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Computable Error Bounds for a Class of Boundary Value Problems: a Coefficients Perturbation Approach

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Abstract

This paper is concerned with computing upper and lower bounds for the error committed when some boundary value problems are approximated by means of numerical techniques based on the Coefficients Perturbation Methods. These computed bounds are expressed in terms of the perturbations introduced in the differential equation and in the prescribed boundary conditions associated with it. Numerical examples demonstrating the sharpness of our results are given.

Keywords: Coefficient perturbation, series expansions, Green's function, boundary value problem.

2000 *Mathematics Subject Classification.* 65L05, 65L70.

1. Introduction

Let $y(x)$ be a twice differentiable function in $I := [x_0, x_1]$ that satisfies the following boundary value problem (b.v.p.):

$$(Dy)(x) := y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad x \in I, \tag{1}$$

$$L_i[y] := c_{i1}y(x_i) + c_{i2}y'(x_i) = y_i, \quad i = 0, 1 \tag{2}$$

where $\{a(x), b(x), f(x)\}$ are given functions and $\{c_{i1}, c_{i2}; i = 0, 1\}$ are some given constants.

Suppose that $\tilde{y}(x)$ is an approximation of $y(x)$ that satisfies the following perturbed version of (1)-(2):

$$(\tilde{D}y)(x) := \tilde{y}''(x) + \tilde{a}(x)\tilde{y}'(x) + \tilde{b}(x)\tilde{y}(x) = \tilde{f}(x); \quad x \in I, \tag{3}$$

$$L_i[\tilde{y}] := c_{i1}\tilde{y}(x_i) + c_{i2}\tilde{y}'(x_i) = y_i, \quad i = 0, 1 \tag{4}$$

where $\{\tilde{a}(x), \tilde{b}(x), \tilde{f}(x)\}$ are approximations of $\{a(x), b(x), f(x)\}$ chosen in such a way that the exact solution $\tilde{y}$ of (3) can be obtained analytically in a closed form.

The natural question that usually arises, when an approximation technique is employed, is how much the generated approximate solution deviates from the unknown exact solution. Results related to the issue of measuring the quality of approximation are reported in a wide range of papers: A possible approach is to find error estimates and error bounds for the solution of linear initial and boundary value problems $e(x) := y(x) - \tilde{y}(x)$ using the Coefficient Perturbation Method (Ixaru 1984, 2000, El-Daou 2002, 2006, El-Daou and Al-Mutawa 2009, El-Daou and Ortiz 1995) or with the Tau method (Ortiz 1969, Khajah & Ortiz 1991, Ojoland & Adeniyi, 2012). In (Akitoshi et al, 2010), a numerical method that provides a guarenteed error bounds was presented. This method is based on estimates of the inverse operator and the Newton-Kantorovich theorem. A computer assist method for generating a posteriori error bounds for b.v.p. is reported in (Birrell, 2015). Therein, the method utilizes a numerically generated approximation to the b.v.p. and the Green's functions. In (Corliss & Rihm 1996 , Lohner 1992, Neher, 1999 and Stetter 1990) enclosure methods and validation techniques were proposed. But the drawbacks of those techniques lie in the fact that they apply only to initial value problems or to boundary conditions of a specific form. Further, some results are not computable because they involve some unknown parameters such as the conditioning number and the Lipschitz constant.

The main concern of this article is to develop practical formulae to compute upper and lower bounds for the error function $e(x) := y - \tilde{y}$ and its derivative $e'(x)$ when boundary conditions of the form (2) are considered with arbitrary $\{c_{ij}; i, j = 1, 2\}$. We find that the most immediate approach to measure the quality of $\tilde{y}(x)$ is to analyze the error equation obtained by subtracting (3) from (1):

$$(De)(x) := e''(x) + a(x)e'(x) + b(x)e(x) = F(x), \tag{5}$$

$$L_i[e] := c_{i1}e(x_i) + c_{i2}e'(x_i) = 0, \quad i = 0, 1 \tag{6}$$

where $e(x) := y(x) - \tilde{y}(x)$ and

$$F(x) := (D\tilde{y}) - f = (f - \tilde{f}) - (a - \tilde{a})\tilde{y}' - (b - \tilde{b})\tilde{y}.$$

In general, solving Equation (5) analytically for $e(x)$ is not easier than solving the original b.v.p. (1). However, in (El-Daou, 2002), $e(x)$ was obtained as an infinite series expansion that involves functions $\{a(x),\ b(x),\ F(x)\}$ and their higher derivatives. This result is recalled now:

Theorem 1. *Let* $\epsilon_0^{(j)} := e^{(j)}(x_0),\ \ j = 0, 1$. *If* $a(x)$ *and* $b(x)$ *are infinitely differentiable functions in* I, *then, for all* $x \in I$,

$$\begin{aligned} e(x) &= \int_{x_0}^{x} A(x,t)F(t)dt + A(x,x_0)\epsilon_0' + B(x,x_0)\epsilon_0, \\ e'(x) &= \int_{x_0}^{x} A'(x,t)F(t)dt + A'(x,x_0)\epsilon_0' + B'(x,x_0)\epsilon_0, \end{aligned}$$

where

$$\begin{aligned} A(x,t) &= \sum_{k\geq 0} \frac{a_k(t)}{k!}(x-t)^k, \qquad & A'(x,t) := \frac{\partial A}{\partial x}, \\ B(x,t) &= \sum_{k\geq 0} \frac{b_k(t)}{k!}(x-t)^k, \qquad & B'(x,t) := \frac{\partial B}{\partial x}, \end{aligned}$$

with $a_0(x) \equiv 0$, $b_0(x) \equiv 1$ *and for all* $k \geq 0$

$$\begin{aligned} a_{k+1}(x) &= a_k'(x) + b_k(x) - a(x)a_k(x), \\ b_{k+1}(x) &= b_k'(x) - b(x)a_k(x). \end{aligned}$$

