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Asymptotic Properties of Longitudinal Weighted Averages for Strongly Mixing Data

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Abstract

We present general results of consistency and normality of a real-valued-longitudinal random variable. We suppose that this random variable is some formed weighted averages of α-mixing data. The results can be applied to within-subject covariance function.

Keywords: longitudinal data, α-mixing data, weighted averages, within-subject covariance function.

1. Introduction

Longitudinal data analysis involves irregularly-spaced and infrequent measurements. So, there is often relatively little information available about each subject. Repeated binary measurements models have been discusses in Heagerty (Heagerty, 1999). The repeated measurements take place on a few scartered observational times points for each subject.

Recent innovation in measurements recorded machine and data collected methods have facillited the collection of longitudinal data. Longitudinal data are observed at sparsely distributed time points and are often subject of experimental error (Diggle, et al., 2002, Yao, 2007).

The case of independent and identically distributed observations using kernel-based estimation has received considerable attention in recent years with contribution (Hart & Wehrly, 1986; Lin & Caroll, 2000; Yao, 2007; Hall, et al., 2008; Degras, 2008; Soro & Hili, 2012).

Yao (Yao, 2007) has proved the asymptotic normality of mean and covariance functions estimators. Also, Degras (Degras, 2008) has proved the asymptotic normality of estimator of the mean function under a mean-square continuous process.

However, the literature on influence of within-subject correlation on asymptotic results is not developped. For instance, see Hart & Wehrly (1986) for the study of Gasser-Müler estimator. Yao (Yao, 2007) has proved that the within-subject correlation can be ignored in deriving the asymptotic variance. His results are obtained for independent data with arguments that the data were formed by weighted averages of longitudinal or functional data. Soro & Hili (Soro & Hili, 2012) extended the results of Yao (Yao, 2007) for a continuous univariate stochastic process.

The main purpose of this article is to extend the results of Soro & Hili (Soro & Hili, 2012) to α-mixing longitudinal data. Our results can be applied to within-subject covariance function introduced by Soro & Hili (Soro & Hili, 2012) with mixing arguments.

We give general asymptotic properties for real-valued function that we assume to be formed from weighted averages of α-mixing data.

The paper is organised as follows. Section 2 contains the definition of the estimator and some assumptions. Sections 3 and 4 are the main results of the paper. They respectively establish the consistency and the asymptotic normality of the estimator.

2. Definition of the Estimator and Some Assumptions

We consider for $1 \leq i \leq n$, N triples $\{(T_{ij}, X_{ij}, Y_{ij}), 1 \leq j \leq N\}$ identically distributed as (T, X, Y) such that the sequence $(X_i, Y_{i.})$ is α-mixing. Y_{ij} is the jth observation of the random variable X_i, measured at the random time T_{ij}. The number of observations $N(n)$ depend on the sample size n. For simplicity, $N(n)$ will be noted N. We assume that X is defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whereas Y is a real random variable. Let ν_i, $1 \leq i \leq 3$ and k_i, $1 \leq i \leq 3$ be some given integers. Denote by ν, k the multi-indices $\nu = (\nu_1, \nu_2, \nu_3)$ and $k = (k_1, k_2, k_3)$. Let $|\nu| = \nu_1 + \nu_2 + \nu_3$, $|k| = k_1 + k_2 + k_3$; $\nu! = \nu_1!\nu_2!\nu_3!$ and $k! = k_1!k_2!k_3!$. As most kernel-based nonparametric estimators can be written as function of averages, then we consider averages (introduced in Soro & Hili (2012)) of the form:

$$\begin{aligned}\Gamma_{\lambda n} &= \Gamma_{\lambda n}(r, s, t)\\ &= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \sum_{i=1}^{n} \sum_{1 \leq j \neq k \neq l \leq N} \gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})\\ &\qquad \times K_3\left(\frac{r - T_{ij}}{h_n}, \frac{s - T_{ik}}{h_n}, \frac{t - T_{il}}{h_n}\right),\end{aligned}$$

for $1 \leq \lambda \leq l$.

For instance, the non-parametric regression model for repeated measurements, which is typically used for longitudinal data treatement, and dose-response curves:

$$Y_{ij} = X_i(T_{ij}) + \varepsilon_{ij}, \;\; 1 \leq i \leq n, 1 \leq j \leq N.$$

Some applications of this model are given in Hart & Wehrly (1986) for biostatistics, Müler (1988) in human growth curve study, Ramsay & Ramsey (2002) for monthly index of nondurable goods production.

Let

$$\begin{aligned}\sigma_\lambda^2 &= \sigma_\lambda^2(r, s, t)\\ &= \|K_3\|^2 \int_{\mathbb{R}^3} \gamma_\lambda^2(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3, \;\; \text{for } 1 \leq \lambda \leq l.\end{aligned}$$

(T_{ij}, Y_{ij}), $i = 1, ..., n$, $j = 1, ..., N$, are assumed to have the joint density $g(t, y)$. The observation times T_{ij} are assumed to be i.i.d. with a marginal density $f(t)$.

Let $f_3(r, s, t)$ be the joint density of (T_{ij}, T_{ik}, T_{il}), $g_3(r, s, t, y_1, y_2, y_3)$ be the joint density of $(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})$ and $g_6(r, s, t, r', s', t', y_1, y_2, y_3, y_1', y_2', y_3')$ be the joint density of the 12-uple $(T_{ij}, T_{ik}, T_{il}, T_{ij'}, T_{ik'}, T_{il'}, Y_{ij}, Y_{ik}, Y_{il}, Y_{ij'}, Y_{ik'}, Y_{il'})$ where $j \neq k \neq l$, and $(j, k, l) \neq (j', k', l')$.

