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Pollution Transfer as Optimal Mass Transport Problem

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Abstract

In this paper, we use mass transportation theory to study pollution transfer in porous media. We show the existence of a L^2-regular vector field defined by a $W^{1,1}-$ optimal transport map. A sufficient condition for solvability of our model, is given by a (non homogeneous) transport equation with a source defined by a measure. The mathematical framework used, allows us to show in some specifical cases, existence of solution for a nonlinear PDE deriving from the modelling. And we end by numerical simulations.

Keywords: porous media, PDE, transport equation, optimal mass transportation, modelling, numerical simulations.

Mathematical classification subject: 34H05, 54C56, 58D25, 65J08, 90B06.

1. Introduction

Pollution problems are interesting and important topic in physics, in mathematical physics, in chemistry, in biology and even in complex sciences. And it is a great challenge to understand the problems in all its dimensions. But there are numerous mathematical obstacles to tackle this challenge even if an acceptable mathematical model is considered. In fact, in general, there are more mathematical unknown than relations (equations inequations, inclusions..). And then, in front of these obstacles, one used to introduce in some cases, physical experimental law in order to reduce the number of variables. Thus, the mathematical study becomes possible with suitable assumptions. We invite the reader to see papers [I. Faye, A. Sy and D. Seck (2008)], [L. Ndiaye, A. Sy and D. Seck (2012)] and the references therein for more details. One of our aim is to propose ways to weaken some hypotheses in such a study. In fact, we shall avoid to use always experimental laws to reduce unknown variables. In this paper, after the modelling of the physical problem, we endeavour to use mass transportation and PDE theories to study pollution in porous media.

Given two distributions μ and ν on $\mathbb{R}^d$ with equal total mass, the classical generic Monge transportation problem consists in finding maps $T : \mathbb{R}^d \to \mathbb{R}^d$ verifying $T_{\#}\mu = \nu, i.e$ (maps transport μ to ν) and minimizing :

$$\min_T I(T), \quad I(T) = \int_{\mathbb{R}^d} c(x, T(x))\, d\mu(x)$$

where c is the cost when moving μ to ν. These maps will be called optimal transport maps. For the existence of solutions, we recommend to see [L. Ambrosio (2000), (2003)], [L. Caffarelli and al. (2002)], [L.C. Evans and W. Gangbo (1999)], [A. Pratelli (2003)]. In particularly Sudakov have studied in [V.N. Sudakov (1979)] the existence of optimal map transportation when $c(x, y) = \|x - y\|$ and μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$.

It is noted that there are many cases where the Monge transportation problem doesn't give a solution, Kantorovich considered the relaxed version of the Monge problem. In this framework, the transportation problem consists in finding among all admissible measures $\gamma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ having μ and ν as marginals, those which solve the minimization problem

$$MK(\mu, \nu, c) = min\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x, y)\, d\gamma(x, y) : \ \pi_1^{\#}\gamma = \mu,\ \pi_2^{\#}\gamma = \nu\}.$$

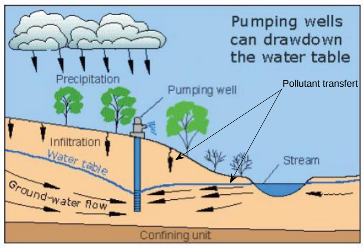
The Monge-Kantorovich problem obtained, depends only on the two distributions μ and ν, and the cost c which may be a function of the path connecting x to y.

When the unknowns of the problem are the distributions μ and ν, the Monge-Kantorovich mass transportation problem can modelise an optimal urban design problem. When the unknown is the transportation network, we can modelise an irrigation problem as well as design public transportation networks.

We also mention the dynamic formulation of mass transportation introduced in [Y. Brenier (2003)] and generalized by G. Buttazzo, C. Jimenez and E. Oudet in [G Buttazzo and al. (2007)].

Let's point out that almost all notions or results we are dealing with in this work could be performed in Riemannian or length spaces frameworks.

The paper is organized as follows: The section 2 is devoted to the modelling of pollutant transfer in porous media. In section 3, we interpret the pollutant transfer as a mass transport problem. This mathematical framework used, allows us to solve the nonlinear system which derives from the modeling and in addition we give the optimality system. In section 4, we give some numerical simulation to illustrate the pollutant transfer in porous media. To end the paper, we give some interesting cases for future developments in section 5.



Example of pollutant transfer in porous media.

2. Modelling of Pollutant Transfer

In this section, we recall to some main steps of the modeling see [I. Faye, A. Sy and D. Seck (2008)], [L. Ndiaye, A. Sy and D. Seck (2012)] Let $\mathcal{D}$ be a porous medium, for $x \in \mathcal{D}$ and $t \in (0, T_1)$ $T_1 > 0$ is a fixed time. For our study, we define the significant following variables:

- $\varepsilon(x, t)$, the effective porosity given by

$$\varepsilon(x, t) = \frac{d\mathcal{V}_l}{d\mathcal{V}_{total}},$$

  where $d\mathcal{V}_l$ is an element of the volume of the fluid and $d\mathcal{V}_{total}$ an element of the total volume ;

- $\sigma(x, t)$, the porosity given by

$$\sigma(x, t) = \frac{d\mathcal{V}_v}{d\mathcal{V}_{total}}$$

  where $d\mathcal{V}_v$ is an element of the volume of the vacuum in the medium.

If we consider $\Omega \in \mathbb{R}^n$ any elementary domain of the porous domain $\mathcal{D}$, then the mass of the considered fluid is given by $M(\Omega, t) = \int_\Omega dm$; where dm is a mass element of the fluid. In fact it is given by the following expression:
$dm = \rho(x, t)\varepsilon(x, t)$; where $\rho(x, t)$ is the density. of the fluid
For our model we shall use these notations: $\rho_s[kg/m^3]$ stands for the density of the solution given by

$$\rho_s = \frac{dm_{solution}}{dv_{solution}}$$

$W(x, t)$ the fraction of the mass (concentration):

$$W(x, t) = \frac{dm_{solute}}{dm_{solution}}.$$

$dm_{solution}$ is a mass element of the solution and dm_{solute} is a mass element of the pollutant.

2.1 Mass Conservation Principle for the Solution

Let

$$dm_{solution} = \rho_s dv_{solution} = \rho_s \frac{dv_{solution}}{dv_{total}} dv_{toal}$$
$$= \rho_s \varepsilon(x,t) dv_{total}$$

then the mass transport equation is given by

$$M_{solution}(\Omega, t) = \int_\Omega dm_{solution} = \int_\Omega \rho_s \varepsilon dx.$$

The balance law claims that the variation of the mass with respect to the time in Ω is equal to the flux exchanged through the boundary of Ω with velocity V. This means

$$\frac{dM_{solution}(\Omega, t)}{dt} = -\int_{\partial\Omega} \rho_s \varepsilon V \nu d\sigma.$$

Hence,

$$\int_\Omega \frac{\partial}{\partial t}(\rho_s \varepsilon) + \int_{\partial\Omega} \rho_s \varepsilon V \nu d\sigma = 0$$

By the Green formula we obtain

$$\int_\Omega (\frac{\partial}{\partial t}(\rho_s \varepsilon) + div(\rho_s \varepsilon V))dx = 0 \quad \forall\ \Omega \subset \mathcal{D}$$

Hence

$$\frac{\partial(\rho_s \varepsilon)}{\partial t} + div(\rho_s q) = 0 \quad \text{in } \mathcal{D} \tag{1}$$

2.2 Mass Conservation Principle for the Pollutant

Here we consider for example that our pollutant liquid is: water mixed to chemical concentration. From the definition of $W(x,t)$ and the expression of $dm_{solution}$, we deduce the following relation

$$dm_{solute} = W(x,t) dm_{solution} = W(x,t)\rho_s(x,t)\varepsilon(x,t) dv_{total}.$$

And then the total mass is given by

$$M(\Omega, t) = \int_\Omega dm_{solute} = \int_\Omega W(x,t)\rho_s(x,t)\varepsilon(x,t)dx.$$

Using the balance law as in the previous subsection for this mass, we have:

$$\int_\Omega (\frac{\partial}{\partial t}(W\rho_s \varepsilon) + div(W\rho_s q + J))dx \ = \ 0,$$

where J is the flux of dispersion diffusion.
And finally we have

$$\frac{\partial}{\partial t}(W\rho_s \varepsilon) + div(W\rho_s q + J) \ = \ 0 \quad \forall\ \Omega \subset \mathcal{D}$$

i.e.

$$\frac{\partial}{\partial t}(W\rho_s \varepsilon) + div(W\rho_s q + J) \ = \ 0 \quad \text{in } \mathcal{D} \tag{2}$$

2.3 Momentum Conservation Principle

This section aims to complete the part devoted to the modeling by giving some basic formulas on the conservation of the momentum. And we sum up them in the following remark.

