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γ-Max Labelings of Graphs

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Abstract

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces an edge-labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ on G defined by

$$f'(e) = |f(u) - f(v)|, \quad \text{for each edge } e = uv \text{ in } E(G).$$

The value of f is defined as

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

The maximum value of a γ-labeling of G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a -labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

In this paper, we give an alternative short proof by mathematical induction to achieve the formulae for $\text{val}_{\max}(K_{r,s})$ and $\text{val}_{\max}(K_n)$.

Keywords: γ-labeling, value of a γ-labeling

1. Introduction

Let G be a graph of order n and size m. A *γ-labeling* of G is defined in (Chartrand, Erwin, VanderJagt & Zhang, 2005) as a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m\}$ that induces an *edge-labeling* $f'\colon E(G) \to \{1, \ldots, m\}$ on G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The *value* of f is defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

If the edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling*. Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a paper of Rosa (Rosa, 1966), who used the term β-valuations. A few years later, Golomb (Golomb, 1972) called these labelings "graceful" and this is the terminology that has been used since then.

Gallian (Gallian, 2009) has written an extensive survey on labelings of graphs. The subject of γ-labelings of graphs was studied in (Bullington, Eroh, & Winters, 2010; Chartrand, Erwin, VanderJagt, & Zhang, 2005; Crosse, Okamoto, Saenpholphat, & Zhang, 2007; Fonseca, Saenpholphat, & Zhang, 2013; Fonseca, Khemmani, & Zhang, 2015; Fonseca, Saenpholphat, & Zhang, 2011; Khemmani & Saduakdee, 2015, 2016).

Obviously, since γ-labeling f of a graph G of order n and size m is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\text{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n - 1$ and every connected graph has a γ-labeling.

The *maximum value* and the *minimum value* of a γ-labeling of G are defined in (Chartrand, Erwin, VanderJagt, & Zhang, 2005) as

$$\text{val}_{\max}(G) = \max\{\text{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\text{val}_{\min}(G) = \min\{\text{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\},$$

respectively. A γ-labeling g of G is a γ-max *labeling* if $\text{val}(g) = \text{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\text{val}(h) = \text{val}_{\min}(G)$. Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

Since $\text{val}(f_1) = 4$ and the size of $P_5 = 4$, it follows that f_1 is a γ-min labeling of P_5. It is shown in (Chartrand, Erwin, VanderJagt, & Zhang, 2005) that the γ-labeling f_9 is a γ-max labeling of P_5.

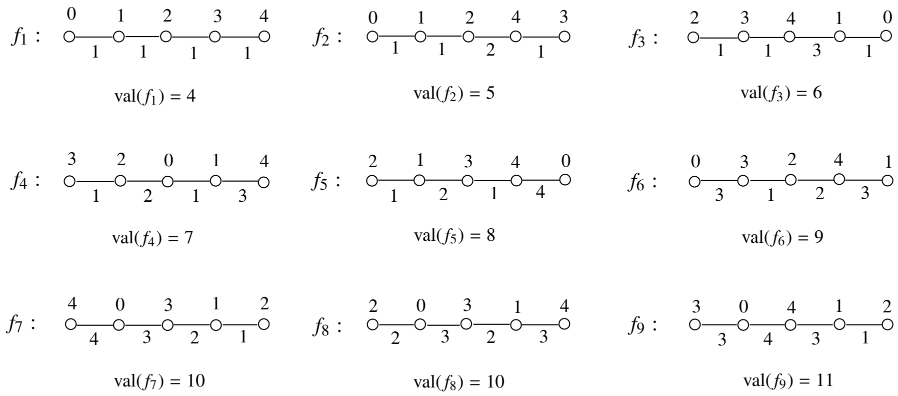

Figure 1. Some γ-labelings of P_5

By the *γ-spectrum* of a graph G, we mean the set

$$\text{spec}(G) = \{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

Observe that $\text{val}_{\min}(G), \text{val}_{\max}(G) \in \text{spec}(G)$ for every graph G.

For integers a and b with $a < b$, let

$$[a, b] = \{a, a+1, \ldots, b\}$$

be a *consecutive set* of integers between a and b.