To establish the properties of our random variable $\Gamma_{\lambda n}$, we need the following assumptions.

Assumptions K.

(K.1) $K_3(., ., .) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is symmetric and has a compact support.

(K.2) $\|K_3\|_2^2 = \int_{\mathbb{R}^3} K_3^2(u, v, w) du dv dw < \infty$.

(K.3) K_3 is a kernel function of order $(|\nu|, |k|)$, that is,

$$\int_{\mathbb{R}^3} u^{\ell_1} v^{\ell_2} w^{\ell_3} K_3(u, v, w) du dv dw = \begin{cases} 0, & 0 \leq |\ell| < |k|, |\ell| \neq |\nu|.\\ (-1)^{|\nu|}\nu!, & |\ell| = |\nu|,\\ C, & |\ell| = |k|.\end{cases} \tag{1}$$

where C is a non null constant.

Assumptions B.

(B.1) $h_n \longrightarrow 0$, $nN(N-1)(N-2)h_n^{|\nu|+3} \longrightarrow \infty$, $nN(N-1)(N-2)h_n^{2|\nu|+3} \longrightarrow a^2$, where a is a positive constant, as $n \longrightarrow +\infty$.

(B.2) $nh_n^{|\nu|+3} \longrightarrow \infty$ and $N(N-1)(N-2)h_n^{|\nu|} \longrightarrow \infty$, as $n \longrightarrow \infty$.

Assumptions D.

The following conditions are assumed, where $N(r, s, t)$ is some neighborhood of $\{(r, s, t)\}$.

(D.1) $\frac{d^{|k|}}{du^{k_1} dv^{k_2} dw^{k_3}} f_3(u, v, w)$ exists and is continuous for $(u, v, w) \in N(r, s, t)$ and $f_3(u, v, w) > 0$ for all arguments $(u, v, w) \in N(r, s, t)$;

(D.2) $g_3(u, v, w, y_1, y_2, y_3)$ is continuous for $(u, v, w) \in N(r, s, t)$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$;

(D.3) $\frac{d^{|k|}}{du^{k_1} dv^{k_2} dw^{k_3}} g_3(u, v, w, y_1, y_2, y_3)$ exists and is continuous for

$(u, v, w) \in N(r, s, t)$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$;

(D.4) $g_6(u, v, w, u', v', w', y_1, y_2, y_3)$ is continuous for (u, v, w, u', v', w')

$\in N(r, s, t)^2$, uniformly for $(y_1, y_2, y_3) \in \mathbb{R}^3$.

The collection $\{\gamma_\lambda\}_{\lambda=1,\dots,l}$ of real functions $\gamma_\lambda : \mathbb{R}^6 \longrightarrow \mathbb{R}$, $\lambda = 1, ..., l$, satisfies:

(D.5) $\gamma_\lambda(r, s, t, y_1, y_2, y_3)$ is continuous for (r, s, t) uniformly for

$(y_1, y_2, y_3) \in \mathbb{R}^3$,

(D.6) $\frac{d^{|k|}}{dr^{k_1} ds^{k_2} dt^{k_3}} \gamma_\lambda(r, s, t, y_1, y_2, y_3)$ exists for all arguments $(r, s, t, y_1, y_2, y_3) \in \mathbb{R}^6$.

The process $\{X_i, Y_i\}$ is strongly mixing:

Let $\mathcal{F}_a^b$ be the sigma algebra generated by the random variables $\{X_i, Y_{i.}\}_{i=a}^b$. Set

$$\alpha(\ell) = \sup_t \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+\ell}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The mixing coefficient satistfies:

Assumption M.

(M.1) $\sum_{\ell=1}^{\infty} \ell^a [\alpha(\ell)]^{1-2/\delta} < \infty$ for some $a > 1 - 2/\delta$, for some $\delta > 2$.

3. Consistency of the Estimator

The folowing theorem gives the consistency of our estimator.

Theorem3.1. If assumptions (K), (B) and (D) are satisfied, we have

$$\Gamma_{\lambda n}(r, s, t) - m_\lambda(r, s, t) \quad \overset{\mathbb{P}}{\longrightarrow} \quad \mathcal{B}(r, s, t), \tag{2}$$

where

$$m_\lambda(r, s, t) = \frac{d^{|\nu|}}{dr^{\nu_1} ds^{\nu_2} dt^{\nu_3}} \int_{\mathbb{R}^3} \gamma_\lambda(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3,$$

$\lambda = 1, ..., l$ and

$$\mathcal{B}(r, s, t) = \frac{(-1)^{|k|}}{k!} \left\{ \int_{\mathbb{R}^3} u^{k_1} v^{k_2} w^{k_3} K_3(u, v, w) du dv dw \right.$$

$$\left. \times \frac{d^{|k|}}{dr^{k_1} ds^{k_2} dt^{k_3}} \int_{\mathbb{R}^3} \gamma_\lambda(r, s, t, y_1, y_2, y_3) g_3(r, s, t, y_1, y_2, y_3) dy_1 dy_2 dy_3 \right\}.$$

Proof.