Remark 2.1. *1. If the porous medium is homogeneous then the Darcy's law is given by*

$$q = -\frac{K}{\lambda}(\nabla p + \rho_s g e_3); \tag{3}$$

where e_3 is third vector of the canonical basis of $\mathbb{R}^3$; p is the pressure, $\vec{g} = ge_3$ is the gravity field, K the intrinsic permeability tensor, λ the dynamic viscosity and K/λ the hydraulic conductivity. For mathematical reasons, we are going to assume, if necessary the following ellipticity condition:
$\frac{K}{\lambda}(x)\xi.\xi \geq \alpha_1 \|\xi\|^2$; α_1 is a positive constant.

2. *If we have some weak concentration the flux of dispersion diffusion J is determined by the Fick law*

$$J = -\rho_s D \nabla W \tag{4}$$

where D be the tensor of dispersion diffusion. In many works it is assumed that the following ellipticity condition: is satisfied:
$D = (d_{ij})_{1\le i,j\le n};\ d_{ij}\xi_i\xi_j \ge \alpha_2\|\xi\|^2$, *where α_2 a positive constant.*

3. *In some interesting experimental cases, it is supposed that the density ρ_s satisfies the following expression:*

$$\rho_s = \rho_0 \exp\left(\beta_T(T - T_0) + \beta_p(p - p_0) + \gamma W\right).$$

ρ_0, T_0, p_0 are respectively the density, temperature, pressure at the initial time, they are data , T and p stand for the temperature and the pressure and they are unknown. And γ, β_T, β_p are given constants.

Finally the proposed model is summarized in a system of equations as follows:

$$\begin{cases} \frac{\partial \varepsilon\rho_s}{\partial t} + div(\rho q) = 0 \\ \frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s W q + J) = 0 \\ J = -\rho_s D\nabla W \\ \rho_s = \rho_0 \exp\left(\beta_T(T - T_0) + \beta_p(p - p_0) + \gamma W\right) \\ q = -\frac{K}{\lambda}(\nabla p + \rho_s g e_3) \end{cases} \tag{5}$$

It is important to underline that the above system of equations is very complicated to solve by direct methods. It is easy to list that $\rho_s, p, T, W, \varepsilon$ are unknown for the two first equations, because from the last three equations some unknowns can be substituted in the two first ones.

We are then going, to make some realistic hypotheses.

2.4 Transport Equation and Mass Conservation Formulas

Before coming to our question let us recall some basic but important results. For additional details about these results and even there refinements see for example [L. Ambrosio (2003)], [L. Ambrosio (2000)] and [C. Villani (2008)].

At first for the mass conservation formula, lets consider a C^1 open set , noted Ω (or more generally a C^1 manifold), and $T \in (0, +\infty]$. Let $\xi(t, x)$ be a measurable vector field defined on $[0, T) \times \Omega$. Let $(\mu_t)_{0\le t\le T}$ be a time dependent family of probability measures on Ω, continuous in time for the weak topology, such that

$$\int_0^T \int_\Omega |\xi(t, x)| d\mu_t(x) dt < +\infty,$$

then the following two statements are equivalent:

1. $\mu = \mu_t(dx)$ is a weak solution of the linear transport partial differential equation

$$\frac{\partial\mu}{\partial t} + \nabla_x(\mu\xi) = 0 \, in \, [0, T) \times \Omega;$$

2. μ_t is the law at time t of a random solution $T_t(x)$ of the following equation

$$\frac{dT_t(x)}{dt} = \xi(t, T_t(x)).$$

If moreover ξ is locally Lipschitz then $(T_t)_{0\le t<T}$ defines a deterministic flow and the above second item can be rewritten as follows

$$\mu_t = (T_t)_\# \mu_0$$

3. If ξ, is Lipschitz, continuous in x, uniformly in t and uniformly bounded, then every solution of the same continuity equation with $\mu_t << \mathcal{L}^d$ for every time is necessarily obtained as $\mu_t = (\Phi_t)_\#\mu_0$, where Φ_t is the flow associated to ξ

Another important reminder for our work is the Moser's technique for coupling smooth positive probability measures. And this technique works in the case of Riemannian manifolds M equipped with a reference probability measure

$$\nu(dx) = \frac{1}{C}e^{-\gamma_0(x)}vol(dx), C = \int e^{-\gamma_0}vol(dx)$$

where $\gamma_0 \in C^1(M)$, for additional details see [B. Dacorogna and J. Moser (1990)], [J. Moser (1965)] and [C. Villani (2008)]. In our case, since we work mainly in $\mathbb{R}^N$, we can use $\gamma_0(x) = \|x\|^2$, the square of the Euclidian norm. In some specific cases we shall use other functions γ_0. For the sake of simplifying the computations we can omit the constant C without loss of generality, it is quite possible to do with.

Let $\mu_0 = \rho_0\nu$, $\mu_1 = \rho_1\nu$ be two probability measures on an open set Ω, assume that ρ_0, ρ_1 are bounded from below by a constant. Further we assume that ρ_0 and ρ_1 are locally Lipschitz and that equation

$$\Delta u - \nabla\gamma_0 \cdot \nabla u = \rho_0 - \rho_1$$

can be solved for some $u \in C^{1,1}_{loc}(\Omega)$ (that is ∇u is locally Lipschitz).
Then, define a locally Lipschitz vector field $\xi(t, x) = \frac{\nabla u(x)}{(1-t)\rho_0(x)+t\rho_1(x)}$ with the associated flow $(T_t(x)_{0\leq t\leq 1})$ and a family $(\mu_t)_{0<t<1}$ of probability measures by $\mu_t = (1-t)\mu_0 + t\mu_1$.
It is easy to see that $\frac{\partial\mu_t}{\partial t} = (\rho_1 - \rho_0)\nu$, $\nabla.(\nabla u\, e^{-\gamma_0}vol) = e^{-\gamma_0}(\Delta u - \nabla\gamma_0 \cdot \nabla u)vol = \nu(\rho_0 - \rho_1)$.
So μ_t satisfies the formula of conservation of mass, therefore $\mu_t = (T_t)_{\#}\mu_0$. In particular T_1 pushes μ_0 forward to μ_1.

Now, we are in situation to discuss on how to transform the general problem of pollution into a transport equation. This will be a guidance for our study of the subject as an optimal mass transport problem.
Let us consider the general system describing the evolution of the pollution in a given porous medium.

$$\begin{cases} \frac{\partial\varepsilon\rho_s}{\partial t} + div(\rho_s q) = 0 \\ \frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s W q + J) = 0 \\ q = \varepsilon V. \end{cases} \tag{6}$$

Expanding the second equation of (6), we have

$$W(\frac{\partial\varepsilon\rho_s}{\partial t} + div(\varepsilon\rho_s V)) + \varepsilon\rho_s\frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W + divJ = 0.$$

And, thanks to the first equation, the above second equation becomes

$$\varepsilon\rho_s\frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W + divJ = 0.$$

The main discussion is to see if it is possible to transform this latter equation into another one expressed by means of a measure μ_t and a vector field ξ to be found.
Let us suppose that J is given and $\mu_t \neq 0$. For the vector field $\xi(t, x)$ equal to $\frac{J}{\mu_t(x)}$, it is easy to see that:

$$div(\mu_t(x)\xi(t, x)) = divJ = -(\varepsilon\rho_s\frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W).$$

To get a positive answer, it suffices that the following equation makes sense in a framework of functional spaces to be precised later.

$$\frac{\partial\mu_t}{\partial t} = \varepsilon\rho_s\frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W.$$

But let us point out that it will be quite possible to give a meaning to $\mu_t(x)$ as a distribution whenever W, $V\,\rho$ and ε, are in the distribution space. The time derivative of the measure μ_t is to be understood in the weak sense.
If we are in the situation where μ_t exists, it will be interesting too, to look for the existence of $\varepsilon\rho_s$ and W.

