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The Fundamental Matrix of the Simple Random Walk with Mixed Barriers

Yao Elikem Ayekple[1], Derrick Asamoah Owusu[1], Nana Kena Frempong[1] & Prince Kwaku Fefemwole[1]

[1] Department Of Mathematics, KNUST, Ghana

Correspondence: Yao Elikem Ayekple, Department Of Mathematics, College of Science KNUST; Ghana. E-mail: jegyaoo@gmail.com

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Abstract

The simple random walk with mixed barriers at state 0 and state n defined on non-negative integers has transition matrix P with transition probabilities p_{ij}. Matrix Q is obtained from matrix P when rows and columns at state 0 and state n are deleted . The fundamental matrix B is the inverse of the matrix $A = I - Q$, where I is an identity matrix. The expected reflecting and absorbing time and reflecting and absorbing probabilities can be easily deduced once B is known. The fundamental matrix can thus be used to calculate the expected times and probabilities of NCD's.

Keywords: simple random walk, reflecting barriers, fundamental matrix, reflecting and absorbing times, reflecting and absorbing probabilities, No claims discounts, mixed barriers

This paper is as a result of motivation from the work of Gunther[1].Gunther's work is based on absorbing barriers of simple random walk rather than a general random walk as stated and this paper is based on mixed barriers of the simple random walk.

The process by which randomly-moving objects wander away from where they started is a random walk. It describes the movements or changes in an object which follows no discernible pattern. Random walk is used in many fields including psychology, physics, chemistry and actuarial science.

Simple random walks with mixed barriers can be used in insurance to compute probabilities and expected times of No-Claim Discounts (NCD's).

NCD refer to a reduction in the premiums of an insurance policy because no claims were made on it.

2. Simple Random Walk with Mixed Barriers

A mixed barrier is a value d such that $P(X_{n+1} = d|X_n = d) = \alpha$ and $P(X_{n+1} = d + 1|X_n = d) = 1 - \alpha$ where $\alpha \in [0, 1]$. In other words, once state d is reached, the random walk remains in this state with probability α or moves to the neighbouring state $d + 1$ with probability $1 - \alpha$ i.e. it is an absorbing barrier with probability α and a reflecting barrier with probability $1 - \alpha$

2.1 Notation

The simple random walk with mixed barriers at 0 and n defined on non-negative integers has transition probabilities $p_{oj} = \delta_{0j}, p_{nj-2} = \delta_{nj-2}, p_{i,i-1} = p_i, p_{i,i+1} = q_{i+1}$ *and* $p_{ii} = r_i$ *where* $p_i + q_{i+1} + r_i = 1, r_0 + q_1 = 1$ *and* $p_n + q_n = 1$ where $i = 0, 1, 2 \ldots\ldots, n$ and $j = 2, 3, \ldots\ldots, n$. If A is an $m \times n$ matrix, then $A(\alpha_1, \ldots\ldots \alpha_h|\beta_1 \ldots\ldots \beta_k)$ indicate the $(m-h) \times (n-k)$ submatrix of A when rows $\alpha_1, \ldots\ldots \alpha_h$ and columns $\beta_1 \ldots\ldots \beta_k$ of A are deleted whereas $A[\alpha_1, \ldots\ldots \alpha_h|\beta_1 \ldots\ldots \beta_k]$ represents the $h \times k$ submatrix of A whose (i, j) entry is a_{α_i,β_j}. If $\alpha_i = \beta_i$ for $i = 1, \ldots\ldots, k$ then notations $A(\alpha_1, \ldots\ldots \alpha_k)$ and $A[\alpha_1, \ldots\ldots \alpha_k]$ will be used respectively.In general, A^{-1} is the inverse of a regular square matrix A with $detA$ as its determinant and I as an identity matrix.