We obtain the consistency of our estimator via the bias-variance decomposition which follows

$$\mathbb{E}[(\Gamma_\lambda(r, s, t) - m_\lambda(r, s, t))^2] \quad = \quad var(\Gamma_\lambda(r, s, t)) + \{\mathbb{E}[\Gamma_\lambda(r, s, t)] - m_\lambda(r, s, t)\}^2. \tag{3}$$

Let prove that the second term in (3) goes to 0 when n goes to $+\infty$. We have

$$\begin{aligned}
\mathbb{E}(\Gamma_{\lambda n}(r,s,t)) &= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\mathbb{E}\left\{\sum_{i=1}^{n}\sum_{1\le j\neq k\le l\neq N}\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})\right.\\
&\qquad \left.\times K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right)\right\}\\
&= \frac{1}{N(N-1)(N-2)h_n^{|\nu|+3}}\left\{\sum_{1\le j\neq k\le l\neq N}\mathbb{E}\left[\gamma_\lambda(T_{1j},T_{1k},T_{1l},Y_{1j},Y_{1k},Y_{1l})\right.\right.\\
&\qquad \left.\left.\times\ K_3\left(\frac{r-T_{1j}}{h_n},\frac{s-T_{1k}}{h_n},\frac{t-T_{1l}}{h_n}\right)\right]\right\}\\
&= \frac{1}{h_n^{|\nu|+3}}E\left\{\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right\}\\
&= m_\lambda(r,s,t)+\frac{(-1)^{|k|}}{k!}\left\{\int_{\mathbb{R}^3}u^{k_1}v^{k_2}w^{k_3}K_3(u,v,w)dudvdw\right.\\
&\quad \left.\times\frac{d^{|k|}}{dr^{k_1}ds^{k_2}dt^{k_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3\times h_n^{|k|-|\nu|}\right\}\\
&\qquad +o(h_n^{|k|-|\nu|}).
\end{aligned}\tag{4}$$

So

$$\mathbb{E}\Gamma_{\lambda n}(r,s,t)-m_\lambda(r,s,t)\quad\longrightarrow\quad\mathcal{B}(r,s,t).\tag{5}$$

Now, we prove that $var(\Gamma_{\lambda n}(r,s,t))\longrightarrow 0$.

$$var(\Gamma_{\lambda n}(r,s,t)) =$$

$$\frac{1}{nN(N-1)(N-2)h_n^{2|\nu|+6}}var\left[\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]+$$

$$\frac{1}{(nN(N-1)(N-2)h_n^{|\nu|+3})^2}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n}\sum_{1\le j\neq k\neq l\le N}\sum_{1\le j'\neq k'\neq l'\le N}$$

$$cov\left\{\gamma_\lambda(T_{ij},T_{ik},T_{il},Y_{ij},Y_{ik},Y_{il})K_3\left(\frac{r-T_{ij}}{h_n},\frac{s-T_{ik}}{h_n},\frac{t-T_{il}}{h_n}\right),\right.$$

$$\left.\gamma_\lambda(T_{i'j'},T_{i'k'},T_{i'l'},Y_{i'j'},Y_{i'k'},Y_{i'l'})K_3\left(\frac{r-T_{i'j'}}{h_n},\frac{s-T_{i'k'}}{h_n},\frac{t-T_{i'l'}}{h_n}\right)\right\}=$$

$$I_1+I_2$$

$$
\begin{aligned}
I_1 &= \frac{1}{nN(N-1)(N-2)h_n^{2|\nu|+6}} var\left[\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right] \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \\
&\times \left\{\mathbb{E}\left[\frac{1}{h_n^{|\nu|+3}}\gamma_\lambda^2(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3^2\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]\right. \\
&- \left.\mathbb{E}^2\left[\frac{1}{h_n^{|\nu|+3}}\gamma_\lambda(T_{11},T_{12},T_{13},Y_{11},Y_{12},Y_{13})K_3\left(\frac{r-T_{11}}{h_n},\frac{s-T_{12}}{h_n},\frac{t-T_{13}}{h_n}\right)\right]\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}} \\
&\times \left\{\frac{1}{h_n^{|\nu|+3}}\int_{\mathbb{R}^6} g_3(t_1,t_2,t_3,y_1,y_2,y_3)\gamma_\lambda^2(t_1,t_2,t_3,y_1,y_2,y_3)\right. \\
&\quad K_3^2\left(\frac{r-t_1}{h_n},\frac{s-t_2}{h_n},\frac{t-t_3}{h_n}\right)dt_1dt_2dt_3dy_1dy_2dy_3 \\
&- \left[\frac{1}{h_n^{|\nu|+3}}\int_{\mathbb{R}^6} g_3(t_1,t_2,t_3,y_1,y_2,y_3)\gamma_\lambda(t_1,t_2,t_3,y_1,y_2,y_3)\right. \\
&\quad \left.\left.K_3\left(\frac{r-t_1}{h_n},\frac{s-t_2}{h_n},\frac{t-t_3}{h_n}\right)dt_1dt_2dt_3dy_1dy_2dy_3\right]^2\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\left\{\frac{1}{h_n^{|\nu|}}\int_{\mathbb{R}^6} g_3(r-h_nu,s-h_nv,t-h_nw,y_1,y_2,y_3)\times\right. \\
&\quad \gamma_\lambda^2(r-h_nu,s-h_nv,t-h_nw,y_2,y_3)K_3^2(u,v,w)\,dudvdwdy_1dy_2dy_3 \\
&- h_n^3\left[\frac{1}{h_n^{|\nu|}}\int_{\mathbb{R}^6} g_3(r-h_nu,s-h_nv,t-h_nw,y_1,y_2,y_3)\right. \\
&\quad \left.\left.\gamma_\lambda(r-h_nu,s-h_nv,t-h_nw,y_2,y_3)K_3(u,v,w)\,dudvdwdy_1dy_2dy_3\right]^2\right\} \\
&= \frac{1}{nN(N-1)(N-2)h_n^{|\nu|+3}}\left\{\sigma_\lambda^2(r,s,t)+o(1)\right\} \\
&\longrightarrow 0, n\longrightarrow +\infty. \qquad (6)
\end{aligned}
$$