Now, let us give two non null concentration distributions: $W(0, x) = W_0(x)$ at the time $t = 0$ and $W(1, x) = W_1(x)$ at the time $t = 1$.
For $\xi(t, x) = \frac{J}{(1-t)W_0(x)+tW_1(x)}$ and $\nu = e^{-\gamma_0(x)}vol(dx)$, we set $\mu_0 = \nu W_0, \mu_1 = \nu W_1$.
In this case, for $\mu_t = (1-t)\mu_0 + t\mu_1$, we have

$$div(\mu_t\xi(t, x)) = div(e^{-\gamma_0}vol(dx)J) = (divJ - \nabla\gamma_0 \cdot J)\nu.$$

Replacing *div J* by its expression, we have:

$$divJ - \nabla\gamma_0 \cdot J = -(\varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V \cdot \nabla W) - \nabla\gamma_0.J.$$

Finally we can claim that:

$$divJ - \nabla\gamma_0 \cdot J = W_0 - W_1$$

whenever it is possible to solve the following system of PDE

$$\begin{cases} \frac{\partial\varepsilon\rho_s}{\partial t} + div(\rho_s\varepsilon V) = 0 \\ -(\varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V \cdot \nabla W) - \nabla\gamma_0.J = W_0 - W_1 \end{cases} \tag{7}$$

And from that situation, we have:

$$div(\mu_t\xi) = -\frac{\partial\mu_t}{\partial t} \tag{8}$$

In the next section, we are going to be more precised by giving details on all what we have just done as discussions in this section.

3. Pollution as Optimal Mass Transport Problem

3.1 Mass Transport Problem

In this section, we suppose that

- the space of measures acting is a time-space domain $Q =]0, T[\times\Omega$ where Ω is a bounded Lipschitz open subset of $\mathbb{R}^d$ with outward normal vector n_Ω. In the sequel we are going to consider $T \equiv 1$.
- the mass density $\rho(t, x)$ at the position x and time t is a Borel measure supported on $\overline{Q}$, i.e. $\rho \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^+)$;
- the velocity field $v(t, x)$ of a particle at (t, x) is a Borel vectorial measure supported on $\overline{Q}$;
- the velocity field $\xi(t, x)$ of the flow at (t, x) is a Borel vectorial measure supported on $\overline{Q}$ (i.e $\xi \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^d)$) and defined by $\xi(t, x) = \rho(t, x)v(t, x)$.

The Monge-Kantorovich mass transportation problem consists in solving the following optimization problem:

$$\min\{\Psi(\rho, \xi) : \quad \rho \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^+), \quad \xi \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^d)\} \tag{9}$$

with the constraints:

$$\begin{cases} -\partial_t\rho - div_x\xi = 0 \ \text{ in } Q, \\ \rho(0, x) = \rho_0(x) \quad \rho(1, x) = \rho_1(x), \\ \xi \cdot n_\Omega = 0 \ \text{ on } \ ]0, 1[\times\partial\Omega. \end{cases} \tag{10}$$

where Ψ is an integral functional on the $\mathbb{R}^{1+d}$-valued measures defined on $\overline{Q}$. Note that (10) is the continuity equation of our mass transportation model.

Theorem 3.1. *Let $W(0, x) = W_0(x)$ at the time $t = 0$ and $W(1, x) = W_1(x)$ at the time $t = 1$ be two concentration measures, where $x \in \overline{Q}$.*
For any $t \in [0, 1]$ $\xi(t, x)$ be a vector in $\mathbb{R}^d$ and for a measure $\nu = e^{-\gamma_0(x)}vol(dx)$, we set $\mu_0 = \nu W_0, \mu_1 = \nu W_1$.
Then the following problem

$$\min\{\Psi(\mu, \xi) : \quad \mu \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^+), \quad \xi \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^d)\} \tag{11}$$

where Ψ is an integral functional on the $\mathbb{R}^{1+d}$-valued measures defined on $\overline{Q}$ by

$$\Psi(\mu, \xi) = \int_{\overline{\Omega}} \int_0^1 |\xi(t, x)|^2 d\mu(t, x),$$

with the constraints:

$$\begin{cases} -\frac{\partial\mu_t}{\partial t} - div_x(\mu_t\xi) = 0 \ \textit{ in } Q, \\ \mu(0, x) = \mu_0(x) \quad \mu(1, x) = \mu_1(x), \\ \xi \cdot n_\Omega = 0 \ \textit{ on } \ ]0, 1[\times\partial\Omega; \end{cases} \tag{12}$$

admits a solution.
The minimum value called $\mathcal{W}(\mu_0, \mu_1)$ *of the minimization problem is the square of the* L^2 *Kantorovich distance i.e.*

$$\mathcal{W}(\mu_0, \mu_1) = (W_2(\mu_0, \mu_1))^2 =$$

$$\min\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x_1 - x_2|^2 \, d\gamma(x_1, x_2) : \ \pi_1^{\#}\gamma = \mu_0 \ and \ \pi_2^{\#}\gamma = \mu_1 \}.$$

W_2 *is the classical 2- Wasserstein distance.*

Proof. Under the hypotheses of the theorem, it suffices to apply the proofs of the main results in [Y. Brenier (2003)] and [G. Buttazzo and al. (2007)]. □

Remark 3.2. *As mentioned in [G. Buttazzo and al. (2007)], mass should move along*

- *straight lines when* Ω *is a convex set;*
- *geodesic curves when* Ω *is not convex.*
- *It is important to point out that the above theorem with if we consider :*
  $\mu(t, x) = \varepsilon(t, x)\rho_s(t, x)$, $\varepsilon(0, x)\rho_s(0, x) = \mu_0(x)$ *and* $\varepsilon(1, x)\rho_s(1, x) = \mu_1(x)$,
  then we can say that the problem

  $$\min\{\Psi(\mu, \xi) : \quad \mu \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^+), \quad \xi \in \mathcal{M}_b(\overline{Q}, \mathbb{R}^d)\} \tag{13}$$

  where Ψ *is an integral functional on the* $\mathbb{R}^{1+d}$*-valued measures defined on* $\overline{Q}$ *by*

  $$\Psi(\mu, \xi) = \int_{\overline{\Omega}} \int_0^1 |\xi(t, x)|^2 d\mu(t, x),$$

  with the constraints:

  $$\begin{cases} -\frac{\partial \mu_t}{\partial t} - div_x(\mu_t \xi) = 0 \ \ in \ Q, \\ \mu(0, x) = \mu_0(x) \quad \mu(1, x) = \mu_1(x), \\ \xi \cdot n_\Omega = 0 \ \ on \ \ ]0, 1[\times\partial\Omega; \end{cases} \tag{14}$$

  admits a solution. This means that $(\mu(t, x), V(t, x))$ *does exist.*
- *For additional details about the existence of the optimal transport and the candidate map, we refer the reader to some references such as [L. Ambrosio and al. (2005)], [F. Santambrosio (2015)] (Theorem 8.1 and the previous needed results), [J. Benamou and Y. Bernier (2005)]. But let us give some hints that we shall connect to system of PDE modelling the pollution problem which is our subject of studies.*

And we can state the following theorem.