Thus for every graph G,

$$\text{spec}(G) \subseteq [\text{val}_{\min}(G),\ \text{val}_{\max}(G)].$$

The *span* of a γ-labeling f of a graph G is defined as

$$\text{span}(f) = \max\{f(v) : v \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

Consequently, if $G \cong P_5$, then $\text{spec}(G) = \{4, 5, 6, 7, 8, 9, 10, 11\}$ and for each γ-labeling f of G, $\text{span}(f) = 4 - 0 = 4$.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\bar{f}\colon V(G) \to \{0, 1, \ldots, m\}$ of f is defined by

$$\bar{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\bar{f}$ a γ-labeling of G as well but $\text{val}(\bar{f}) = \text{val}(f)$. This gives us the following.

Observation 1 (Chartrand, Erwin, VanderJagt, & Zhang, 2005) *Let f be a γ-labeling of a graph G. Then f is a γ-max labeling (γ-min labeling) of G if and only if $\bar{f}$ is a γ-max labeling (γ-min labeling) of G.*

The following result appeared in (Fonseca, Khemmani, & Zhang, 2015) is useful to us.

Theorem 1 (Fonseca, Khemmani, & Zhang, 2015) *If f is a γ-max labeling of a nontrivial graph G of order n and size m, then $\{0, m\} \subseteq f(V(G))$.*

In (Bullington, Eroh & Winters, 2010; Chartrand, Erwin, VanderJagt, & Zhang, 2005), the maximum and minimum values of a γ-labeling of path P_n, cycle C_n, complete graph K_n, double star $S_{p,q}$ and complete bipartite graph $K_{r,s}$ are determined.

For any positive integers n, m, Δ with $\Delta \geq m$, let G be a nontrivial graph of order n and size m, a γ^{Δ}*-labeling* of G is defined in (Fonseca, Saenpholphat, & Zhang, 2013) as a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m, m+1, \ldots, \Delta\}$ that induces an *edge-labeling* $f' : E(G) \to \{1, 2, \ldots, \Delta\}$ on G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The *value* of f is defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

The *maximum value* of a γ^{Δ}-labeling of G is

$$\mathrm{val}^{\Delta}_{\max}(G) = \max\{\mathrm{val}(f) : \ f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}.$$

The *minimum value* of a γ^{Δ}-labeling of G is

$$\mathrm{val}^{\Delta}_{\min}(G) = \min\{\mathrm{val}(f) : \ f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}.$$

A γ^{Δ}-labeling g of G is a γ^{Δ}-max *labeling* if $\mathrm{val}(g) = \mathrm{val}^{\Delta}_{\max}(G)$ and a γ^{Δ}-labeling h is a γ^{Δ}-min *labeling* if $\mathrm{val}(h) = \mathrm{val}^{\Delta}_{\min}(G)$.
Note that $\mathrm{val}_{\max}(G) = \mathrm{val}^{\Delta}_{\max}(G)$ and $\mathrm{val}_{\min}(G) = \mathrm{val}^{\Delta}_{\min}(G)$ when $\Delta = m$.

We first make the following observation for γ^{Δ}-max and γ^{Δ}-min labelings of graphs.

Observation 2 *Let f be a γ^{Δ}-labeling of a graph G. Then f is a γ^{Δ}-*max *labeling (γ^{Δ}-*min *labeling) of G if and only if $\bar{f}$ is a γ^{Δ}-*max *labeling (γ^{Δ}-*min *labeling) of G.*

In 2010, the explicit formula for $\mathrm{val}_{\max}(K_{r,s})$ and the standard form for γ-max labeling of complete bipartite graph $K_{r,s}$ were determined by Bullington, Eroh and Winters (Bullington, Eroh, & Winters, 2010). Later, Fonseca, Khemmani and Zhang (Fonseca, Khemmani, & Zhang, 2015) in 2015 presented the alternative proof of the formula for $\mathrm{val}_{\max}(K_{r,s})$ that employs γ-min labelings of complete graphs, as we state next.