Let consider I_2. We use the fact that triples $\{Y_{ij},Y_{ik},Y_{il}\}$ and $\{Y_{ij'},Y_{ik'},Y_{il'}\}$ are independent and equidistributed.

$$
\begin{aligned}
I_2 = \frac{[N(N-1)(N-2)]^2}{[nN(N-1)(N-2)h_n^{|\nu|+3}]^2}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} \\
cov\left\{\gamma_\lambda(T_{i1}^{(1)},T_{i1}^{(2)},T_{i1}^{(3)},Y_{i1}^{(1)},Y_{i1}^{(2)},Y_{i1}^{(3)})K_3\left(\frac{r-T_{i1}^{(1)}}{h_n},\frac{s-T_{i1}^{(2)}}{h_n},\frac{t-T_{i1}^{(3)}}{h_n}\right),\right. \\
\left.\gamma_\lambda(T_{i'2}^{(1)},T_{i'2}^{(2)},T_{i'2}^{(3)},Y_{i'2}^{(1)},Y_{i'2}^{(2)},Y_{i'2}^{(3)})K_3\left(\frac{r-T_{i'2}^{(1)}}{h_n},\frac{s-T_{i'2}^{(2)}}{h_n},\frac{t-T_{i'2}^{(3)}}{h_n}\right)\right\} \\
= \frac{1}{n^2h_n^{2|\nu|+6}}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} cov(R_{\lambda,i},R_{\lambda,i'}).
\end{aligned}
$$

Let $S=\{(i,i'):0\leq|i-i'|<d_n, i,i'=1,...,n, i\neq i'\}$.

$$
\begin{aligned}
I_2 &= \frac{1}{n^2h_n^{2|\nu|+6}}\sum_{\substack{i=1\\ i\neq i'}}^{n}\sum_{i'=1}^{n} cov(R_{\lambda,i},R_{\lambda,i'}) \\
&= \frac{1}{n^2h_n^{2|\nu|+6}}\left\{\sum_{i,i'=1}^{n}\sum_{(i,i')\in S} cov(R_{\lambda,i},R_{\lambda,i'}) + \sum_{i,i'=1}^{n}\sum_{(i,i')\notin S} cov(R_{\lambda,i},R_{\lambda,i'})\right\} \\
&= I_{21}+I_{22}.
\end{aligned}
$$

By Holder inequality,

$$|cov(R_{\lambda,i}, R_{\lambda,i'})| \leq (\mathbb{E}[R^2_{\lambda,i}]\mathbb{E}[R^2_{\lambda,i'}])^{1/2} + [\mathbb{E}|R_{\lambda,i'}|]^2,$$

so

$$\begin{aligned}|I_{21}| &\leq \frac{1}{n^2 h_n^{|\nu|+3}} \sum_{i,i'=1}^{n} \sum_{(i,i')\in S} \left\{ \frac{1}{h_n^{|\nu|+3}} (\mathbb{E}[R^2_{\lambda,i}]\mathbb{E}[R^2_{\lambda,i'}])^{1/2} + \frac{1}{h_n^{|\nu|+3}} [\mathbb{E}|R_{\lambda,i'}|]^2 \right\} \\ &= \frac{1}{n^2 h_n^{|\nu|+3}} \sum_{i,i'=1}^{n} \sum_{(i,i')\in S} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\}.\end{aligned}$$

Since $Card(S) \leq nd_n$, we obtain

$$\begin{aligned}|I_{21}| &\leq \frac{nd_n}{n^2 h_n^{|\nu|+3}} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\} \\ &\leq \frac{d_n}{n h_n^{|\nu|+3}} \left\{ \sigma^2_\lambda(r,s,t) + o(1) \right\}.\end{aligned}$$

Choosing $d_n = (\ln\ln n)^2 \ln n$, $h_n = \frac{\ln\ln n}{\ln n}$, it comes,

$$d_n \longrightarrow \infty, \; h_n \longrightarrow 0, \; n h_n^{|\nu|+3} \longrightarrow \infty \text{ and } \frac{d_n}{n h_n^{|\nu|+3}} \longrightarrow 0.$$

Hense

$$I_{21} \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \tag{7}$$

Now consider I_{22}. By Davydov's lemma (see Hall & Heyde, Corrollary A.2), and (K.1) we have

$$\begin{aligned}|cov(R_{\lambda,i}, R_{\kappa,i'})| &\leq 8[\mathbb{E}|R_{\lambda,i}|^\delta]^{2/\delta}[\alpha(|i-i'|)]^{1-2/\delta} \\ &\leq 8C\left[h_n^{|\nu|+3}\right]^{2/\delta}[\alpha(|i-i'|)]^{1-2/\delta}.\end{aligned}$$

It follows that

$$\begin{aligned}|I_{22}| &\leq \frac{8C\left[h_n^{|\nu|+3}\right]^{2/\delta}}{n^2 h_n^{2|\nu|+6}} \sum_{i,i'=1}^{n} \sum_{(i,i')\notin S} [\alpha(|i-i'|)]^{1-2/\delta} \\ &\leq \frac{8C}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{i,i'=1}^{n} \sum_{(i,i')\notin S} [\alpha(|i-i'|)]^{1-2/\delta}.\end{aligned}$$

Reducing the double sum above to a single sum, it follows that

$$\begin{aligned}|I_{22}| &\leq \frac{8C}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{n} \ell^a[\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8Cn}{n^2 h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{n} \ell^a[\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8C}{n h_n^{(2|\nu|+3)(3-1/\delta)}} \sum_{\ell=d_n+1}^{\infty} \ell^a[\alpha(\ell)]^{1-2/\delta}.\end{aligned}$$

Since $\delta \geq 2$, then $(3 - 1/\delta) > 0$ and from assumption (M), one has

$$I_{22} \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \tag{8}$$

Combining (6), (7) and (8), we conclude that $var(\Gamma_\lambda(r,s,t))$ goes to 0 as n goes to $+\infty$. So Theorem 3.1 is proved. □

4. Asymptotic Normality of the Estimator

The asymptotic normality of our estimator is given by the following theorem.