Theorem 3.3. *Let us consider the following functional environment defined by:*
$\mathcal{V}(\mu_0, \mu_1)$ *be the set of all* $(\mu, \xi) = (\mu(t, .), \xi(t, .))$ *such that*

$$\begin{cases} \mu \in C([0, 1] : w^* - \mathcal{P}_{ac}(\mathbb{R}^d)); \\ \xi \in L^2(d\mu(t, x)); \\ \cup_{0\leq t\leq 1} spt(\mu(t, .)) \ \ is \ bounded; \\ -\frac{\partial \mu_t}{\partial t} - div_x(\mu_t \xi) = 0 \ \ in \ Q, \\ \mu(0, x) = \mu_0(x) \quad \mu(1, x) = \mu_1(x), \end{cases} \tag{15}$$

where $\mathcal{P}_{ac}(\mathbb{R}^d))$ *is the space of absolutely continuous (with respect of Legesgue measure) probability measures on* $\mathbb{R}^d$ *; in fact it can be identified with a subspace of* $L^1(\mathbb{R}^d)$ *and* $w^* - \mathcal{P}_{ac}(\mathbb{R}^d))$ *stands for the set* $\mathcal{P}_{ac}(\mathbb{R}^d))$ *endowed with the weak* $-*$ *topology.*
Then, under the above functional framework, to solve the following system of PDE standing for a model of pollution

$$\begin{cases} \frac{\partial \varepsilon\rho_s}{\partial t} + div(\rho q) = 0 \\ \frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s W q + J) = 0 \\ q = \varepsilon V. \end{cases} \tag{16}$$

it suffices to be able to solve the following transport equation

$$\begin{cases} \frac{\partial \mu_t}{\partial t} = \varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W \ in \ \ ]0,T[\times\Omega \\ W(0,x) = W_0(x) \ in \ \Omega \end{cases} \tag{17}$$

where the existence of $V, \varepsilon\rho_s$, *and* μ_t *is ensured by the existence of the optimal transport maps.*

Before proving this theorem by optimal mass transportation theory, let's give some foundamental and important notions and results (see for instance [W. Gangbo (2004)]).

Let's suppose that in a region (place) $X \subset \mathbb{R}^d$, we have sand piles and in another region Y in $\mathbb{R}^d$, we have holes to be filled by the sand.

In optimal mass tranport the aim is to minimize the cost to move the sand from X to Y.

For this, we shall introduce a model expressed as follows:

Let μ_o be the mass density of the sand and μ_1 the mass density of the holes.

Suppose that:

for any $(x, y) \in X \times Y$ we associate $c(x, y) > 0$ the transport cost from the point x to the point y. Let T be the strategy of transport used for which x is moved to Tx.

T satisfies the conservation of the mass expressed as follows:

$$\int_{T^{-1}[B]} \mu_o(x)\, dx = \int_B \mu_1(y)\, dy \ \ \forall B \subset \mathbb{R}^d.$$

Whenever this above equality is satisfied, we say that T is a strategy to move μ_o onto μ_1 and the notation used, is $T_\# \mu_o = \mu_1$.

Let $x \in X$, if $\mu_o(x)\, dx$ is a mass density in a neighborhood of x, then the transport cost of $\mu_o(x)\, dx$ from the initial position (neighborhood of x)onto a neighborhood of Tx is $c(x, Tx)\, \mu_o(x)\, dx$.

And then the total cost to transport the sand with density equal to μ_o onto the holes with capacity (density) μ_1 is given by:

$$cost\ [T] = \int_{\mathbb{R}^n} c(x, Tx)\, \mu_o(x)\, dx.$$

The Monge's problem consists in finding a solution of the following problem

$$\inf_T \left\{ \int_{\mathbb{R}^n} c(x, Tx)\, \mu_o(x)\, dx\ ,\ T_\# \mu_o = \mu_1 \right\}.$$

Relaxation of the Monge's Formulation: The Kantorovich's Formulation

Let X and Y be the supports respectively of the measures μ_o et μ_1, and let $T : X \longrightarrow 2^Y$ be a plan (relation) where $2^Y = \{A, A \subset Y\}$.

For any $x \in X$, we associate the measure γ_x supported by the set Tx which explains us how the mass located in x is distributed through Tx. Then the transport cost from x to Tx is given by:

$$\int_{\{y\in Y\ /\exists x\in X\ ;\ y=Tx\}} c(x, y)\, d\gamma_x(y).$$

The total transport cost from μ_o to μ_1 is equal to

$$\bar{I}[T, \{\gamma_x\}_{x\in X}] = \int_X \Big[\int_{\{y\in Y\ /\exists x\in X\ ;\ y=Tx\}} c(x, y)\, d\gamma_x(y) \Big]\, d\mu_o(x).$$

It is convenient to define a measure γ on $X \times Y$ which contains the information coded in $[T, \{\gamma_x\}_{x\in X}]$ by :

$$\int_{X\times Y} F(x, y)\, d\gamma(x, y) = \int_X \left[\int_{\{y\in Y\ /\exists x\in X\ ;\ y=Tx\}} F(x, y)\, d\gamma_x(y) \right]\, d\mu_o(x)$$

and the measure γ has to satisfy the following conservation mass conditions:

$$\mu_o[A] = \gamma[A \times Y] \ \text{ and } \ \gamma[X \times B] = \mu_1[B]$$

$\forall A \subset X$ and $B \subset Y$ borelian sets.

The Kantorovich's problem, in terms of γ is a relaxation of the Monge's problem.

And it is quite possible to extend the set $\mathcal{T}(\mu_o, \mu_1)$ of plans defined by $T : X \longmapsto Y$ such that $T_{\#}\mu_o = \mu_1$ to a bigger set $\Gamma(\mu_o, \mu_1)$ and the function defined by $I : T \longmapsto I[T] = \int_X c(x, Tx)\, \mu_o(x)\, dx$ to the function $\bar{I}$ defined on $\Gamma(\mu_o, \mu_1)$.

Thus if $\mathcal{T}(\mu_o, \mu_1) \neq \emptyset$, then we have

$$\inf_{\mathcal{T}(\mu_o,\mu_1)} I \geq \inf_{\Gamma(\mu_o,\mu_1)} \bar{I}.$$

In fact

$$\inf_{\mathcal{T}(\mu_o,\mu_1)} I = \inf_{\Gamma(\mu_o,\mu_1)} \bar{I}.$$

Definition 3.4. *Let μ_o and μ_1 be two measures defined respectively on X and Y.*

i) The map $T : X \longrightarrow Y$carries μ_o to μ_1 and one notes

$$T_{\#}\mu_o = \mu_1 \;\; if \;\; \mu_1[B] = \mu_o[T^{-1}(B)]\; \forall B \subset Y.$$

ii) Let γ be a measure defined on $X \times Y$, then the projection $proj_{X\gamma}$ $(resp.proj_{Y\gamma})$ is a measure defined on X (resp.on Y) as follows:

$$proj_X\gamma(A) = \gamma[A \times Y] \quad \forall A \subset X \tag{18}$$

$$(resp\ proj_Y\gamma(B) = \gamma[X \times B] \quad \forall B \subset Y) \tag{19}$$

iii) μ_o and μ_1 are to be said marginals of a measure γ on $X \times Y$ if $\mu_o = proj_X\gamma$ and $\mu_1 = pro_Y\gamma$ and we say that $\gamma \in \Gamma(\mu_o, \mu_1)$ and γ is the transport scheme from μ_o in μ_1.

$$\Gamma(\mu_o, \mu_1) = \{\gamma : X \times Y \longrightarrow \mathbb{R}^n\ ,\ \gamma_{\#}\mu_o = \mu_1\}.$$

Let 's review the Monge's problem:

The aim is to find a minimum for the following functional

$$\int_{\mathbb{R}^n} c(x, Tx)\, d\mu_o(x); \tag{20}$$

where $T \in \mathcal{T}(\mu_o, \mu_1)$ i.e $T_{\#}\mu_o = \mu_1$.
And

the Kantorovich's problem consists in finding minimum for the following functional

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} c(x, y)\, d\gamma(x, y); \tag{21}$$

where $\gamma \in \Gamma(\mu_o, \mu_1)$.

Remark 3.5. *It is not easy to show the existence of the minimum for the Monge's problem. But for the Kantorovich's, it is possible most of the time to show easier the existence of the minimum.*

We can sum up one of the first main results in optimal mass transportation as follows:

Theorem 3.6. *Given μ and ν probability measures on a compact domain $\Omega \subset \mathbb{R}^d$ there exists an optimal transport plan γ for the cost $c(x, y) = h(x - y)$ with h strictly convex. It is unique and of the form $(id, T)_{\#}\mu$, provided μ is absolutely continuous and $\partial\Omega$ is negligible. Moreover, there exists a Kantorovich potential ϕ, and T and the potentials ϕ are linked by*

$$T(x) = x - (\nabla h)^{-1}(\nabla\phi(x)).$$

For the proof see for instance [F. Santambrosio 2015].