We see then that to construct error estimations for the function and the derivative we can simply replace the infinite series $A(x,t)$ and $B(x,t)$ in $e(x)$ and $e'(x)$ expressions above, by their respective nth partial sums:

$$A_n(x,t) := \sum_{k=0}^{n} \frac{a_k(t)}{k!}(x-t)^k \quad \text{and} \quad B_n(x,t) := \sum_{k=0}^{n} \frac{b_k(t)}{k!}(x-t)^k,$$

for an arbitrary large $n \geq 1$. Once an estimation $\tilde{e}(x) \approx e(x)$ is obtained, one expects to gain an improved approximation $\tilde{\tilde{y}} = \tilde{y} + \tilde{e} \approx y$. If this procedure is continued in an iterative fashion as in (Auzinger et al. 2004), a further improvement can be achieved.

In this paper, rather than finding error estimates, we attempt to obtain error enclosures $e_l(x)$ and $e_u(x)$ such that $e_l(x) \leq e(x) \leq e_u(x)$. This guarantees that the exact solution y will satisfy the double inequality $\tilde{y} - e_l \leq y \leq \tilde{y} - e_u$. Such error bounds were obtained in (El-Daou 2002, 2006) for the special case where the linear functionals (2) define initial conditions; that is $y(x_0) = y_0$ and $y'(x_0) = y_1$. This result is recalled in Section 2. In section 3 we extend our analysis to treat a more general case. Numerical examples illustrating our results are given in Section 4.

2. Error Bounds for Initial Value Problems

The application of Cauchy's inequalities (Davis 1975) to $A(x,t)$ (resp. $B(x,t)$), as a real-analytic function of x, implies that for any given real $\rho > 0$, there exist two positive real numbers γ_1 and γ_2 such that for all $k > n$

$$\left\|\frac{a_k}{k!}\right\| \leq \gamma_1\rho^k \quad \text{(resp.} \quad \left\|\frac{b_k}{k!}\right\| \leq \gamma_2\rho^k), \tag{7}$$

where $\|u\| := \sup\{|u(t)| : t \in I\}$. We have then the following proposition of which the proof is given in (El-Daou 2002):

Proposition 2. *Let* $R_n[A]$ *and* $R_n[B]$ *stand for the nth remainders of* $A(x)$ *and* $B(x)$ *respectively,*

$$R_n[A] := \sum_{k>n} \frac{a_k(t)}{k!}(x-t)^k \quad \textit{and} \quad R_n[B] := \sum_{k>n} \frac{b_k(t)}{k!}(x-t)^k.$$

Let $\rho < min(1, \frac{1}{x_1 - x_0})$ *and let* $n \geq 1$. *Then for all* $x \in I := [x_0, x_1]$ *we have*

$$R_n[A] \leq \gamma_1\phi_{n,\rho}(x,t) \textit{ and } R_n[B] \leq \gamma_1\phi_{n,\rho}(x,t)$$

where $\{\gamma_1, \gamma_2\}$ *are given in (7) and*

$$\phi_{n,\rho}(x,t) := \frac{(x-t)^{n+1}\rho^{n+1}}{1 - \rho(x-t)}. \tag{8}$$

Theorem 1 and Proposition 2 yield Theorem 3 whose the proof is given in (El-Daou 2002). Note that whenever notation $H^{(i)}$ appears throughout the paper, it refers to the ith derivative of $H(x,t)$ with respect to x, for any single or two variables function H; in particular $H^{(0)} = H$.

Theorem 3. *Let $\rho < min(1, \frac{1}{x_1 - x_0})$ and let $n \geq 1$. Then for all $x \in I$ we have*

$$\left| e^{(i)}(x) - E_n^{(i)}(x) \right| \leq W_n^{(i)}(x); \quad (i = 0, 1), \tag{9}$$

where $E_n^{(i)}(x)$, the error estimation, and $W_n^{(i)}(x)$, the estimate deviation, are expressed as:

$$E_n^{(i)}(x) \;:=\; \int_{x_0}^{x} A_n^{(i)}(x,t)F(t)dt + A_n^{(i)}(x,x_0)\epsilon_0^{'} + B_n^{(i)}(x,x_0)\epsilon_0, \tag{10}$$

$$W_n^{(i)}(x) \;:=\; \gamma_1 \int_{x_0}^{x} \phi_{n,\rho}^{(i)}(x,t)|F(t)|dt + [\gamma_1|\epsilon'_0| + \gamma_2|\epsilon_0|]\phi_{n,\rho}^{(i)}(x,x_0), \tag{11}$$

where $A_n^{(i)}(x,t)$ and $B_n^{(i)}(x,t)$ are the nth partial sums of $A^{(i)}(x,t)$ and $B^{(i)}(x,t)$ respectively.