Theorem 2 (Bullington, Eroh & Winters, 2010) *For any two positive integers $r \geq s$,*

$$\mathrm{val}_{\max}(K_{r,s}) = rs\left(rs - \frac{1}{2}(r+s) + 1\right).$$

Theorem 3 (Bullington, Eroh, & Winters, 2010) *Let f be a γ-labeling of complete bipartite graph $K_{r,s}$ with partite sets V_r and V_s of cardinality r and s, respectively. Then f is a γ-*max *labeling of $K_{r,s}$ if and only if*

$$1. f(V_r) = [0, r-1] \ \textit{and} \ f(V_s) = [rs - (s-1), rs], \ \textit{or}$$

$$2. f(V_r) = [rs - (r-1), rs] \ \textit{and} \ f(V_s) = [0, s-1]\,.$$

In 2005, the maximum value of γ-labeling of complete graph K_n was determined by Chartrand et al. (Chartrand, Erwin, VanderJagt, & Zhang, 2005). As well, the authors (Fonseca, Khemmani, & Zhang, 2015) characterized the γ-max labeling of complete graph K_n in 2015.

Theorem 4 (Chartrand, Erwin, VanderJagt, & Zhang, 2005) *For every positive integer n,*

$$\mathrm{val}_{\max}(K_n) = \begin{cases} \frac{(n^2-1)(3n^2-5n+6)}{24} & \textit{if } n \textit{ is odd} \\ \frac{n(3n^3-5n^2+6n-4)}{24} & \textit{if } n \textit{ is even.} \end{cases}$$

Theorem 5 (Fonseca, Khemmani, & Zhang, 2015) *Let f be a γ-labeling of a complete graph K_n. Then f is a γ-*max *labeling of K_n if and only if*

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] & \textit{if } n \textit{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] \cup \{k\} & \textit{if } n \textit{ is odd}, \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$.

The goal of this paper is to present an alternative approach to formulae for $\mathrm{val}_{\max}(K_{r,s})$ and $\mathrm{val}_{\max}(K_n)$ proved by mathematical induction.

The reader is referred to Chartrand and Zhang (Chartrand & Zhang, 2005) for basic definitions and terminology not mentioned here.

2. γ^{Δ}-max Labelings of Graphs

In this section, we begin our investigation for γ^{Δ}-max labeling of any nontrivial graph by presenting a useful lemma.

Lemma 1 *Let f be a γ^{Δ}-*max *labeling of a nontrivial graph G of order n and size m. Let $u, w \in V(G)$ with $f(u) = \min\{f(v)\colon v \in V(G)\}$ and $f(w) = \max\{f(v)\colon v \in V(G)\}$. Then neighborhoods of u and w are not empty.*

Proof. For any nontrivial connected graph G, it is obvious that $|N(u)|$ and $|N(w)|$ are not empty. Let G be a disconnected graph. We will show that $|N(u)| \neq 0$ and $|N(w)| \neq 0$ Assume, to the contrary, that $|N(u)| = 0$ or $|N(w)| = 0$.

Case 1. $|N(u)| = 0$.
Then u is an isolated vertex of G. Since G has a γ^{Δ}-max labeling, $\Delta \geq \mathrm{m} \geq \mathrm{n} - 1$. Therefore, there is a component G_1 of G with $|V(G_1)| \geq 2$. Let $x \in V(G_1)$ with $f(x) = \min\{f(v)\colon v \in V(G_1)\}$. Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} f(x) & \text{if } v = u \\ f(u) & \text{if } v = x \\ f(v) & \text{if } v \neq u, x. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (g(v) - g(x)) \\ &= \mathrm{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (f(v) - f(u)) \\ &= \mathrm{val}(f) + |N(x)|(f(x) - f(u)) \\ &> \mathrm{val}(f), \end{aligned}$$

which is a contradiction.

Case 2. $|N(w)| = 0$.
By a similar argument, this leads to a contradiction with the maximum value of a γ^{Δ}-labeling of G. □

We now show formula for span of γ^{Δ}-max labelings of graphs.

Proposition 1 *Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-*max *labeling of G, then* $\mathrm{span}(f) = \Delta$.

Proof. Let f be a γ^{Δ}-max labeling of G. Let $u, w \in V(G)$ with $f(u) = \min\{f(v)\colon v \in V(G)\}$ and $f(w) = \max\{f(v)\colon v \in V(G)\}$. Then $f(u) \geq 0$ and $f(w) \leq \Delta$. Assume, to the contrary, that $span(f) < \Delta$. Then $f(w) - f(u) < \Delta$. Therefore, $f(u) > 0$ or $\Delta - \mathrm{f(w)} > 0$.