Proof. of the theorem 3.3

Existence of the two optimal transport maps T_t and S_t :

From this step, we shall deduce the existence of $\varepsilon\rho_s; V$.

It suffices to apply the Benamou-Brenier theorem and techniques in the proof see for instance [J. Benamou and Y. Bernier (2000)], and [F. Santambrosio (2015)] to each of the following compatibility equations:

$$\frac{\partial \rho_t}{\partial t} + div(\rho_t \Psi) = 0;$$

$$\frac{\partial \varepsilon\rho_s}{\partial t} + div(\rho\varepsilon V) = 0$$

As a consequence of the above theorem 3.6, we can particularize the above theorem to the quadratic case $c(x, y) = \frac{1}{2}|x-y|^2$, thus getting the existence of an optimal transport map

$$T(x) = x - \nabla\phi(x) = \nabla(\frac{x^2}{2} - \phi(x)) = \nabla u(x)$$

for a convex function u.

For $t \in [0, 1]$, let us set $T_t(x) = (1 - t)x + tT(x)$ then $\mu_t = (T_t)_{\#}\mu_0$.
$\xi_1(t, x) = \Psi(t, x) = (T - Id)oT_t^{-1}$ if $\mu_t \neq 0$, and we define $\xi_1(t, x)$ to be 0 whenever $\mu_t \equiv 0$.

And in the same way we have

$\xi_2(t, x) = (S - Id)oS_t^{-1}$ if $m_t \neq 0$, and we define $\xi_2(t, x)$ to be 0 whenever $m_t \equiv 0$.

Let us note that $\xi_1, \xi_2 \in L^2(d\mu(t, x))$. In fact: $||\xi_1||_{L^2(\mu(t,.))} = |\mu^{'}|(t)$ where $|\mu^{'}|(t)$ denotes the metric derivative at time t of the curve $t \mapsto \mu(t, .)$ with respect to the Wasserstein distance $d = W_2$.

The above equality holds also for ξ_2.

The general definition for the metric derivative, can be expressed as follows:

Let $\mathcal{P}_2(\Omega) = \{\mu \in \mathcal{P}(\Omega) : \int_\Omega |x|^2 d\mu(x) < +\infty\}$.

If $w : [0, 1] \to \mathcal{P}_2(\Omega)$ is a curve valued in the metric space (X, d) we define the metric derivative of w at time t, denoted by $|w^{'}|(t)$through

$$|w^{'}|(t) := \lim_{h\to 0} \frac{d(w(t + h), w(t))}{|h|},$$

provided this limit exists.

And finally, having at hands $\mu_t, \varepsilon\rho_s$ and V.

Existence of the Two Optimal Transport Maps W :

To find W, we have to look at the following PDE , $\frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s Wq + J) = 0$.

After expansion, it is equivalent to :

$$\varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W + divJ = 0$$

Let us set now $\mu_t\xi_1 = J$, then $\frac{\partial \mu_t}{\partial t} + divJ = 0$.
Combining the two last equalities, we get

$$\frac{\partial \mu_t}{\partial t} = \varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W(**)$$

From the first part of the proof, μ_t is known and then $\frac{\partial \mu_t}{\partial t}$ is a datum. By the hypothesis $(**)$ we get W. □

This theorem makes sense if it is possible to solve the above transport equation which is not an easy task.

3.1.1 Discussion

Let set $m_t(x) := m(t,x) = \varepsilon(t,x)\rho_s(t,x)$. The Cauchy problem becomes

$$\begin{cases} \frac{\partial \mu_t}{\partial t} = \varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V.\nabla W \text{ in }]0,T[\times\Omega \\ W(0,x) = W_0(x) \text{ in } \Omega \end{cases} \tag{22}$$

It is important to point out that the transport equation si still a great of interest for the mathematician community working on PDE topic and and physicists.

Let's suppose at first that $m_t \neq 0$, and set $f_t = \frac{1}{m_t}\frac{\partial \mu_t}{\partial t}$ and $V = b_t$.

The question is to see if under BV or Sobolev regularity of f_t and b_t, the above Cauchy problem gets solution. And if yes is it possible to get information about the uniqueness, to define a flow?

The authors are not aware that the answer is always positive in very weak hypotheses or if the question is closed. But they know that relevant works due to [R. Diperma and P.L. Lions (1989)], [L. Ambrosio (2004)], [L. Ambrosio (2005)] and [C. De Lellis (2007)] on these questions have been done.

In [R. Diperma and P.L. Lions (1989)], DiPerna and Lions showed existence, uniqueness and stability of regular Lagrangian flows for Sobolev vector fields with bounded divergence.

In [L. Ambrosio (2004)], Ambrosio extended to BV coefficients, see Theorem 3.5 (renormalization property) (pp 14).

These two relevant papers have been followed by other very interesting ones bringing great progress in the topic such as [L. Ambrosio and G. Crippa (2008)], [L. Ambrosio and G. Crippa (2013)] , [G. Crippa and C. De Lellis (2008)], [G. Crippa and C. De Lellis (2008) (bis)] .

Let's now give some particular cases for the solvability of our transport equation:

- If V satisfies the hypotheses given in [R. Diperma and P.L. Lions (1989)], pp 4 and $f_t \in L^1(0,T;L^p(\mathbb{R}^d)), T > 0, p \in [0,+\infty]$ and $W_0 \in L^p(\mathbb{R}^d)$, then Proposition II.1 and remark (see pp 4 [R. Diperma and P.L. Lions (1989)]) ensure us existence of solution $W \in L^\infty(0,T;L^p(\mathbb{R}^d))$. And the flow exists and is unique for

$$\begin{cases} \frac{dX}{dt} = V(X), X \in \mathbb{R}^d \\ X(0)) = x \end{cases} \tag{23}$$

- When $m_t := m(x)$ does not depend on t and $m(x) > 0$, then if V satisfies the same hypotheses as in the above first item, formally we should have $W(t,X(t)) = \frac{\mu_t(X(t))}{m(x)} + W_0(x)$ in the distribution sense; where $\dot{X} = V, X(0) = x$.

After these two particular cases, let us point out that the regularity of V is a priori $L^2(d\mu(t,x))$ and f_t is a measure and may be singular.

Another important fact is the regularity of the optimal transport map. In fact, since in our case the cost function c is in quadratic form as explained above, for $\mu_0 = f(x)dx; \mu_1 = g(y)dy$ such that $0 < C_1 \leq f, g \leq C_2$ where C_1, C_2 are constants then $T \in W^{1,1}$; for additional details the proof see [G. De Phillips and A. Figalli (2013)], [G. De Phillips and A. Figalli (2013) (bis)], [G. De Phillips and A. Figalli (2013) (ter)] and [T. Schmidt (2013)].

We end this discussion by claiming that the questions seem to be not easy according and it should be interesting to come back to this problem in next works.

In the sequel of this paper we shall be particularily interested by the case where $\xi(t,x) = \frac{J}{(1-t)W_0(x)+tW_1(x)}$ and the linear interpolation for the measure: $\mu_t = (1-t)\mu_0 + t\mu_1$. And we shall study the following system of PDE combined with initial and boundary condition if necessary

$$\begin{cases} \frac{\partial \varepsilon\rho_s}{\partial t} + div(\rho_s\varepsilon V) = 0 \\ \varepsilon\rho(0,x) = m_0 \text{ a convenient initial data} \\ -(\varepsilon\rho_s\frac{\partial W}{\partial t} + \varepsilon\rho_s V\cdot\nabla W) - \nabla\gamma_0.J = W_0 - W_1 = \frac{1}{\nu}\frac{\partial \mu_t}{\partial t}, \end{cases} \tag{24}$$

A particular non linear boundary value problem is studied too.

3.2 Partial Differential Equations

In this section we are going to solve one type of non linear partial equations deriving from basic model (5).