Expressions (9)-(10)-(11) form a one-step explicit algorithm for enclosing the error, in the sense that the knowledge of the *exact* initial values of ϵ_0 and $\epsilon_0^{'}$ is sufficient to evaluate $E_n^{(i)}(x)$ and $W_n^{(i)}(x)$ for any $x \in I = [x_0, x_1]$. In (El-Daou 2006), algorithm (9)-(10)-(11) was modified to accept *bounds* of ϵ_0 and $\epsilon_0^{'}$ without jeopardizing the efficiency of the algorithm. The modified version is stated in Theorem 4 in which we stress the dependence of $E_n^{(i)}(x)$ (resp. $W_n^{(i)}(x)$) on the initial errors ϵ_0 and $\epsilon_0^{'}$ by writing $E_n^{(i)}[x; |\epsilon_0|, |\epsilon_0^{'}|]$ for $E_n^{(i)}(x)$, (resp. $W_n^{(i)}[x; |\epsilon_0|, |\epsilon_0^{'}|]$).

Theorem 4. *Suppose that the assumptions of Theorem 3 hold. Suppose further that there exist four real numbers $\{\ell_\pm,\ \ell_\pm^{'}\}$ such that*

$$\ell_-^{(i)} \leq \epsilon_0^{(i)} \leq \ell_+^{(i)}, \quad i = 0, 1.$$

Then for $n \geq 1$ and for all $x \in I$, the exact error in function and in its derivative satisfy the inequalities

$$E_n^{(i)}[x; |\ell_{\bar{\beta}^{(i)}}|, |\ell_{\bar{\alpha}^{(i)}}^{'}|] - W_n^{(i)}[x; |\ell_s|, |\ell_{s'}^{'}|] \leq e^{(i)}(x) \leq E_n^{(i)}[x; |\ell_{\beta^{(i)}}|, |\ell_{\alpha^{(i)}}^{'}|] + W_n^{(i)}[x; |\ell_s|, |\ell_{s'}^{'}|], \tag{12}$$

where, for $i = 0, 1$

$$\begin{aligned} \alpha^{(i)} \equiv \alpha_n^{(i)}(x) &:= \text{sign}[A_n^{(i)}(x,x_0)], \quad \bar{\alpha}^{(i)} = -\alpha^{(i)}, \\ \beta^{(i)} \equiv \beta_n^{(i)}(x) &:= \text{sign}[B_n^{(i)}(x,x_0)], \quad \bar{\beta}^{(i)} = -\beta^{(i)}, \\ s^{(i)} &:= \text{sign}[\epsilon_0^{(i)}]. \end{aligned}$$

3. Error Bounds for Boundary Value Problems

This section is devoted to treat the case of b.v.p's. The main task is to find computable bounds for ϵ_0 and $\epsilon_0^{'}$, as required by Theorem 4, when the boundary conditions are defined as in (2). In order that the b.v.p. makes sense, we first assume that

$$|c_{01}| + |c_{02}| > 0 \text{ and } |c_{11}| + |c_{12}| > 0. \tag{13}$$

In accordance with (13), we can assume that c_{01} and c_{11} are nonzero and write

$$\epsilon_0 = (-\frac{c_{02}}{c_{01}})\epsilon_0^{'} \equiv \lambda_0\epsilon_0^{'} \quad \text{and} \quad \epsilon_1 = (-\frac{c_{12}}{c_{11}})\epsilon_1^{'} \equiv \lambda_1\epsilon_1^{'}. \tag{14}$$

Therefore, $e(x)$ can be written as

$$e(x) = \delta(x) + \eta(x)\epsilon_0^{'} \tag{15}$$

where

$$\delta(x) := \int_{x_0}^{x} A(x,t)F(t)dt \text{ and } \eta(x) := A(x,x_0) + \lambda_0 B(x,x_0). \tag{16}$$

In order to formulate the main results of this paper let:

$$\begin{aligned} g(t) &:= a^{'}(t) - b(t) \\ K(x,t) &:= (x-t)g(t) - a(t) \\ r_i &:= 1 + \lambda_i a(x_i), \ (i = 0, 1) \end{aligned}$$

$$\begin{aligned} G(x) &:= \lambda_0 + r_0(x - x_0) + a(x_1)\eta(x_1)x + \int_{x_0}^{x} K(x,t)\eta(t)dt \\ \rho(x) &:= a(x_1)\delta(x_1)x + \int_{x_0}^{x}(x-t)F(t)dt + \int_{x_0}^{x} K(x,t)\delta(t)dt. \end{aligned}$$

Note that

$$\begin{aligned} G^{'}(x_1) &= r_0 + \int_{x_0}^{x_1} g(t)\eta(t)dt \\ \rho^{'}(x_1) &= \int_{x_0}^{x_1} F(t)dt + \int_{x_0}^{x_1} g(t)\delta(t)dt. \end{aligned} \tag{17}$$

We are able now to give formal expressions for $\epsilon_0^{'}$ and ϵ_0:

Proposition 5. *Suppose that* $\Delta := [G(x_1) - \tilde{p}\eta(x_1)]r_1 - \lambda_1 G^{'}(x_1) \neq 0$ *where* $\tilde{p} := x_1 a(x_1)$. *Then*

$$\epsilon_0^{'} = \frac{\Delta_0}{\Delta} \quad and \quad \epsilon_1^{'} = \frac{\Delta_1}{\Delta}, \tag{18}$$

where

$$\begin{aligned} \Delta_0 &= \lambda_1\rho^{'}(x_1) - [\rho(x_1) - \tilde{p}\delta(x_1)]r_1, \\ \Delta_1 &= [G(x_1) - \tilde{p}\eta(x_1)]\rho^{'}(x_1) - [\rho(x_1) - \tilde{p}\delta(x_1)]G^{'}(x_1). \end{aligned}$$

Proof. Let us integrate (5) from x_0 to x:

$$\begin{aligned} & e^{'}(x) - e^{'}(x_0) + \int_{x_0}^{x} a(t)e^{'}(t)dt + \int_{x_0}^{x} b(t)e(t)dt = \int_{x_0}^{x} F(t)dt \\ & e^{'}(x) - e^{'}(x_0) + a(x)e(x) - a(x_0)e(x_0) - \int_{x_0}^{x} a^{'}(t)e(t)dt + \\ & \qquad + \int_{x_0}^{x} b(t)e(t)dt = \int_{x_0}^{x} F(t)dt \\ & e^{'}(x) - \epsilon_0^{'} + a(x)e(x) - a(x_0)e(x_0) - \int_{x_0}^{x} g(t)e(t)(t)dt = \int_{x_0}^{x} F(t)dt \\ & e^{'}(x) - \epsilon_0^{'} + a(x)e(x) - a(x_0)\lambda_0\epsilon_0^{'} - \int_{x_0}^{x} g(t)e(t)(t)dt = \int_{x_0}^{x} F(t)dt \\ & e^{'}(x) - [1 + a(x_0)\lambda_0]\epsilon_0^{'} + a(x)e(x) - \int_{x_0}^{x} g(t)e(t)(t)dt = \int_{x_0}^{x} F(t)dt \\ & e^{'}(x) - r_0\epsilon_0^{'} + a(x)e(x) - \int_{x_0}^{x} g(t)e(t)(t)dt = \int_{x_0}^{x} F(t)dt \end{aligned}$$

and therefore

$$e^{'}(x) = r_0\epsilon_0^{'} - a(x)e(x) + \int_{x_0}^{x} F(t)dt + \int_{x_0}^{x} g(t)e(t)dt. \tag{19}$$

When $x = x_1$, (19) becomes

$$\begin{aligned} \epsilon_1^{'} &= r_0\epsilon_0^{'} - a(x_1)\epsilon_1 + \int_{x_0}^{x_1} F(t)dt + \int_{x_0}^{x_1} g(t)e(t)dt \\ &= r_0\epsilon_0^{'} - a(x_1)\lambda_1\epsilon_1^{'} + \int_{x_0}^{x_1} F(t)dt + \int_{x_0}^{x_1} g(t)e(t)dt \\ &= r_0\epsilon_0^{'} - a(x_1)\lambda_1\epsilon_1^{'} + \int_{x_0}^{x_1} F(t)dt + \int_{x_0}^{x_1} g(t)\delta(t)dt + \int_{x_0}^{x_1} g(t)\eta(t)\epsilon_0^{'}dt \end{aligned}$$

which implies that

$$r_1\epsilon_1^{'} - \left\{ r_0 + \int_{x_0}^{x_1} g(t)\eta(t)dt \right\} \epsilon_0^{'} = \int_{x_0}^{x_1} F(t)dt + \int_{x_0}^{x_1} g(t)\delta(t)dt. \tag{20}$$

A second integration of (19) yields:

$$\begin{aligned} e(x)-\epsilon_0 =&(x-x_0)r_0\epsilon_0{}' + \int_{x_0}^{x}\int_{x_0}^{t} F(s)dsdt + \int_{x_0}^{x}\int_{x_0}^{t} g(s)e(s)dsdt - \int_{x_0}^{x} a(t)e(t)dt \\ =&(x-x_0)r_0\epsilon_0{}' + \int_{x_0}^{x}(x-t)F(t)dt + \int_{x_0}^{x}(x-t)g(t)e(t)dt - \int_{x_0}^{x} a(t)e(t)dt \\ =&(x-x_0)r_0\epsilon_0{}' + \int_{x_0}^{x}(x-t)F(t)dt + \int_{x_0}^{x} K(x,t)e(t)dt. \end{aligned}$$

That is

$$e(x) = \epsilon_0 + (x-x_0)r_0\epsilon_0{}' + \int_{x_0}^{x}(x-t)F(t)dt + \int_{x_0}^{x} K(x,t)e(t)dt. \tag{21}$$

Setting $x = x_1$ in (21), we get

$$\epsilon_1 - \epsilon_0 = (x_1-x_0)r_0\epsilon_0^{'} + \int_{x_0}^{x_1}(x_1-t)F(t)dt + \int_{x_0}^{x_1} K(x_1,t)\delta(t)dt + \left\{\int_{x_0}^{x_1} K(x_1,t)\eta(t)dt\right\}\epsilon_0^{'}.$$

That is,

$$\lambda_1\epsilon_1^{'} - \left\{\lambda_0 + (x_1-x_0)r_0 + \int_{x_0}^{x_1} K(x_1,t)\eta(t)dt\right\}\epsilon_0^{'} = \int_{x_0}^{x_1}(x_1-t)F(t)dt + \int_{x_0}^{x_1} K(x_1,t)\delta(t)dt. \tag{22}$$