Case 1. $f(u) > 0$.
Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} 0 & \text{if } v = u \\ f(v) & \text{if } v \neq u. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v \in N(u)} (f(v) - f(u)) + \sum_{v \in N(u)} (g(v) - g(u)) \\ &= \mathrm{val}(f) - \sum_{v \in N(u)} (f(v) - f(u)) + \sum_{v \in N(u)} (f(v) - 0) \\ &= \mathrm{val}(f) + |N(u)|(f(u) - 0) \\ &> \mathrm{val}(f) \quad \text{(by Lemma 1)}, \end{aligned}$$

which is a contradiction.

Case 2. $\Delta - \mathrm{f(w)} > 0$.
A similar argument to the one used in Case 1 leads to a contradiction with the maximum value of a γ^{Δ}labeling of G. □

This also provides the following corollary.

Corollary 1 *Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-*max *labeling of G, then* $\{0, \Delta\} \subseteq f(V(G))$.

3. γ-max Labelings of Complete Bipartite Graphs

We define the γ^{Δ}-*spectrum* of a graph G by

$$\text{spec}^{\Delta}(G) = \{\text{val}(f)\colon\ f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}.$$

Consequently, $\{\text{val}^{\Delta}_{\min}(G), \text{val}^{\Delta}_{\max}(G)\} \subseteq \text{spec}^{\Delta}(G)$ for every graph G. As an illustration, we now establish the γ^{Δ}-spectrum of a star $K_{1,s}$.

Proposition 2 *For positive integers s, Δ with $\Delta \geq s$,*

$$\text{spec}^{\Delta}(K_{1,s}) = \left\{\binom{\Delta - k + 1}{2} - \binom{\Delta - s + 1}{2} + \binom{k+1}{2} \colon 0 \leq k \leq \Delta\right\}.$$

Proof. Let $K_{1,s}$ be a star with $V(K_{1,s}) = \{v\} \cup V_s$ where v is a central vertex and $V_s = \{v_1, v_2, \ldots, v_s\}$ and f a γ^{Δ}-labeling of a graph $K_{1,s}$ with $f(v) = k$ where $0 \leq k \leq \Delta$.
If $k = 0$, then we may assume that $f(v_i) = \Delta - (s - i)$ for all $1 \leq i \leq s$. Then

$$\text{val}(f) = \sum_{i=1}^{s} |f(v_i) - f(v)| = \sum_{i=1}^{s}(\Delta - (s - i)) = \binom{\Delta + 1}{2} - \binom{\Delta - s + 1}{2}.$$

If $k = \Delta$, then by Observation 2,

$$\text{val}(f) = \binom{\Delta + 1}{2} - \binom{\Delta - s + 1}{2}.$$

If $0 < k < \Delta$, then we may assume that

$$f(v_i) = \begin{cases} i - 1 & \text{if } 1 \leq i \leq k \\ \Delta - (s - i) & \text{if } k + 1 \leq i \leq s\,. \end{cases}$$

Therefore,

$$\begin{aligned} \text{val}(f) &= (k + (k-1) + \cdots + 1) + ((\Delta - (s-1)) + (\Delta - (s-2)) + \cdots + (\Delta - k)) \\ &= \tbinom{k+1}{2} + \tbinom{\Delta-k+1}{2} - \tbinom{\Delta-s+1}{2}, \end{aligned}$$

as desired. □

In Proposition 2, we considered γ^{Δ}-spectrum of a star $K_{1,s}$. We are now ready to compute the maximum value of a γ^{Δ}-labeling of $K_{1,s}$.

Corollary 2 *For positive integers s, Δ with $\Delta \geq s$,*

$$\text{val}^{\Delta}_{\max}(K_{1,s}) = \binom{\Delta + 1}{2} - \binom{\Delta - s + 1}{2}.$$

Moreover, let f be a γ^{Δ}-labeling of $K_{1,s}$ with

$$f(v) = 0 \quad \text{and} \quad f(V_s) = [\Delta - (s-1), \Delta].$$

*Then f and $\bar{f}$ are only γ^{Δ}-*max *labelings of $K_{1,s}$.*

Next, we show an alternative and yet simple proof employing mathematical induction of Theorem 3 which is proposed by Bullington, Eroh and Winters (Bullington, Eroh & Winters, 2010) in 2010 and by Fonseca, Khemmani and Zhang (Fonseca, Khemmani & Zhang, 2015) in 2015. In order to do this, first, let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, where $V_r = \{u_1, u_2, \ldots, u_r\}$ and $V_s = \{v_1, v_2, \ldots, v_s\}$ and then we discuss γ^{Δ}-max labelings of $K_{r,s}$ as follows.