Theorem 4.1. If assumptions (K), (B), (D) and (M) are satisfied, we have

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) \quad \longrightarrow \quad \mathcal{N}(0, \sigma_\lambda^2(r,s,t)). \tag{9}$$

Proof.

First, recall that

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \frac{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}{nN(N-1)(N-2)h_n^{|\nu|+3}} \sum_{i=1}^{n}$$

$$\sum_{1\neq j\neq k\neq l\leq N} \left[\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right]$$

$$= \frac{1}{\sqrt{nN(N-1)(N-2)h_n^3}} \sum_{i=1}^{n}$$

$$\sum_{1\leq j\neq k\neq l\leq N} \left[\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right]$$

$$= \sum_{i=1}^{n} \sum_{1\leq j\neq k\neq l\leq N} \frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}$$

$$\left[\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right.$$

$$\left. - \mathbb{E}\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right)\right].$$

Denote

$$Z_{ijkl} = \frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right).$$

Then

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \sum_{i=1}^{n} \sum_{1\leq j\neq k\neq l\leq N} (Z_{ijkl} - \mathbb{E}Z_{ijkl}).$$

Denote $Z_{n,i} = \sum_{1\leq j\neq k\neq l\leq N}(Z_{ijkl} - \mathbb{E}Z_{ijkl})$. Hence

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n} - \mathbb{E}\Gamma_{\lambda n}) = \sum_{i=1}^{n} Z_{n,i}.$$

We now introduce Bernstein's big-block and small-block decomposition. We partition the set $\{1, 2, ..., n\}$ into $2k_n + 1$ subsets with large blocks of size u_n and small blocks of size v_n and we set $k_n = \left\lfloor \frac{n}{u_n+v_n} \right\rfloor$, where $u_n = \left\lfloor nN(N-1)(N-2)h_n^{|\nu|+3} \right\rfloor$ and $v_n = o(nN(N-1)(N-2)h_n^{|\nu|+3})$. The symbol $\lfloor . \rfloor$ is integer part. Using (B.2), one has

$$\frac{v_n}{u_n} \longrightarrow 0, \frac{u_n}{n} \longrightarrow 0 \quad , \quad \frac{nN(N-1)(N-2)}{u_n h_n^3} \longrightarrow 0, \frac{n}{u_n}\alpha(v_n) \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \tag{10}$$

Let U_m, V_m and W_m be defined as follows:

$$U_m = \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} Z_{n,i}, \;\; 0 \le m \le k_n - 1 \tag{11}$$

$$V_m = \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} Z_{n,i}, \;\; 0 \le m \le k_n - 1 \tag{12}$$

$$W_m = \sum_{i=k_n(u_n+v_n)+1}^{n} Z_{n,i}. \tag{13}$$

Then, we obtain the decomposition

$$T_n = \sum_{i=1}^{n} Z_{n,i} = \sum_{m=0}^{k_n-1} U_m + \sum_{m=0}^{k_n-1} V_m + W_m \tag{14}$$

$$= S_{n,1} + S_{n,2} + S_{n,3}. \tag{15}$$

Now, let start the proof of theorem 4.1.

The main idea is to show that as $n \longrightarrow \infty$,

$$A_1 = \mathbb{E}[S_{n,2}^2] \longrightarrow 0 \tag{16}$$

$$A_2 = \mathbb{E}[S_{n,3}^2] \longrightarrow 0 \tag{17}$$

$$A_3 = \left| \mathbb{E}[\exp(iuS_{n,1}) - \prod_{m=0}^{k_n-1} \mathbb{E}[\exp(iuU_m)] \right| \longrightarrow 0 \tag{18}$$

$$A_4 = \mathbb{E}[U_m^2] \longrightarrow \sigma_\lambda^2(r,s,t) \tag{19}$$

$$A_5 = \sum_{m=0}^{k_n-1} \mathbb{E}\left[U_m^2 I\{|U_m| > \varepsilon\sigma_\lambda(r,s,t)\}\right] \longrightarrow 0, \forall\varepsilon > 0. \tag{20}$$

Remark: Relations (16) and (17) imply that $S_{n,2}$ and $S_{n,3}$ are asymptotically negligeable; (18) shows that the summands $\{U_m\}$ in $S_{n,1}$ are asymptotically independent; (19) and (20) are Lindeberg-Feller conditions for asymptotic normality of $S_{n,1}$ under dependence. Expressions (16)-(18) entail the asymptotic normality

$$T_n \xrightarrow{L} \mathcal{N}(0, \sigma_\lambda^2(r,s,t)) \tag{21}$$

(i) **Proof of (16)**

$$\begin{aligned}\mathbb{E}[S_{n,2}^2] &= var\left(\sum_{m=0}^{k_n-1} V_m\right) \\ &= \sum_{m=0}^{k_n-1} var(V_m) + \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} cov(V_m, V_{m'}) \\ &= A_{11} + A_{12}.\end{aligned} \tag{22}$$