2.2 Stochastic Matrix

The simple random walk with mixed barriers is a time homogeneous Markov chain $(X_k : k \geq 0)$ with state space

$\{0, 1, \ldots\ldots n\}$ and transition matrix

$$P = \begin{pmatrix} r_0 & q_1 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\ p_1 & r_1 & q_2 & 0 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & p_2 & r_2 & q_3 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & p_3 & r_3 & q_4 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & p_4 & r_4 & q_5 & \ldots & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & p_{n-2} & r_{n-2} & q_{n-1} & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & p_{n-1} & r_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_n & r_n \end{pmatrix}$$

such that $P\{X_{k+1} = j|X_k = i\} = p_{ij}$ for $i, j = 0, 1, \ldots\ldots, n$. If $Q = P(O, n)$ then Q results from P when rows 0 *and* n and columns 0 and n are deleted such that $A = I - Q$. $B = A^{-1}$ is now the fundamental matrix affiliated with P.

3. Determining The Fundamental Matrix

The inverse of a regular square matrix can be obtained using different methods. The adjugate of $A = (I - Q)$ is used to determine its inverse in this paper.

If $A : d \times d$ is a regular square matrix, the adjugate of matrix A; $adj(A)$ is the matrix whose $(ij)th$ entry is $(-1)^{j+i} detA(j|i)$. With $B = A^{-1} = adj(A)/det(A)$, we obtain

$$b_{ij} = (-1)^{i+j} \frac{detA(j|i)}{detA}$$

for $i, j = 1, 2, \ldots\ldots, d$. An elementary expression for the determinant of matrix A is needed in order to proceed.

Lemma *P is the transition matrix of the simple random walk with mixed barriers at state 0 and state n. We let $Q = P(0, n)$ and define $A_d = I - Q$ with $d = n - 1$. $detA_d$ is given by*

$$detA_d = \sum_{k=0}^{d} \left(\prod_{i=1}^{d-k} p_i \right) \left(\prod_{j=d-k+2}^{d+1} q_j \right)$$

$\forall d \geq 1$

Proof. If $n = 2$, then $d = 1$ with $Q = P(0, 2) = [r_1]$

$$A_1 = I - Q = 1 - [r_1] = p_1 + q_2 \text{ and } detA_d = p_1 + q_2$$

Since

$$detA_1 = \sum_{k=0}^{1} \left(\prod_{i=1}^{1-k} p_i \right) \left(\prod_{j=3-k}^{2} q_j \right) = p_1 + q_2$$

This proposition thus holds for $d = 1$

$$If\ n = 3, d = 2. \qquad Q = P(0, 3) = \begin{pmatrix} r_1 & q_2 \\ p_2 & r_2 \end{pmatrix}$$

$$A_2 = I - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} r_1 & q_2 \\ p_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 - r_1 & -q_2 \\ -p_2 & 1 - r_2 \end{pmatrix} = \begin{pmatrix} p_1 + q_2 & -q_2 \\ -p_2 & p_2 + q_3 \end{pmatrix}$$

$$detA_2 = det \begin{pmatrix} p_1 + q_2 & -q_2 \\ -p_2 & p_2 + q_3 \end{pmatrix} = p_1 p_2 + p_1 q_3 + q_2 q_3$$

Since

$$detA_2 = \sum_{k=0}^{2} \left(\prod_{i=1}^{2-k} p_i \right) \left(\prod_{j=4-k}^{3} q_j \right) = p_1 p_2 + p_1 q_3 + q_2 q_3$$

, the proposition also holds for $d = 2$.

$$If\ n = 4, d = 3. \quad Q = P(0,4) = \begin{pmatrix} r_1 & q_2 & 0 \\ p_2 & r_2 & q_3 \\ 0 & p_3 & r_3 \end{pmatrix}$$

$$A_3 = I - Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} r_1 & q_2 & 0 \\ p_2 & r_2 & q_3 \\ 0 & p_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1-r_1 & -q_2 & 0 \\ -p_2 & 1-r_2 & -q_3 \\ 0 & -p_3 & 1-r_3 \end{pmatrix}$$

$$detA_3 = det \begin{pmatrix} 1-r_1 & -q_2 & 0 \\ -p_2 & 1-r_2 & -q_3 \\ 0 & -p_3 & 1-r_3 \end{pmatrix} = p_1p_2p_3 + p_1p_2q_4 + p_1q_3q_4 + q_2q_3q_4$$