Theorem 6 *Let f be a γ^{Δ}-labeling of a complete bipartite graph $K_{r,s}$ with*

$$f(V_r) = [0, r-1] \quad \text{and} \quad f(V_s) = [\Delta - (s-1), \Delta]\,,$$

*where $\Delta \geq rs$. Then f and $\bar{f}$ are only two γ^{Δ}-*max *labelings of $K_{r,s}$.*

Proof. We proceed by induction on $r+s$. The result is certainly true for $r+s=2$. Assume that $r+s \geq 3$ and the result holds for $K_{r',s'}$ when $2 \leq r'+s' < r+s$. By Corollary 2, hence the theorem holds when $r=1$. Suppose that $r \geq 2$. Let f be a γ^{Δ}-max labeling of $K_{r,s}$ with $f(u_1) < f(u_2) < \cdots < f(u_r)$ and $f(v_1) < f(v_2) < \cdots < f(v_s)$.

Assume that $f(u_1) < f(v_1)$. By Corollary 1, $f(u_1) = 0$. Furthermore, for each $j \in \{1, 2, \ldots, s\}$ it follows that $f(v_j) \leq \Delta-(s-j)$. Let $K_{r-1,s}$ be a complete bipartite graph with vertex set $V(K_{r-1,s}) = V(K_{r,s})-\{u_1\}$ and partite sets $V_{r-1} = V_r-\{u_1\}$ and V_s. Consequently, let f_1 be a $\gamma^{\Delta-1}$-labeling of $K_{r-1,s}$ defined by

$$f_1(u) = f(u) - 1 \quad \text{for each} \quad u \in V(K_{r-1,s}) .$$

Then

$$\mathrm{val}(f_1) = \sum_{\substack{2\leq i\leq r\\1\leq j\leq s}} f_1'(u_iv_j) \leq \mathrm{val}_{\max}^{\Delta-1}(K_{r-1,s}) .$$

Let g_1 be a $\gamma^{\Delta-1}$-max labeling of $K_{r-1,s}$. Since $\Delta - 1 \geq (r-1)s$, by induction hypothesis, we have

$$g_1(V_{r-1}) = [0, r-2] \quad \text{and} \quad g_1(V_s) = [(\Delta-1)-(s-1), (\Delta-1)] .$$

We can extend g_1 to a γ^{Δ}-labeling g of $K_{r,s}$ defined by

$$g(u) = \begin{cases} 0 & \text{if } u = u_1 \\ g_1(u) + 1 & \text{otherwise} . \end{cases}$$

Since

$$\begin{aligned} \mathrm{val}_{\max}^{\Delta}(K_{r,s}) &= \mathrm{val}(f) \\ &= \sum_{\substack{2\leq i\leq r\\1\leq j\leq s}} f'(u_iv_j) + \sum_{j=1}^{s} |f(v_j) - f(u_1)| \\ &\leq \mathrm{val}(f_1) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &\leq \mathrm{val}_{\max}^{\Delta-1}(K_{r-1,s}) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &= \mathrm{val}(g_1) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &= \mathrm{val}(g) \\ &\leq \mathrm{val}_{\max}^{\Delta}(K_{r,s}) , \end{aligned}$$

it follows that

$$\sum_{j=1}^{s} |f(v_j) - f(u_1)| = \sum_{j=1}^{s} |\Delta - (s-j) - 0| \tag{1}$$

and

$$\mathrm{val}(f_1) = \mathrm{val}_{\max}^{\Delta-1}(K_{r-1,s}) . \tag{2}$$

From (1), we have

$$f(V_s) = [\Delta - (s-1), \Delta].$$

From (2), we have $f_1(V_{r-1}) = [0, r-2]$, hence

$$f(V_{r-1}) = [1, r-1]$$

and we have $f(u_1) = 0$. Therefore,

$$f(V_r) = [0, r-1] \quad \text{and} \quad f(V_s) = [\Delta - (s-1), \Delta] .$$

On the other hand, if $f(v_1) < f(u_1)$, then a similar argument to the one used shows that

$$f(V_s) = [0, s-1] \quad \text{and} \quad f(V_r) = [\Delta - (r-1), \Delta] .$$

□

The following result is the consequence of Theorem 6 when $\Delta = rs$.