In terms of $G(x)$ and $\rho(x)$, equations (20)-(22) form the algebraic system

$$\begin{bmatrix} -G^{'}(x_1) & r_1 \\ -G(x_1)+\tilde{p}\eta(x_1) & \lambda_1 \end{bmatrix}\begin{bmatrix} \epsilon_0^{'} \\ \epsilon_1^{'} \end{bmatrix} = \begin{bmatrix} \rho^{'}(x_1) \\ \rho(x_1)-\tilde{p}\delta_1 \end{bmatrix},$$

of which the solution is $\{\epsilon_0^{'},\ \epsilon_1^{'}\}$ as required. This completes the proof of the proposition. □

Now we obtain bounds for ϵ_0 and $\epsilon_1^{'}$. To this end we need the following technical lemma:

Lemma 6. *Suppose that $\sum_{i=0}^{\infty} p_i$ and $\sum_{i=0}^{\infty} q_i$ are two convergent numerical series and for all $n \geq 1$ let $P_n \in \mathbf{R}^+$ and $Q_n \in \mathbf{R}^+$ such that $|\sum_{i>n}^{\infty} p_i| \leq P_n$ and $|\sum_{i>0}^{\infty} q_i| \leq Q_n$. Then for appropriately chosen n we have the following enclosure:*

$$\min\left\{\frac{(\sum_{i=0}^{n} p_i) \pm P_n}{(\sum_{i=0}^{n} q_i) \pm Q_n}\right\} \leq \frac{\sum_{i=0}^{\infty} p_i}{\sum_{i=0}^{\infty} q_i} \leq \max\left\{\frac{(\sum_{i=0}^{n} p_i) \pm P_n}{(\sum_{i=0}^{n} q_i) \pm Q_n}\right\}.$$

Clearly, Δ and Δ_0, introduced in Proposition 5, have infinite series expansions because their expressions involve $A(x,t)$ and $B(x,t)$. Then we can write their nth residuals $R_n[\Delta]$ and $R_n[\Delta_0]$ as:

$$\begin{aligned} R_n[\Delta] &= R_n\left[G(x_1)-\tilde{p}\eta(x_1)]r_1 - \lambda_1 G^{'}(x_1)\right] \\ &= r_1R_n[G(x_1)] - \tilde{p}r_1R_n[\eta(x_1)] - \lambda_1R_n[G^{'}(x_1)] \\ &= r_1\left[x_1a(x_1)R_n[\eta(x_1)] + \int_{x_0}^{x_1} K(x_1,t)R_n[\eta(t)]dt\right] - \\ &\quad \tilde{p}r_1R_n[\eta(x_1)] - \lambda_1\int_{x_0}^{x_1} g(t)R_n[\eta(t)]dt \\ &= \int_{x_0}^{x_1}\left[r_1K(x_1,t)-\lambda_1 g(t)\right]R_n[\eta(t)]dt = \int_{x_0}^{x_1}\Psi(t)R_n[\eta(t)]dt. \end{aligned} \tag{23}$$

$$\begin{aligned} R_n[\Delta_0] &= \lambda_1R_n[\rho^{'}(x_1)] - r_1R_n[\rho(x_1)] + r_1\tilde{p}R_n[\delta(x_1)] \\ &= \lambda_1\int_{x_0}^{x_1} g(t)R_n[\delta(t)]dt - r_1x_1a(x_1)R_n[\delta(x_1)] - \\ &\quad r_1\int_{x_0}^{x_1} K(x_1,t)R_n[\delta(t)]dt + r_1\tilde{p}R[\delta(x_1)] = -\int_{x_0}^{x_1}\Psi(t)R_n[\delta(t)]dt. \end{aligned} \tag{24}$$

From (16), $\eta(x) = A(x,x_0) + \lambda_0B(x,x_0)$ and $\delta(x) = \int_{x_0}^{x} A(x,t)F(t)dt$. So

$$R_n[\eta(x)] = R_n[A(x,x_0)] + \lambda_0R_n[B(x,x_0)] \text{ and } R_n[\delta(x)] = \int_{x_0}^{x} R_n[A(x,t)]F(t)dt.$$

Therefore, (23) becomes

$$R_n[\Delta] = \int_{x_0}^{x_1} \Psi(t)\,(R_n[A(t,x_0)] + \lambda_0 R_n[B(t,x_0)])\,dt,$$

which, owing to Proposition 5, implies that

$$|R_n[\Delta]| \le \int_{x_0}^{x_1} |\Psi(t)|\left(\gamma_1 + |\lambda_0|\gamma_2)\phi_{n,\rho}(t,x_0)\right)dt \equiv M_n[\Delta]. \tag{25}$$

Similarly, it follows from (24) that

$$R_n[\Delta_0] \;=\; -\int_{x_0}^{x_1}\int_{x_0}^{t} \Psi(t)R_n[A(t,s)]F(s)dsdt$$

and therefore

$$|R_n[\Delta_0]| \le \gamma_1 \int_{x_0}^{x_1}\int_{x_0}^{t} |\Psi(t)F(s)|\phi_{n,\rho}(t,s)dsdt \equiv M_n[\Delta_0]. \tag{26}$$