Since

$$detA_3 = \sum_{k=0}^{3} \left(\prod_{i=1}^{3-k} p_i \right) \left(\prod_{j=5-k}^{4} q_j \right) = p_1p_2p_3 + p_1p_2q_4 + p_1q_3q_4 + q_2q_3q_4$$

The proposition suffices for $d = 1$, $d = 2$ & $d = 3$. But assuming the hypothesis is true for $d-1$ and $d \geq 2$, $\det A_{d+1}$ can be formulated in terms of A_{d-1} and A_d such that:

$$\begin{aligned} detA_{d+1} &= det \begin{pmatrix} A_d & -q_{d+1} \\ -p_{d+1} & p_{d+1} + q_{d+2} \end{pmatrix} \\ &= det \begin{pmatrix} A_{d-1} & -q_d & \\ -p_d & p_d + q_{d+1} & -q_{d+1} \\ & -p_{d+1} & p_{d+1} + q_{d+2} \end{pmatrix} \\ &= (p_{d+1} + q_{d+2}) detA_d + p_{d+1} det \begin{pmatrix} A_{d-1} & \cdots & 0 \\ & & \vdots \\ & -p_d & -q_{d+1} \end{pmatrix} \\ &= (p_{d+1} + q_{d+2}) detA_d - p_{d+1} q_{d+1} detA_{d-1} \\ &= q_{d+2} detA_d + p_{d+1}(detA_d - q_{d+1} detA_{d-1}) \end{aligned} \tag{1}$$

Thus

$$detA_{d+1} = q_{d+2} detA_d + p_{d+1}(detA_d - q_{d+1} detA_{d-1}) \tag{2}$$

By hypothesis,

$$q_{d+2} detA_d = q_{d+2} \sum_{k=0}^{d} \left(\prod_{i=1}^{d-k} p_i \right) \left(\prod_{j=d-k+2}^{d+1} q_j \right) = \sum_{k=1}^{d+1} \left(\prod_{i=1}^{(d+1)-k} p_i \right) \left(\prod_{j=(d+1)-k+2}^{d+2} q_j \right) \tag{3}$$

$$q_{d+1} detA_{d-1} = \sum_{k=1}^{d} \left(\prod_{i=1}^{d-k} p_i \right) \left(\prod_{j=d-k+2}^{d+1} q_j \right) \tag{4}$$

$$\begin{aligned} p_{d+1}(detA_d - q_{d+1} detA_{d-1}) &= p_{d+1} \left[\sum_{k=0}^{d} \left(\prod_{i=1}^{d-k} p_i \right) \left(\prod_{j=d-k+2}^{d+1} q_j \right) - \sum_{k=1}^{d} \left(\prod_{i=1}^{d-k} p_i \right) \left(\prod_{j=d-k+2}^{d+1} q_j \right) \right] \\ &= p_{d+1} \prod_{i=1}^{d} p_i \quad = \prod_{i=1}^{d+1} p_i \end{aligned} \tag{5}$$

Substituting (3), (4) and (5) into (2)

$$\begin{aligned} detA_{d+1} &= \sum_{k=1}^{d+1} \left(\prod_{i=1}^{(d+1)-k} p_i \right) \left(\prod_{j=(d+1)-k+2}^{d+2} q_j \right) + \prod_{i=1}^{d+1} p_i \\ &= \sum_{k=0}^{d+1} \left(\prod_{i=1}^{d+1-k} p_i \right) \left(\prod_{j=(d+1)-k+2}^{d+2} q_j \right) \end{aligned} \tag{6}$$

which is the result wanted.

4. Elementary Expressions For $A(j|i)$

An elementary expression is needed for the determinant of $A(j|i)$. Submatrix $A(j|i)$ results after deleting row j and column i from the tridiagonal matrix A. If $i < j$, we obtain a submatrix in the form of a lower triangular block, upper triangular block form when $i > j$ and diagonal block form when $i = j$. Every block is a square submatrix of A and the determinant of the block matrices is the product of the diagonal blocks.The following expressions is derived when determining determinants of these block matrices.