To control A_{11}, we get

$$\begin{aligned}var(V_m) &= var\left(\sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} Z_{n,i}\right) \\ &= \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} var(Z_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+u_n+1\\ i\neq i'}}^{(m+1)(u_n+v_n)} \sum_{i'=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} cov(Z_{n,i}, Z_{n,i'})\end{aligned} \tag{23}$$

and using the second-order stationarity and the fact that $\{Z_{ijkl}\}$ and $\{Z_{ij'k'l'}\}$ are independent,

$$\begin{aligned}
var(V_m) &= \sum_{i=1}^{v_n} var(Z_{n,i}) + \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n} cov(Z_{n,i}, Z_{n,i'}) \\
&= v_n var(Z_{n,1}) + \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n} cov(Z_{n,i}, Z_{n,i'}) \\
&= \frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1)). \qquad (24)
\end{aligned}$$

Because

$$\begin{aligned}
var(Z_{n,1}) &= var\left(\sum_{1\leq j\neq k\neq l\leq N}(Z_{1jkl} - \mathbb{E}Z_{1jkl})\right) \\
&= var\left(\sum_{1\leq j\neq k\neq l\leq N}(Z_{jkl} - \mathbb{E}Z_{jkl})\right) \\
&= \sum_{1\leq j\neq k\neq l\leq N} var\left(Z_{jkl} - \mathbb{E}Z_{jkl}\right) \\
&= N(N-1)(N-2)var(Z_{111} - \mathbb{E}Z_{111})
\end{aligned}$$

$$\begin{aligned}
&= N(N-1)(N-2)\left\{\mathbb{E}(Z_{111} - (\mathbb{E}Z_{111})^2\right\} \\
&= N(N-1)(N-2)\left\{\frac{\sigma_\lambda^2(r,s,t)}{nN(N-1)(N-2)}(1+o(1))\right\} \\
&= \frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)).
\end{aligned}$$

And also,

$$\begin{aligned}
|cov(Z_{n,i}, Z_{n,i'})| &\leq \frac{N(N-1)(N-2)}{n}\left|\frac{1}{h_n^{|\nu|+3}}cov(R_{\lambda,i}, R_{\lambda,i'})\right| \\
&\leq \frac{N(N-1)(N-2)}{n}\left\{\sigma_\lambda^2(r,s,t) + o(1)\right\} \\
\sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i'=1}^{v_n}|cov(Z_{n,i}, Z_{n,i'})| &\leq \frac{v_n}{n}\left\{v_n N(N-1)(N-2)[\sigma_\lambda^2(r,s,t) + o(1)]\right\} \\
&= \frac{v_n}{n}\{o(n)\} \\
&= v_n o(1).
\end{aligned}$$

Then, we get

$$\begin{aligned}
|A_{11}| &\leq \sum_{m=0}^{k_n-1}\left\{v_n\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + v_n o(1)\right\} \\
&= k_n\left\{v_n\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + v_n o(1)\right\} \\
&= k_n v_n\left\{\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1)) + o(1)\right\} \\
&= k_n v_n\left\{\frac{\sigma_\lambda^2(r,s,t)}{n}(1+o(1))\right\} \\
&= k_n\frac{v_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1))
\end{aligned}$$

$$\begin{aligned}
&= \left\lfloor \frac{n}{u_n+v_n} \right\rfloor \frac{v_n}{n}\sigma^2_\lambda(r,s,t)(1+o(1)) \\
&\sim \frac{n}{u_n}\frac{v_n}{n}\sigma^2_\lambda(r,s,t)(1+o(1)) \\
&= \frac{v_n}{u_n}\sigma^2_\lambda(r,s,t)(1+o(1)) \\
&\longrightarrow 0, \quad \text{by } (10).
\end{aligned}$$

Now

$$\begin{aligned}
A_{12} &= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1} cov(V_m, V_{m'}) \\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1} \sum_{\substack{i=m(u_n+v_n)+u_n+1\\ i\neq i'}}^{(m+1)(u_n+v_n)} \sum_{i=m'(u_n+v_n)+u_n+1}^{(m'+1)(u_n+v_n)} cov(Z_{n,i}, Z_{n,i'}) \\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1} \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i=1}^{v_n} cov(Z_{n,m(u_n+v_n)+u_n+i}, Z_{n,m'(u_n+v_n)+u_n+i'}) \\
&= \sum_{\substack{m=0\\ m\neq m'}}^{k_n-1}\sum_{m'=0}^{k_n-1} \sum_{\substack{i=1\\ i\neq i'}}^{v_n}\sum_{i=1}^{v_n} cov(Z_{n,\lambda_m+i}, Z_{n,\lambda_{m'}+i'})
\end{aligned}$$

since $|\lambda_m - \lambda_{m'} + i - i'| \geq u_n$ then we reduce the sums and we write

$$\begin{aligned}
|A_{12}| &\leq \sum_{\substack{i=1\\ |i-i'|\geq u_n}}^{n}\sum_{i=1}^{n} |cov(Z_{n,i}, Z_{n,i'})| \\
&\leq \frac{N(N-1)(N-2)}{nh_n^{|\nu|+3}} 8C[h_n^{|\nu|+3}]^{2/\delta}\sum_{\ell=1}^{\infty}\ell^a[\alpha(\ell)]^{1-2/\delta}. \\
&= \frac{8CN(N-1)(N-2)}{nh_n^{(|\nu|+3)(1-2/\delta)}}\sum_{\ell=1}^{\infty}\ell^a[\alpha(\ell)]^{1-2/\delta} \\
&= o(1).
\end{aligned}$$

$$\text{Therefore } A_{12} \longrightarrow 0, \text{ as } n\longrightarrow+\infty. \tag{25}$$

Combining (23) and (24), it follows that $\mathbb{E}[S^2_{n,2}] \longrightarrow 0$ and

$$S_{n,2} \longrightarrow 0 \text{ in probability.}$$

This achieves the proof of (16).