We have proved then the main result of this paper which is summarized in following theorem:

Theorem 7. *The assumptions and notation of Proposition 5 hold. Then for appropriately chosen n we have the following enclosure:*

$$\min\left\{\frac{S_n[\Delta_0] \pm M_n[\Delta_0]}{S_n[\Delta] \pm M_n[\Delta]}\right\} \le \epsilon_0' \le \max\left\{\frac{S_n[\Delta_0] \pm M_n[\Delta_0]}{S_n[\Delta] \pm M_n[\Delta]}\right\} \tag{27}$$

and

$$\min\left\{\lambda_0\frac{S_n[\Delta_0] \pm M_n[\Delta_0]}{S_n[\Delta] \pm M_n[\Delta]}\right\} \le \epsilon_0 \le \max\left\{\lambda_0\frac{S_n[\Delta_0] \pm M_n[\Delta_0]}{S_n[\Delta] \pm M_n[\Delta]}\right\} \tag{28}$$

where $S_n[\Delta]$ and $S_n[\Delta_0]$ are the nth partial sums of Δ and Δ_0 respectively and $M_n[\Delta]$ and $M_n[\Delta_0]$ are defined by (25) and (26).

The results given in Theorem 7 provide computable upper and lower bounds for the initial errors ϵ_0 and ϵ_0'. Combining Theorem 7 with algorithm (12) we can find sufficiently sharp bounds for the error $e(x)$ at any x in the interval of interest.

Finally, we show how to treat the case where $c_{02} = c_{12} = 0$. Then $\epsilon_0 = e(x_0) = e(x_1) = \epsilon_1 = 0$ and therefore (21) becomes

$$e(x) = (x - x_0)\epsilon_0' + \int_{x_0}^{x}(x-t)F(t)dt + \int_{x_0}^{x} K(x,t)e(t)dt.$$

Evaluate this at $x = x_1$ and note that $e(x_1) = 0$,

$$(x_1 - x_0)\epsilon_0' + \int_{x_0}^{x_1}(x_1-t)F(t)dt + \int_{x_0}^{x_1} K(x_1,t)e(t)dt = 0.$$

Since $e(x) = \delta(x) + \eta(x)\epsilon_0'$ with $\delta(x) = \int_{x_0}^{x} A(x,t)F(t)dt$ and $\eta(x) = A(x,x_0)$,

$$(x_1 - x_0)\epsilon_0' + \int_{x_0}^{x_1}(x_1-t)F(t)dt + \int_{x_0}^{x_1} K(x_1,t)\left[\delta(t) + \eta(t)\epsilon_0'\right]dt = 0$$

which implies that

$$\left[(x_1 - x_0) + \int_{x_0}^{x_1} K(x_1,t)\eta(t)dt\right]\epsilon_0' + \int_{x_0}^{x_1}\left[(x_1-t)F(t) + K(x_1,t)\delta(t)\right]dt = 0.$$

Consequently,

$$\epsilon_0' = \frac{\int_{x_0}^{x_1}\left[(t-x_1)F(t) - K(x_1,t)\delta(t)\right]dt}{(x_1-x_0) + \int_{x_0}^{x_1} K(x_1,t)\eta(t)dt} \equiv \frac{\Delta_0}{\Delta}. \tag{29}$$

To find the residuals we proceed as in the proof of Theorem 7:

$$R_n[\Delta_0] = -\int_{x_0}^{x_1} K(x_1,t)R_n[\delta(t)]dt = -\int_{x_0}^{x_1} K(x_1,t)\int_{x_0}^{t} R_n[A(t,s)]F(s)dsdt.$$

Then

$$|R_n[\Delta_0|| \leq \gamma_1 \int_{x_0}^{x_1} \int_{x_0}^{t} |K(x_1,t)F(s)|\phi_n(t,s)dsdt \equiv M_n(\Delta_0). \tag{30}$$

Similarly

$$R_n[\Delta] = \int_{x_0}^{x_1} K(x_1,t)R_n[\eta(t)]dt = \int_{x_0}^{x_1} K(x_1,t)A(x_1,t)dt,$$

and therefore

$$|R_n[\Delta]| \leq \gamma_1 \int_{x_0}^{x_1} |K(x_1,t)|\phi_n(t,x_0)dt \equiv M_n(\Delta). \tag{31}$$

As a result we have this corollary:

Corollary 8. *If $c_{02} = c_{12} = 0$ then the estimates (27)-(28) apply with Δ and Δ_0 given by (29) and $M_n[\Delta]$ and $M_n[\Delta_0]$ defined by (30)-(31).*

4. Numerical Examples

We present now some numerical results which illustrate the sharpness of the error bounds established in this paper.

Example 1. Consider the boundary value problem:

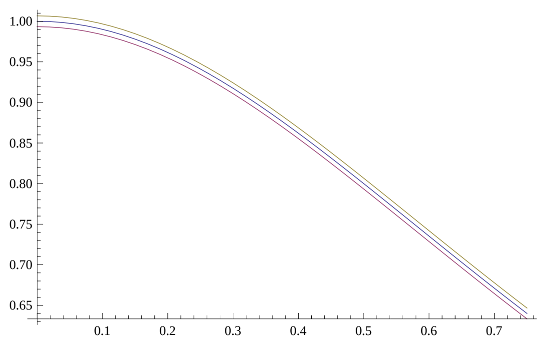
$$\begin{aligned} y''(x) + \frac{3x}{(1+x^2)}y'(x) + \frac{2}{(1+x^2)^2}y(x) &= 0\ ;\quad x \in [0, \frac{3}{4}] \\ \frac{1}{2}y(0) - y'(0) &= 1/2 \\ y(\frac{3}{4}) + y'(\frac{3}{4}) &= 0.0256 \end{aligned}$$

the exact solution of which is $y = (1+x^2)^{-1}$.