$$i) detA(j|i) = det(A[1,\dots,i-1])det(A[i,\dots,j-1|i+1,\dots,j])det(A[j+1,\dots,d])$$

if $1 \le i < j \le d = n-1$,

$$ii) detA(j|i) = det(A[1,\dots,j-1])det(A[j+1,\dots,i|j,\dots,i-1])det(A[i+1,\dots,d])$$

if $1 \le j < i \le d$ and

$$iii) detA(j|i) = det(A[1,\dots,i-1])det(A[j+1,\dots,d])$$

if $1 \le i = j \le d$. As a convention, if $u > v$, then $det(A[u,\dots,v]) = 1$

4.1 Determinants of Block Matrices

Given a matrix of form$A[1,\dots\dots,\ell]$, elementary expressions can be obtained from the Lemma. The Lemma can also be used to derive elementary expressions for the determinants of matrices of type $A[\ell+1,\dots\dots,d]$ and $A[1,\dots\dots,d-\ell]$. We must also consider that the indices have the offset ℓ. We obtain

$$A[\ell+1,\dots\dots,d] = \sum_{k=0}^{d-\ell}\left(\prod_{u=\ell+1}^{d+1-k} p_u\right)\left(\prod_{j=d-k+2}^{d+1} q_v\right)$$

If $1 \le i < j \le d$ then $det(A[i,\dots,j-1|i+1,\dots,j])$ dwindles to a lower triangular matrix and when $1 \le j < i \le d$ then $det(A[j+1,\dots,i|j,\dots,i-1])$ is an upper triangular matrix. We can further say

$$det(A[i,\dots,j-1|i+1,\dots,j]) = (-1)^{j-i}\prod_{k=i+1}^{j} q_k \qquad (1 \le i < j \le d)$$

and

$$det(A[j+1,\dots,i|j,\dots,i-1]) = (-1)^{i-j}\prod_{k=j+1}^{k} p_k \qquad (1 \le j < i \le d)$$

4.2 The Fundamental Matrix B

Theorem 1 *For the simple random walk with mixed barriers at state 0 and state n,* $B : (n-1)\times(n-1)$ *is the fundamental matrix .* b_{ij} *are the entries of the matrix B and are obtained by*

$$b_{ij} = \frac{\left[\sum_{k=0}^{i-1}\left(\prod_{u=1}^{i-k-1} p_u\right)\left(\prod_{v=i-k+1}^{i} q_v\right)\right]\times\left[\prod_{k=i+1}^{j} q_k\right]\times\left[\sum_{k=0}^{n-j-1}\left(\prod_{u=j+1}^{n-k-1} p_u\right)\left(\prod_{v=n-k+1}^{n} q_v\right)\right]}{\left[\sum_{k=0}^{n-1}\left(\prod_{u=1}^{n-k-1} p_u\right)\left(\prod_{v=n-k+1}^{n} q_v\right)\right]}$$

for $1 \le i \le j \le d = n-1$.

$$b_{ij} = \frac{\left[\sum_{k=0}^{j-1}\left(\prod_{u=1}^{j-k-1} p_u\right)\left(\prod_{v=j-k+1}^{j} q_v\right)\right]\times\left[\prod_{k=i}^{j+1} p_k\right]\times\left[\sum_{k=0}^{n-i-1}\left(\prod_{u=i+1}^{n-k-1} p_u\right)\left(\prod_{v=n-k+1}^{n} q_v\right)\right]}{\left[\sum_{k=0}^{n-1}\left(\prod_{u=1}^{n-k-1} p_u\right)\left(\prod_{v=n-k+1}^{n} q_v\right)\right]}$$

for $d \geq i > j \leq 1$

From the Theorem , we can obtain the reflecting and absorbing time and probability through

$$E[T|X_0 = i] = \sum_{j=1}^{n-1} b_{ij} \qquad \text{and} \qquad P\{X_T = n|X_0 = i\} = b_{i,n-1}.q_{n-1}$$

respectively with $i = 1, \ldots\ldots n-1$.

5. Conclusion

The elementary matrix theory is used to determine the closed form expressions for the expected time and probabilities of a simple random walk with mixed barriers. If only few entries of the fundamental matrix must be determined when the reflecting and absorbing time and reflecting and absorbing probabilities for a specified initial state is of interest, this method will be very useful. It can thus be used to estimate the expected times and probabilities of the lowest and highest claims on NCDs.

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