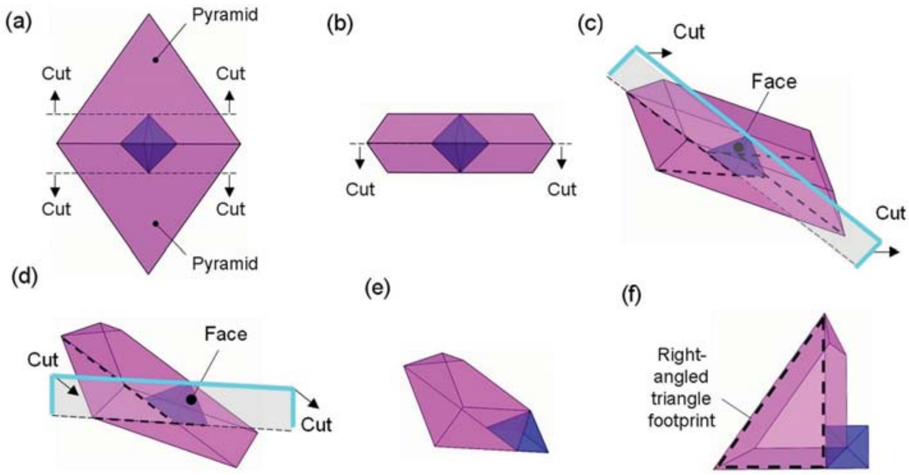

Figure 8. Formation of the revolving element.

The result is the shape of the rotating element in three dimensions (Figure 8e), which is the 3D equivalent to the right-angled triangle in 2D. Figure 8f shows it from a top view, where its compliance with the base right-angled triangle is observable. Note that the 3D Pythagoras' theorem needs to comply with the rules of the 2D Pythagoras' theorem.

Now to construct the new assembly of Z-octahedron, Figure 9a-d shows the gradual addition of four revolving elements around the central octahedron. Placing the pyramid on top completes the upper half of the Z-octahedron in Figure 9e. Adding the lower half completes the Z-octahedron (Figure 9f).

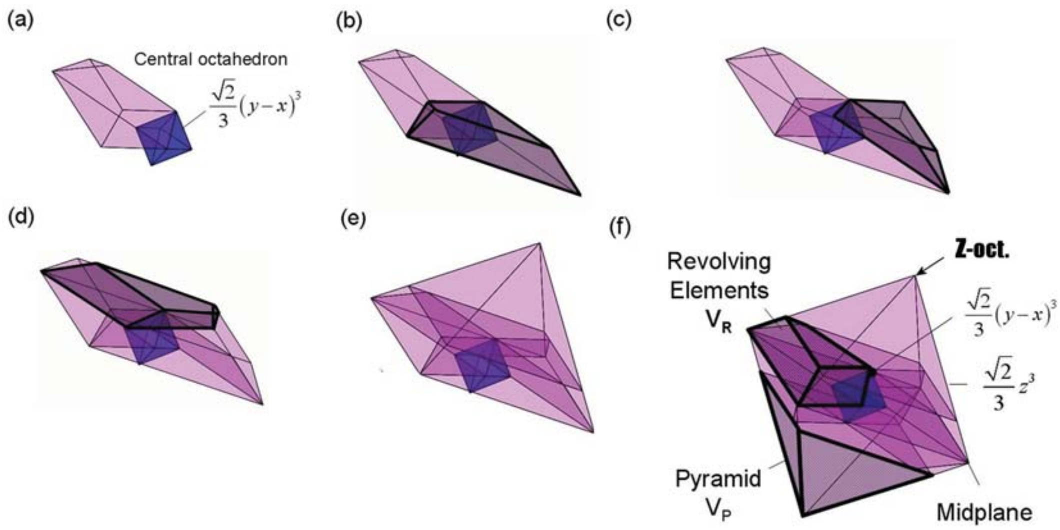

Figure 9. Assembly of the right hand side of $x^3 + y^3 = z^3$.