In this example we have $a(x) = 3x(1+x^2)^{-1}$, $b(x) = 2(1+x^2)^{-2}$ and $f(x) = 0$. Let us replace the coefficients $a(x)$ and $b(x)$ by their values at the midpoint, $a(\frac{3}{8})$ and $b(\frac{3}{8})$ respectively. Then we solve the perturbed problem

$$\begin{aligned} \tilde{y}''(x) + a(\frac{3}{8})\tilde{y}'(x) + b(\frac{3}{8})\tilde{y} &= 0\ ;\quad x \in [0, \frac{3}{4}] \\ \tilde{y}(0) - 2\tilde{y}'(0) &= 1 \\ \tilde{y}(\frac{3}{4}) + \tilde{y}'(\frac{3}{4}) &= 0.0256 \end{aligned}$$

whose the exact solution can be found analytically. Table 1 displays the computed function and derivative values and their upper and lower bounds at some points x_i in $[0, \frac{3}{4}]$. These bounds were computed using formulas (25-26-27-28). In this example $n = 40$, $\rho = 0.80$, $\gamma_1 = 0.79$ and $\gamma_2 = 1$. The same results are plotted in Fig. 1.



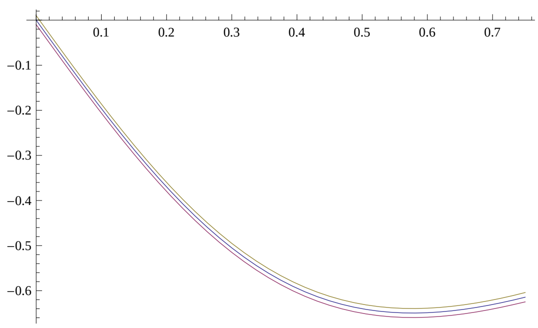


Figure 1. (Example 1). Plot of the exact solution, its upper and lower bounds for Example 1: (Left) the function, (Right) the first derivative

Table 1. This example was solved by the constant coefficient perturbation method. This table lists the exact errors in the function and its derivative y_i, $y_i^{'}$ at some points x_i as well as their respective upper and lower bounds.

x_i	$\overline{y}_i$ y_i $\underline{y}_i$	$\overline{y}_i - \underline{y}_i$	$\overline{y'}_i$ y_i^1 $\underline{y'}_i$	$\overline{y'}_i - \underline{y'}_i$
0	1.0000001		4.723E-8	
	1.0000000		0	
	0.99999991	1.814E-7	-4.350E-8	9.072E-8
0.05	0.99750633		-0.099501813	
	0.99750623		-0.099501869	
	0.99750615	1.855E-7	-0.099501921	1.082E-7
0.1	0.99009911		-0.19605915	
	0.99009901		-0.19605921	
	0.99009892	1.886E-7	-0.19605927	1.241E-7
0.15	0.97799521		-0.28694226	
	0.97799511		-0.28694233	
	0.97799502	1.908E-7	-0.28694240	1.379E-7
0.2	0.96153856		-0.36982241	
	0.96153846		-0.36982249	
	0.96153837	1.920E-7	-0.36982256	1.493E-7
0.25	0.94117657		-0.44290650	
	0.94117647		-0.44290657	
	0.94117638	1.923E-7	-0.44290665	1.582E-7
0.3	0.91743129		-0.50500791	
	0.91743119		-0.50500800	
	0.91743110	1.918E-7	-0.50500808	1.646E-7
0.35	0.89086870		-0.55555272	
	0.89086860		-0.55555280	
	0.89086851	1.905E-7	-0.55555288	1.684E-7
0.4	0.86206906		-0.59453024	
	0.86206897		-0.59453032	
	0.86206888	1.885E-7	-0.59453041	1.700E-7
0.45	0.83160093		-0.62240387	
	0.83160083		-0.62240395	
	0.83160074	1.860E-7	-0.62240403	1.694E-7
0.5	0.80000010		-0.63999992	
	0.80000000		-0.64000000	
	0.79999991	1.830E-7	-0.64000008	1.671E-7
0.55	0.76775441		-0.64839128	
	0.76775432		-0.64839136	
	0.76775423	1.795E-7	-0.64839145	1.648E-7
0.6	0.73529421		-0.64878882	
	0.73529412		-0.64878893	
	0.73529403	1.766E-7	-0.64878904	2.178E-7
0.65	0.70298780		-0.64244833	
	0.70298770		-0.64244921	
	0.70298760	1.974E-7	-0.64245017	1.842E-6
0.7	0.67114133		-0.63058363	
	0.67114094		-0.63060223	
	0.67114053	7.924E-7	-0.63062246	3.884E-5
0.75	0.64000618		-0.61404169	
	0.64000000		-0.61440000	
	0.63999332	1.286E-5	-0.61478715	7.454E-4