Remark 3.7. 1. ***Case 1:*** *ρ_s and ε are constants:*
For $\xi(t,x) = \frac{-\rho_s D\nabla W}{(1-t)W_0(x)+tW_1(x)}$ and $\nu = e^{-\gamma_0(x)}vol(dx)$, we set $\mu_0 = \nu W_0, \mu_1 = \nu W_1$.
Suppose that $\rho_s = \varepsilon = 1$ and the diffusion coefficient D is given, the system of PDE is reduced as follows

$$\begin{cases} divV = 0 \\ \frac{\partial W}{\partial t} + V\cdot\nabla W - \nabla\gamma_0.D\nabla W = W_1 - W_0 \end{cases} \tag{25}$$

2. ***Case 2:*** *ε is the only constant:*

By taking into account the conservation of the mass equation the following equation

$$\frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s Wq + J) = 0$$

is reduced as follows

$$\rho_s\frac{\partial W}{\partial t} + \rho_s V\cdot\nabla W + div(\frac{J}{\varepsilon}) = 0.$$

For $\mu_0 = W_0\nu$ and $\mu_1 = W_1\nu$ where ν is taken as above in the previous case, $J = \mathcal{F}(\nabla_x W, W)$ where $\mathcal{F}$ is a given function and $\xi(t,x) = \frac{\mathcal{F}(\nabla_x W)}{(1-t)W_0(x)+tW_1(x)}$, we have:

$$\begin{cases} \frac{\partial\rho_s}{\partial t} + \Delta\zeta = 0 \\ \nabla\zeta = \rho_s V \\ \rho_s\frac{\partial W}{\partial t} + \nabla\zeta\cdot\nabla W - \nabla\gamma_0.J = W_1 - W_0 \\ div(\nu J) = \mu_0 - \mu_1 \end{cases} \tag{26}$$

3.2.1 Combination with an Experimental Law

This subsection is devoted to deal with a particular case considering an experimental law with suitable hypotheses which are expressed as follows:

- **H-1** the fluid density of the solution $\rho_s > 0$ and is a constant;
- **H-2** the hydraulic conductivity tensor $\frac{K}{\lambda}$ and D are positive constants;
- **H-3** the evolution is isotherm i.e. the temperature T in the medium is a constant.

Because of the hypothesis **H-1**, the equations (1) and (2) become

$$\frac{\partial\varepsilon}{\partial t} + divq = 0 \text{ in } Q; \tag{27}$$

$$\frac{\partial}{\partial t}(\varepsilon W) + div(Wq + \frac{J}{\rho_s}) = 0 \text{ in } Q; \tag{28}$$

Using hypothesis **H-3**, we can establish a relation between the pressure p and the concentration W:

$$\ln\frac{\rho_s}{\rho_0} = \beta_p(p - p_0) + \gamma W.$$

Then

$$\beta_p\nabla p + \gamma\nabla W = 0.$$

So,

$$\triangle W = -\frac{\beta_p}{\gamma}\triangle p. \tag{29}$$

Replacing J and q by its values, we have:

$$\begin{cases} \frac{\partial\varepsilon}{\partial t} - \frac{K}{\lambda}\triangle p = 0 \\ \varepsilon\frac{\partial W}{\partial t} + W\frac{\partial\varepsilon}{\partial t} + W\nabla\cdot q - \frac{K}{\lambda}\nabla W\cdot\nabla p - \rho_s g\frac{K}{\lambda}\frac{\partial W}{\partial z} - D\triangle W = 0 \end{cases}$$

The equation (27) implies that $Wdivq = -W\frac{\partial\varepsilon}{\partial t}$ and therefore

$$\varepsilon\frac{\partial W}{\partial t} - \frac{K}{\mu}\nabla W\cdot\nabla p - \rho_s g\frac{K}{\lambda}\frac{\partial W}{\partial z} - D\triangle W = 0 \tag{30}$$

Without loosing of generality in the reasoning , we give up the term $-\rho_s g\frac{K}{\mu}\frac{\partial W}{\partial z} = 0$ in the above equation. Finally we are going to consider in the sequel, the below Cauchy problem translating the evolution of the pollution:

$$\begin{cases} \varepsilon\frac{\partial W}{\partial t} - \frac{K}{\lambda}\nabla W\cdot\nabla p + \frac{D\beta_p}{\gamma}\triangle p = 0 \quad on \quad]0,T[\times\Omega \\ W(0,x) = W_0(x) \quad on \quad \Omega \end{cases} \tag{31}$$

Let us set $a = \frac{\gamma K}{\beta_p\mu D}$ and $b = \frac{\gamma}{D\beta_p}$, then the transport equation becomes:

$$\begin{cases} \triangle p - a\nabla W\cdot\nabla p + b\varepsilon\frac{\partial W}{\partial t} = 0 \quad in \quad]0,T[\times\Omega \\ W(0,x) = W_0(x) \end{cases} \tag{32}$$

Remark 3.8. *Let's suppose that the space is equipped with a probability measure*

$$\nu(dx) = \exp(-a\,\gamma_0)vol(dx).$$

Given concentrations $W(0,x) = W_0(x)$ at the time $t = 0$ and $W(1,x) = W_1(x)$ at the time $t = 1$, let us set

$$\mu_0 = b\varepsilon W_0\nu \quad and \quad \mu_1 = b\varepsilon W_1\nu.$$

If we define a vector field that assumed locally Lipschitz by:

$$\xi(t,x) = \frac{\nabla p}{[(1-t)W_0 + tW_1]},$$

with associated flow $(X_t(x))_{0\le t\le 1}$ and a family $(\mu_t)_{0\le t\le 1}$ of probability measures given by:

$$\mu_t = [(1-t)\mu_0 + t\mu_1].$$

Then we remark that

$$\frac{\partial\mu_t}{\partial t} = (\mu_1 - \mu_0) = b(W_1 - W_0)\varepsilon\nu$$

and

$$\begin{aligned} \nabla\cdot(\mu_t\xi(t,\cdot)) &= \nabla\cdot(\nabla p\exp(-a\,\gamma_0)vol(dx)) \\ &= b\nu(\varepsilon\triangle p - a\varepsilon\nabla p\cdot\nabla\gamma_0 + \nabla\varepsilon\cdot\nabla p) \\ &= b(W_0 - W_1)\varepsilon\nu \end{aligned}$$

And finally we have the following relation translating the continuity equation:

$$(W_0 - W_1)\varepsilon = \varepsilon\triangle p - a\varepsilon\nabla p\cdot\nabla\gamma_0 + \nabla\varepsilon\cdot\nabla p \tag{33}$$

Next we are going to focus the following type of PDE

$$\triangle p + \nabla W\cdot\xi + \frac{\partial W}{\partial t} = 0 \quad in \quad]0,T[\times\Omega$$

when $\xi = f(t,x)\nabla W$, where f is a bounded function with respect to the time t and the variable x. It is easy to remark that these types of equations translate a family of equations translating the evolution of the pollution.

3.3 Existence and Uniqueness of the Solution

Let us consider

$$\begin{cases} \frac{\partial W(t,x)}{\partial t} + \xi(t,x).\nabla W(t,x) - \frac{D}{\varepsilon}\Delta W(t,x) &= 0 \quad in\ (0,1)\times\Omega. \\ W(x,o) &= W_0(x) \quad in\ \ \Omega \\ \frac{\partial W(t,x)}{\partial\nu} &= W_1(t,x) \quad on\ \ (0,1)\times\partial\Omega \end{cases} \tag{34}$$

with

$$\xi(t,x) = \frac{1}{\underbrace{(1-t)\mu_0(x) + t\mu_1(x)}_{f(t,x)}} \nabla W(t,x) \tag{35}$$

Replacing the expression (35) in (34), we obtain in the system

$$\begin{cases} \frac{\partial W(t,x)}{\partial t} + f(t,x)|\nabla W(t,x)|^2 - \frac{D}{\varepsilon}\Delta W(t,x) & = & 0 & in\ (0,1)\times\Omega \\ W(x,0) & = & W_0(x) & in\ \ \Omega \\ \frac{\partial W(t,x)}{\partial \nu} & = & W_1(t,x) & on\ \ (0,1)\times\partial\Omega \end{cases} \tag{36}$$

Theorem 3.9. *Let $W(t,x)$ be the solution of (36). Setting $u(t,x) := \psi(W(t,x))$, then $u(t,x)$ is the solution of the following partial differential equation,*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \frac{\theta(x)}{\varepsilon}\Delta u(x,t) & = & 0 & in\ \Omega\times(o,1) \\ u(x,0) & = & u_0(x) & \Omega\times\{t=0\} \\ \frac{\partial u(x,t)}{\partial \nu} & = & u_1(x,t) & on\ \partial\Omega\times(0,1) \end{cases} \tag{37}$$

with $\psi(x)$ solution of the following ordinary differential equation.