(ii) **Proof of (17)** Using the same arguments as in the proof of (16), one has

$$\begin{aligned}
\mathbb{E}[S^2_{n,3}] &= var(\sum_{m=0}^{k_n-1} U_m) \\
&\leq \frac{u_n+v_n}{n}\left\{\sigma^2_\lambda(r,s,t)+o(1)\right\}. \\
&\sim \frac{u_n}{n}\left\{\sigma^2_\lambda(r,s,t)+o(1)\right\}. \\
&\longrightarrow 0.
\end{aligned} \tag{26}$$

(iii) **Proof of (18)** The proof is based on the Lemma of Volkonskii & Rozanov (1959).

Here note that U_m is $\{\mathcal{F}_{i_1,\ldots,i_{u_n}}\}$-mesurable with $i_1 = m(u_n+v_n)+1$ and $i_{u_n} = m(u_n+v_n)+u_n$ and taking $V_m = \exp(iuU_m)$

as in the Lemma of Volkonskii & Rozanov, we have

$$\begin{aligned}\left|\mathbb{E}[\exp(iuS_{n,1}) - \mathbb{E}[\exp(iuU_m)]\right| &\leq 16k_n\alpha(v_n+1)\\ &\sim 16\frac{n}{u_n}\alpha(v_n+1)\\ &\longrightarrow 0 \text{ by } (10).\end{aligned} \tag{27}$$

(iv) **Proof of (19)** Replacing u_n by v_n we have

$$\begin{aligned} var(U_m) &= var\left(\sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} Z_{n,i}\right)\\ &= \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} var(Z_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+1\\ i\neq i'}}^{m(u_n+v_n)+u_n}\sum_{i'=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} cov(Z_{n,i}, Z_{n,i'})\\ &= u_n\sigma_\lambda^2(r,s,t)(1+o(1)).\end{aligned} \tag{28}$$

So that

$$\begin{aligned}\sum_{m=0}^{k_n-1}\mathbb{E}[U_m^2] &= k_n\frac{u_n}{n}\sigma_\lambda^2(r,s,t)(1+o(1))\\ &\sim \frac{u_n}{u_n}\sigma_\lambda^2(r,s,t)(1+o(1))\\ &\longrightarrow \sigma_\lambda^2(r,s,t).\end{aligned}$$

(v) **Proof of (20)** We need a truncation argument. Let τ_n be a fixed truncation point. We can replace $\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})$ with the truncated process
$\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})I(|\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})| \leq \tau_n)$ in (Y_{ij}, Y_{ik}, Y_{il}). Denote

$$\begin{aligned} Z_{ijkl}^{\tau_n} &= \frac{1}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})I(|\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})| \leq \tau_n)\\ &\quad K_3\left(\frac{r-T_{ij}}{h_n}\frac{s-T_{ik}}{h_n}, \frac{t-T_{il}}{h_n}\right),\\ Z_{n,i}^{\tau_n} &= \sum_{1\leq j\neq k\neq l\leq N}(Z_{ijkl}^{\tau_n} - \mathbb{E}Z_{ijkl}^{\tau_n}).\end{aligned}$$

Define $T_n^{\tau_n} = \sum_{i=1}^n Z_{n,i}^{\tau_n}$ and

$$T_n^{*\tau_n} = \sum_{i=1}^n (Z_{n,i} - Z_{n,i}^{\tau_n}) = \sum_{i=1}^n Z_{n,i}I(|\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})| > \tau_n). \tag{29}$$

Since $|\gamma_\lambda(T_{ij}, T_{ik}, T_{il}, Y_{ij}, Y_{ik}, Y_{il})| \leq \tau_n$ and from (K.1), it follows that

$$|Z_{n,i}^{\tau_n}| \leq 2C\frac{N(N-1)(N-2)\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}$$

and

$$\max_{0\leq m\leq k_n-1}|U_m^{\tau_n}| \leq 2C\frac{N(N-1)(N-2)u_n\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}.$$

Therefore if we take τ_n and u_n such that

$$u_n\tau_n = \frac{n^{1/2}h_n^{|\nu|+3}}{(N(N-1)(N-2))^{1/2}},$$

then,

$$\max_{0\leq m\leq k_n-1}|U_m^{\tau_n}|\leq 2C\frac{N(N-1)(N-2)u_n\tau_n}{\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}}\longrightarrow 0.$$

Hence, for n sufficiently large, the set $\{|U_m^{\tau_n}|>\epsilon\sigma_\lambda^2(r,s,t)\}$ becomes empty for all $\varepsilon>0$. Thus, $\mathbb{P}(|U_m^{\tau_n}|>\epsilon\sigma_\lambda^2(r,s,t))=0$ for large n, for all $\varepsilon>0$ so

$$\sum_{m=0}^{k_n-1}\mathbb{E}\left[U_m^2 I\{|U_m|>\varepsilon\sigma_r(r,s,t)\}\right]=0,\ \text{for all}\ \ \varepsilon>0.$$

Hence

$$T_n^{\tau_n}\ \xrightarrow{L}\ \mathcal{N}(0,\sigma^2_{\lambda,\tau_n}(r,s,t)).\tag{30}$$