Theorem 7 *Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, let f be a γ-labeling of $K_{r,s}$ with*

$$f(V_r) = [0, r-1] \quad \text{and} \quad f(V_s) = [rs - (s-1), rs] .$$

*Then f and $\bar{f}$ are only two γ-*max *labelings of $K_{r,s}$.*

4. γ-max Labelings of Complete Graphs

The γ-max labelings of complete graphs K_n were characterized in (Fonseca, Khemmani & Zhang, 2015). In this section, we present characterization of γ^{Δ}-max labelings and γ-max labeling of complete graphs K_n, by applying a similar fashion to the one used in the proof of Theorem 6.

Theorem 8 *Let f be a γ^{Δ}-labeling of a complete graph K_n with*

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

*where $\Delta \geq \binom{n}{2}$ and $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \Delta - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ^{Δ}-*max *labelings of K_n.*

Proof. Let K_n be a complete graph with $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Assume that n is even. We use mathematical induction on n. When $n = 2$, the result is obvious. Assume that $n \geq 4$ and the result holds for $K_{n'}$ when n' is even and $2 \leq n' < n$. Let f be a γ^{Δ}-max labeling of K_n with $f(u_1) < f(u_2) < \cdots < f(u_n)$. By Corollary 1, $f(u_1) = 0$ and $f(u_n) = \Delta$. Let f_1 be a $\gamma^{\Delta-2}$-labeling of a complete graph K_{n-2} with vertex set $V(K_{n-2}) = \{u_2, u_3, \ldots, u_{n-1}\}$ defined by

$$f_1(u_i) = f(u_i) - 1 \quad \text{for each} \quad 2 \leq i \leq n-1 .$$

Let g_1 be a $\gamma^{\Delta-2}$-max labeling of K_{n-2}. Since $\Delta - 2 \geq \binom{n-2}{2}$, by induction hypothesis, we have

$$g_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2)\right] .$$

We can extend g_1 to a γ^{Δ}-labeling g of K_n defined by

$$g(u) = \begin{cases} 0 & \text{if } u = u_1 \\ \Delta & \text{if } u = u_n \\ g_1(u) + 1 & \text{if } u \neq u_1, u_n . \end{cases}$$

Since

$$\begin{aligned} \mathrm{val}^{\Delta}_{\max}(K_n) &= \mathrm{val}(f) \\ &= \mathrm{val}(f_1) + \sum_{i=2}^{n-1} (f(u_i) - f(u_1)) + \sum_{i=2}^{n-1} (f(u_n) - f(u_i)) + (f(u_n) - f(u_1)) \\ &\leq \mathrm{val}^{\Delta-2}_{\max}(K_{n-2}) + \sum_{i=2}^{n-1} (\Delta - 0) + (\Delta - 0) \\ &= \mathrm{val}(g_1) + \sum_{i=2}^{n-1} (g(u_i) - g(u_1)) + \sum_{i=2}^{n-1} (g(u_n) - g(u_i)) + (g(u_n) - g(u_1)) \\ &= \mathrm{val}(g) \\ &\leq \mathrm{val}^{\Delta}_{\max}(K_n) , \end{aligned}$$

it follows that

$$\mathrm{val}(f_1) = \mathrm{val}^{\Delta-2}_{\max}(K_{n-2}) .$$

Thus,

$$f_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2)\right].$$

Hence

$$f(V(K_{n-2})) = \left[1, \left\lfloor \frac{n-2}{2} \right\rfloor\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 2, (\Delta-1)\right]$$

and we have $f(u_1) = 0$, $f(u_n) = \Delta$. Therefore,

$$f(V(K_n)) = \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right].$$

On the other hand, if n is odd, then by a similar argument, this shows that

$$f(V(K_n)) = \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] \cup \{k\}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \Delta - \left\lfloor \frac{n}{2} \right\rfloor\right]$. □

The following result is the consequence of Theorem 8 when $\Delta = \binom{n}{2}$.

Theorem 9 *Let f be a γ-labeling of a complete graph K_n with*

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

*where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ-*max *labelings of K_n.*

5. Open Question

The characterization of γ-max labelings of $K_{r,s}$ and K_n were determined. The main open question is to characterize γ-min labelings of those graphs.

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