4.7 Final Assembly

The central octahedron connects the left hand side and right hand side. In three dimensions, the Y-octahedron contains symmetry elements, just like in two dimensions the Y-square contains symmetry rectangles. Similarly, in three dimensions, the Z-octahedron contains revolving elements, just like in two dimensions the Z-square contains revolving triangles. Merging the central octahedron on the left hand side (Figure 6f) to the central octahedron on the right hand side (Figure 9f) completes the geometry of $x^3 + y^3 = z^3$, shown in Figure 10 alongside with the geometric representation of the other dimensions, i.e. $x + y = z$ and $x^2 + y^2 = z^2$.

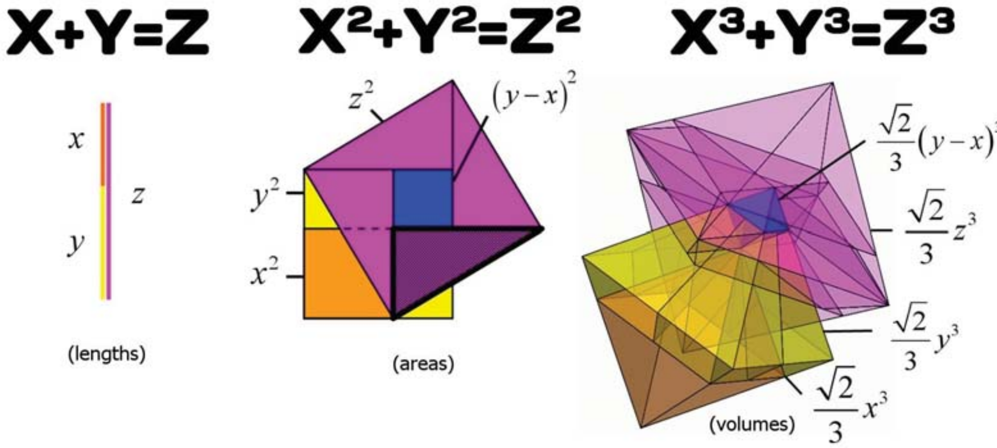

Figure 10. The 1D, 2D and 3D Pythagoras' theorem.

5. Conclusion

The three dimensional version of the Pythagoras' theorem is derived by geometric manipulation of the two dimensional version. Consequentially, they are interconnected. The 1D Pythagoras' theorem transforms to 2D in two steps: a 90-degree rotation, and a perpendicular extrusion. The 2D Pythagoras' theorem transforms to 3D using the same steps. Both the two and three dimensional version are governed by a symmetry and rotation axis, identified using the central square method. An octahedron emerges naturally during the process. Hence, just like squares in 2D, octahedrons are the basic elements that govern the geometry of the 3D Pythagoras' theorem. Each two dimensional element has a direct three dimensional equivalent. The geometry of the 3D Pythagoras' theorem is a natural evolution of the 1D and 2D. This evolution begs the question – Is there a bigger theorem at play that encompasses all three?

References

Einstein, A. (1961). *Relativity: The Special and the General Theory*, (Vol. 2). Crown Trade Paperbacks.

Friberg, J. (1981). Methods and Traditions of Babylonian Mathematics. *Historia Mathematica, 8*, 227–318. https:/doi.org/10.1016/0315-0860(81)90069-0

Kelly, B. S. and Splittgerber, A. G. (2005). The Pythagorean Theorem and the Solid State, *Journal of Chemical Education, 82*, 756

Maor, E. (2007). *The Pythagorean Theorem: A 4,000-year History*, Princeton University Press.

Roy, R. (2003). Babylonian Pythagoras' theorem, the Early History of Zero and a Polemic on the Study of the History of Science, *Resonance Journal, 8*, 30–40

Strogaz, S. (2015). *Einstein's Boyhood Proof of the Pythagorean Theorem*, The New Yorker, Retrieved from http://www.newyorker.com/tech/elements/einsteins-first-proof-pythagorean-theorem

Teia, L. (2015). Pythagoras' triples explained by central squares, *Australian Senior Mathematical Journal, 29*, 7–15

Teia, L. (2015). $X^3 + Y^3 = Z^3$*: The Proof*, self-published with Amazon.

Teia, L. (2016). Anatomy of the Pythagoras' tree, *Australian Senior Mathematical Journal, 30*, 38–47

Copyrights