$$\frac{\psi''(s)}{\psi'(s)} = -\frac{\varepsilon\, f(x,t)}{D} \tag{38}$$

Proof. Let $u(x,t) = \psi(W(x,t))$, then

$$\frac{\partial u}{\partial t} = \psi'(W)\frac{\partial W}{\partial t}, \quad \nabla u = \psi'(W)\nabla W, \quad \Delta u = \psi''(W)|\nabla W|^2 + \psi'(W)\Delta W.$$

$$\frac{\partial u}{\partial t} = \psi'(W)\frac{\partial W}{\partial t} = -\psi'(W)\left[f(x,t)|\nabla W(x,t)|^2 - \frac{D}{\varepsilon}\Delta W(x,t)\right]$$

$$\frac{\partial u}{\partial t} = -\psi'(W)f(x,t)|\nabla W(x,t)|^2 + \frac{D}{\varepsilon}\psi'(W)\Delta W(x,t)$$

$$\frac{\partial u}{\partial t} = -\psi'(W)f(x,t)|\nabla W(x,t)|^2 + \frac{D}{\varepsilon}\Delta u - \frac{D}{\varepsilon}\psi''(W)|\nabla W|^2$$

$$\Longleftrightarrow \frac{\partial u}{\partial t} - \frac{D}{\varepsilon}\Delta u = -|\nabla W(x,t)|^2\left[\psi'(W)f(x,t) + \frac{D}{\varepsilon}\psi''(W)\right]$$

It follows that $\frac{\partial u}{\partial t} - \frac{D}{\varepsilon}\Delta u = 0$ since ψ solves the ordinary differential equation

$$\psi'(W)f(x,t) + \frac{D}{\varepsilon}\psi''(W) = 0 \tag{39}$$

In order to compute ψ, let us suppose that the integral

$$\int_0^x \frac{\varepsilon f(s,t)}{D} ds < \infty. \quad \forall\ t\ \in\ [0,1]$$

and we set

$$K(x,t) = \int_0^x \frac{\varepsilon f(s,t)}{D} ds$$

The expression (39) implies

$$\int \frac{\psi''(s)}{\psi'(s)} ds = \log(\psi'(x)) = -K(x,t) \Longleftrightarrow \psi(x) = \int_0^x e^{-K(s,t)} ds \tag{40}$$

Conversely, if ψ is in the form (40) and $u(x,t)$ the solution of (37), the function $W(x,t) = \psi^{-1}(u(x,t))$ is solution of (36), see [I? Faye, A. Sy and D. Seck (2008)] for details. □

4. Numerical Simulations

In the numerical results, we suppose that the porosity $\varepsilon = 1$ so that,

$$\psi'(x) = e^{-xf(x,t)} \quad \Longrightarrow \quad \psi(x,t) = -\frac{1}{f(t)}e^{-xf(x,t)}.$$

Our process to realize numerical simulations is organized as follows:

1. We solve the below boundary value problem by Matlab software;

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{D}{\varepsilon}\Delta u(t,x) & = & 0 & (0,1)\times\Omega \\ \frac{\partial u(0,x)}{\partial \nu} & = & u_0(x) & \Omega \\ u(0,x) & = & u_1(x) & \Omega \end{cases} \tag{41}$$

   It is followed by the numerical computation of

2. $$W(t,x) = \psi^{-1}(u(t,x)) = -\frac{1}{f(t,x)}\log(-u(t,x)f(t,x)) \tag{42}$$

   and

3. $\xi(t,x)$ is given by

$$\xi(t,x) = \frac{1}{(1-t)\mu_0(x) + t\mu_1(x)}\nabla W(t,x)$$

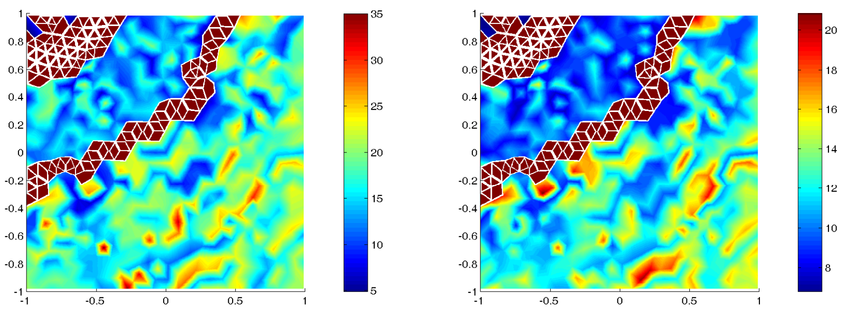
And finally we do some numerical representation in order to illustrate the theoretical study of our subject.

4.1 Example 1

In this example, the following data are used: $\mu_0 = \xi_0 = 1,\ \mu_1 = 2$

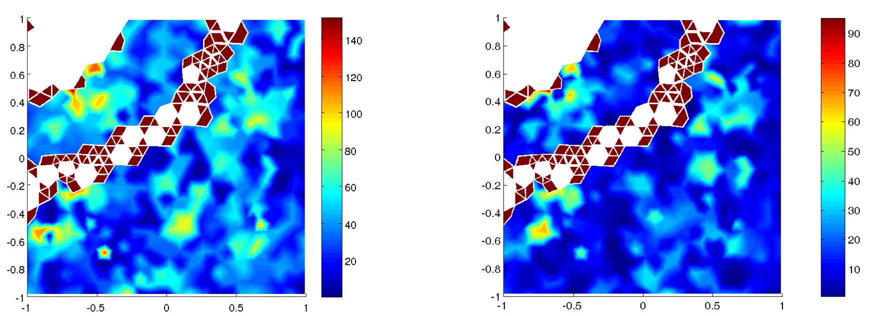
$\Rightarrow\ f(x,t) = f(t) = \frac{1}{1+t}$ and a Neumann non-homogeneous boundary condition on Γ_N an a Neumann homogeneous boundary condition on Γ

$W(x,t)$:



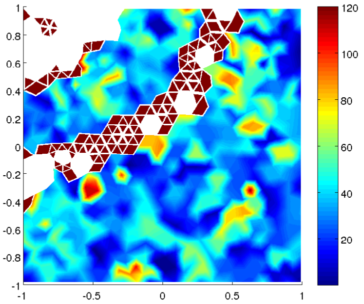
(a) $\frac{\partial u(x,0)}{\partial \nu} = x,$ (b) $\frac{\partial u(x,0)}{\partial \nu} = -1$

The first components of $\xi(x,t)$

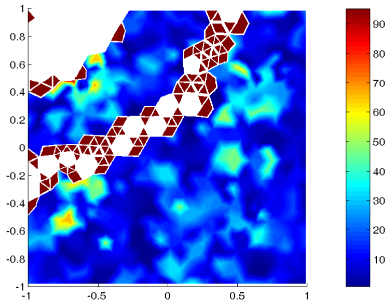


(a) $\frac{\partial u(x,0)}{\partial \nu} = x,$ (b) $\frac{\partial u(x,0)}{\partial \nu} = -1$

The second components of $\xi(x, t)$



(a) $\frac{\partial u(x,0)}{\partial \nu} = x,$

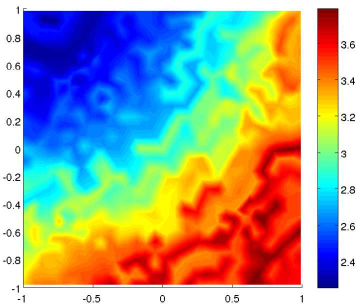


(b) $\frac{\partial u(x,0)}{\partial \nu} = -1$

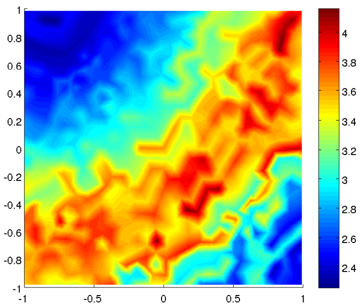
4.2 Example 2

In this example, the following data are used: $\mu_0 = \xi_0 = 1,\ \mu_1 = |x|$
$\Rightarrow f(x, t) = \frac{1}{(1-t)+t|x|}$ and a Dirichlet homogeneous conditions on the boundary Γ_N and a Neumann homogeneous boundary condition on Γ