In order to complete the proof, namely to establish. (21) for the general case, it suffices to show that as first $n\longrightarrow+\infty$ and $\tau_n\longrightarrow+\infty$ (see Masry, 2005 or Fan & Masry, 1992) we have

$$var(T_n^{*\tau_n})\quad\longrightarrow\quad 0.\tag{31}$$

Indeed,

$$\left|\mathbb{E}\exp(iuT_n)-\exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right|$$

$$\begin{aligned}
=\ & \left|\mathbb{E}\exp(iu(T_n^{\tau_n}+T_n^{*\tau_n}))-\exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)\right.\\
& \left.+\exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)-\exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right|\\
\leq\ & \left|\mathbb{E}\exp(iuT_n^{\tau_n})-\exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)\right|+\mathbb{E}\left|\exp(iu(T_n^{*\tau_n})-1\right|\\
& +\left|\exp(-u^2\sigma^2_{\lambda,\tau_n}(r,s,t)/2)-\exp(-u^2\sigma_\lambda^2(r,s,t)/2)\right|.
\end{aligned}$$

Letting $n\longrightarrow+\infty$, the first term goes to zero by (30), for every $\tau_n>0$; the second term converges to zero by (31), because first $n\longrightarrow+\infty$ and then $\tau_n\longrightarrow+\infty$; the third term goes to zero as $\tau_n\longrightarrow+\infty$ by the dominated convergence theorem.

Therefore, it remains to prove (31). Note that by (29), $T_n^{*\tau_n}$ has the same structure as $T_n^{\tau_n}$ except that $Z_{n,i}^{\tau_n}$ is replace by $(Z_{n,i}-Z_{n,i}^{\tau_n})$. Applying the Lemma 2.3 in Fan & Masry (1992) or the same arguments as in Masry (2005) we conclued that, for all fixed $\tau_n>0$, one has (31).

Then, it suffices to choose τ_n sufficiently large, such that the non-troncated part becomes asymptotically negligeable. □

Theorem 4.2. Under assumptions of theorems 3.1 and 4.1, we have

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(\Gamma_{\lambda n}-m_{\lambda n})\quad\longrightarrow\quad\mathcal{N}(\mathcal{B}(r,s,t),\sigma_\lambda^2(r,s,t)).\tag{32}$$

Proof. Theorem 4.2 follows from theorem 3.1 and theorem 4.1.□

Under the assumptions of theorem 3.1 and theorem 4.1, we rewrite theorem 2.1 in Soro & Hili (2012) with mixing arguments.

Let $H:\mathbb{R}^l\longrightarrow\mathbb{R}$ be a function with continuous second order derivatives. We denote the gradient vector $(\frac{\partial H}{\partial x_1}(v),...,\frac{\partial H}{\partial x_l}(v))^T$ by $DH(v)$.

Let

$$m_\lambda=m_\lambda(r,s,t)=\frac{d^{|\nu|}}{dr^{\nu_1}ds^{\nu_2}dt^{\nu_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3,$$

$1\leq\lambda\leq l$,

$$\begin{aligned}
\mathbf{B}(r,s,t)\quad=\quad & \frac{(-1)^{|k|}a}{k!}\sum_{\lambda=1}^{l}\left\{\int_{\mathbb{R}^3}u^{k_1}v^{k_2}w^{k_3}K_3(u,v,w)dudvdw\right.\\
& \left.\times\frac{d^{|k|}}{dr^{k_1}ds^{k_2}dt^{k_3}}\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3\right\}\\
& \times\left\{\frac{\partial H}{\partial m_\lambda}(m_1,...,m_l)^T\right\}
\end{aligned}$$

and

$$\begin{aligned}\delta_{\lambda k} &= \delta_{\lambda k}(r,s,t)\\ &= \|K_3\|^2\int_{\mathbb{R}^3}\gamma_\lambda(r,s,t,y_1,y_2,y_3)\gamma_k(r,s,t,y_1,y_2,y_3)g_3(r,s,t,y_1,y_2,y_3)dy_1dy_2dy_3,\\ \Xi &= (\delta_{k\lambda})_{1\le\lambda,k\le l}\ \text{the variance-covariance matrix,}\end{aligned}$$

Theorem 4.3. Assume assumptions of theorems 3.1 and 4.1 hold. Then

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}[H(\Gamma_{1n},...,\Gamma_{ln})-H(m_1,...,m_l)]$$

$$\overset{L}{\longrightarrow}\ \mathcal{N}(\mathbf{B}(r,s,t),[DH(m_1,...,m_l)]^T\Xi[DH(m_1,...,m_l)]). \tag{33}$$

Proof.

A l-dimensional Taylor expansion of H around $(m_1,...,m_l)^T$ of order 1 combined with (2) gives

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}[H(\mathbb{E}\Gamma_{1n},...,\mathbb{E}\Gamma_{ln})-H(m_1,...,m_l)]\ \overset{\mathbb{P}}{\longrightarrow}\ \mathbf{B}(r,s,t). \tag{34}$$

Applying the Cramér-Wold device to (9) it comes

$$\sqrt{nN(N-1)(N-2)h_n^{|\nu|+3}}(H(\Gamma_{1n},...,\Gamma_{ln})-H(\mathbb{E}\Gamma_{1n},...,\mathbb{E}\Gamma_{ln}))$$

$$\longrightarrow\ \mathcal{N}(0,[DH(m_1,...,m_l)]^T\Xi[DH(m_1,...,m_l)]). \tag{35}$$

Finally, (34) and (35) lead to (33). □

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