$W(x, t)$

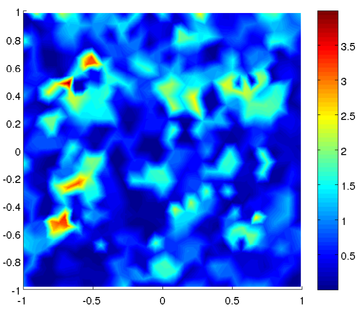


(a) $\frac{\partial u(x,0)}{\partial \nu} = x,$

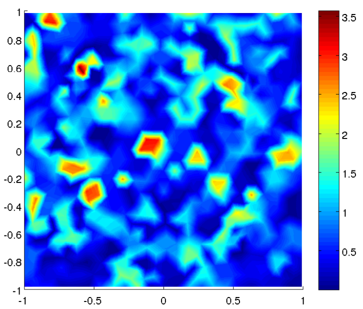


(b) $\frac{\partial u(x,0)}{\partial \nu} = -1.$

The first components of $\xi(x, t)$

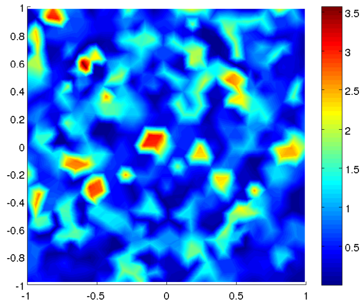


(a) $\frac{\partial u(x,0)}{\partial \nu} = x,$

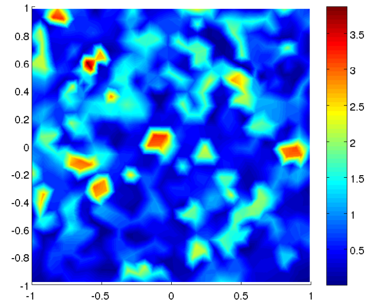


(b) $\frac{\partial u(x,0)}{\partial \nu} = -1$

The second components of $\xi(x, t)$



(a) $\frac{\partial u(x,0)}{\partial \nu} = x$,



(b) $\frac{\partial u(x,0)}{\partial \nu} = -1$.

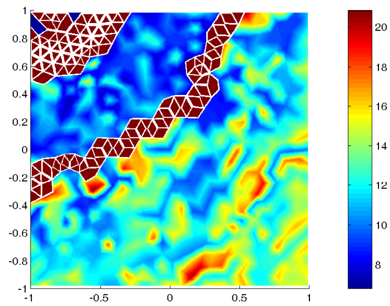
4.3 Example 3

In this example we shall treat two cases: for $\gamma_0 = \|x\|^2$ and for $\gamma_0 = 0$. ρ_s and ε are constants

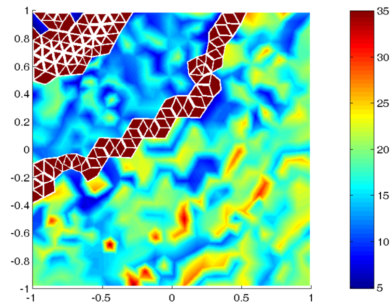
- $\gamma_0 = \|x\|^2$:
  For $\xi(t,x) = \frac{-\rho_s D\nabla W}{(1-t)W_0(x)+tW_1(x)}$ and $\nu = e^{-\gamma_0(x)} vol(dx)$, we set $\mu_0 = \nu W_0, \mu_1 = \nu W_1$.
  Suppose that $\rho_s = \varepsilon = 1$ and the diffusion coefficient D is given, the system of PDE is reduced as follows

$$\left\{ \begin{array}{rcl} divV & = & 0 \\ \frac{\partial W}{\partial t} + V \cdot \nabla W - \nabla\gamma_0.D\nabla = W_1 - W_0 & & \end{array} \right. \quad (43)$$

$W(x,t)$ for $W_0 = 1$ $W_1 = \pi$ and $D = 1$.



(a) $\frac{\partial u(x,0)}{\partial \nu} = x$,

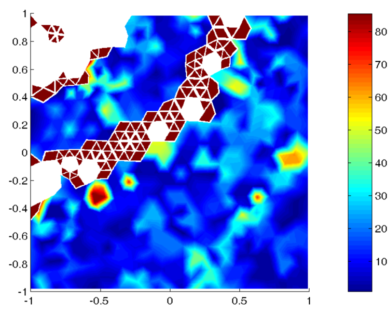


(b) $\frac{\partial u(x,0)}{\partial \nu} = -1$.

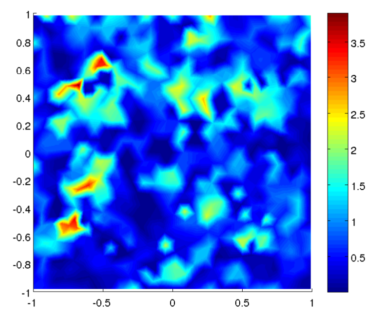
- $\gamma_0 = 0$

$$\left\{ \begin{array}{rcl} divV & = & 0 \\ \frac{\partial W}{\partial t} + V \cdot \nabla W = W_1 - W_0 & & \end{array} \right. \quad (44)$$

$W(x,t)$ for $W_0 = 1$ $W_1 = \pi$ and $D = 1$.



(a) $\frac{\partial u(x,0)}{\partial \nu} = x$,



(b) $\frac{\partial u(x,0)}{\partial \nu} = -1$.

Remark 4.1. *In the case of Neumann homogeneous boundary condition on Γ_N, the product $u(x,t).f(x,t)$ which appear in (42) admits some singular points. This explains why $\nabla W(x,t)$ goes to infinity somewhat. This appear clearly in nmerical simulations.*

5. Extensions

This section is devoted to set issues as perspectives for this paper.

1. Instead of limiting numerical simulations in the case of linear interpolation we think that it should be interesting to try numerical analysis in the case of optimal mass transportation by considering:

$$\begin{cases} \frac{\partial \varepsilon\rho_s}{\partial t} + div(\rho_s \varepsilon V) = 0 \\ \varepsilon\rho(0,x) = m_0 \text{ a convenient initial data} \\ -(\varepsilon\rho_s \frac{\partial W}{\partial t} + \varepsilon\rho_s V \cdot \nabla W) - \nabla\gamma_0.J = \frac{1}{\nu}\frac{\partial \mu_t}{\partial t}, \end{cases} \tag{45}$$

   and minimal additional hypotheses needed, like in [J. Benamou and Y. Bernier (2000)], [L. Ambrosio and al. (2005)] and [F. Santambrosio (2015)].

2. In the case where $\gamma_0 = 0$, ε is the only constant:
   by taking into account mass conservation principle the following equation is reduced to:

$$\frac{\partial(\varepsilon\rho_s W)}{\partial t} + div(\rho_s W q + J) = 0$$

   is reduced as follows

$$\rho_s \frac{\partial W}{\partial t} + \rho_s V \cdot \nabla W + div(\frac{J}{\varepsilon}) = 0.$$

   For $\mu_0 = W_0\nu$ and $\mu_1 = W_1\nu$ where ν is taken as above, $J = \mathcal{F}(\nabla_x W, W)$ where $\mathcal{F}$ is a given function and $\xi(t,x) = \frac{\mathcal{F}(\nabla_x W)}{(1-t)W_0(x) + tW_1(x)}$, we have:

$$\begin{cases} \frac{\partial \rho_s}{\partial t} + \Delta\zeta = 0 \\ \nabla\zeta = \rho_s V \\ \rho_s \frac{\partial W}{\partial t} + \nabla\zeta \cdot \nabla W - div(J) = 0 \end{cases} \tag{46}$$

   It would be interesting to investigate the two following cases:

   - if $\mathcal{F}(\nabla_x W, W) = |\nabla_x W|^{p-2}\nabla_x W$, $1 \le p < +\infty$ we have in the last equation of the above system, the p-Laplacian equation ;
   - if $\mathcal{F}(\nabla_x W, W) = \rho_s \nabla_x W^m$, $m \ge 1$ we have at hands, the familiar term of the porous media equation.

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