Example 2. Consider the boundary value problem:

$$y''(x) + \frac{\ln(x+1)}{\sqrt{16-x^2}} y'(x) + \frac{\cos x}{\sqrt{16-x^2}} y(x) = f(x); \quad x \in [0,\ 0.8] \tag{32}$$

$$y(0) = y(0.8) = 0 \tag{33}$$

(34)

$f(x)$ is chosen so that the exact solution is $y(x) = x(x-0.8)e^{-x^2+4x}$. In this example we used the collocation method at the zeros of $T_6(x)$, the Chebyshev polynomial of degree 6 shifted to $[0,\ 0.8]$. Table 2 displays the exact error in the function and in the derivatives as well as the error upper and lower bounds that are computed using formulas (25-26-27-28). In this example $n = 16$, $\rho = 0.999$, $\gamma_1 = 3.9E-5$ and $\gamma_2 = 1.2E-5$. The same results are plotted in Fig. 2.

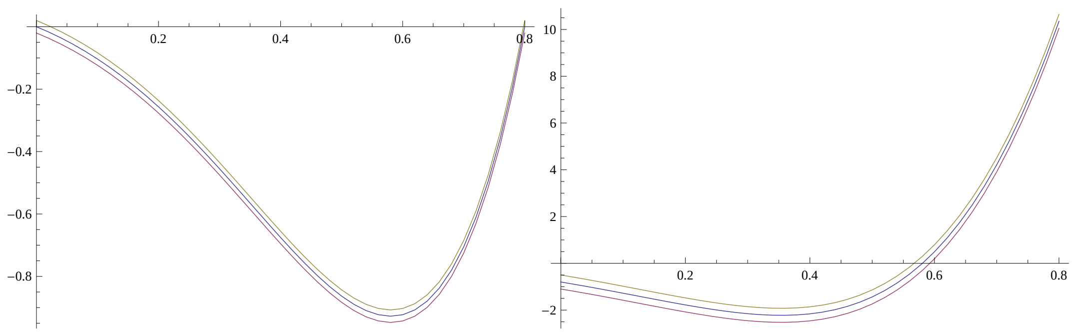


Figure 2. (Example 2). This problem was solved by collocation at Chebyshev $T_6(x)$. Plot of the exact solution, its upper and lower bounds for Example 2: (Left) the function, (Right) the first derivative

Example 3. Consider the boundary value problem:

$$y''(x) - (3 + 4x + 4x^2)y = 0; \quad x \in [0, 1] \tag{35}$$

$$y(0) = 1, \quad y(1) = \mathrm{e}^2 \tag{36}$$

the exact solution of which is $y = \mathrm{e}^{x+x^2}$.

In this example, $a(x) = 0$, $b(x) = -(3 + 4x + 4x^2)$ and $f(x) = 0$. Let us approximate $y(x)$ by means of the recursive Tau Method (see Ortiz, 1969). In this method, we solve the following perturbed problem:

$$Y''(x) - (3 + 4x + 4x^2)Y = \sum_{i=0}^{3} \tau_i L^*_{6+i}(x); \quad x \in [0, 1] \tag{37}$$

$$Y(0) = 1, \quad Y(1) = e^2 \tag{38}$$

where $L^*_k(x)$ is the kth Legendre polynomial shifted to $[0, 1]$, and $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ are Tau parameters. Having made the necessary calculations we found the approximate polynomial

$$\begin{aligned} Y(x) = {} & 1.0000000000000 + 1.0000006832351x + 1.4450450966195x^2 + 1.8901755592870x^3 \\ & -2.2971153408288x^4 + 7.8247500851272x^5 - 6.9878438810860x^6 + 3.5140464238074x^7 \end{aligned}$$

and the Tau parameters

$$\tau_0 = -0.13324498990945, \quad \tau_1 = -0.026881173106684,$$
$$\tau_2 = -0.0038350929137077, \quad \tau_3 = -0.00028910295547572$$

The error bounds are computed using algorithm (25-26-27-28) with the following data: $n = 25$, $\gamma_1 = 2.3477676E-7$, $\gamma_2 = 8.370487E-7$and $\rho = 0.99$. Table 3 displays the computed bounds along with the exact counterparts for comparison. Note that $err'(0) = -6.83235E-7$. The same results are plotted in Fig. 3.

Table 2. (Example 2). This problem was solved by collocation at the zeros of Chebyshev $T_6(x)$. This table lists the exact errors in the function and its derivative y_i, $y_i^{'}$ at some points x_i as well as their respective upper and lower bounds.

x_i	$\bar{y}_i$ y_i $\underline{y}_i$	$\bar{y}_i - \underline{y}_i$	$\overline{y'}_i$ y_i^1 $\underline{y}_i^{'}$	$\overline{y'}_i - \underline{y}_i^{'}$
0	0		-0.79999999999016	
	0		-0.80000000000000	
	0	0	-0.80000000000990	1.974E-11
0.04	-0.0356176965483		-0.98319839398	
	-0.0356176965487		-0.98319839399	
	-0.0356176965491	7.896E-13	-0.98319839400	1.973E-11
0.08	-0.0788165159118		-1.17839448679	
	-0.0788165159126		-1.17839448680	
	-0.07881651591334	1.579E-12	-1.17839448681	1.971 E-11
0.12	-0.12998632621488		-1.38081160961	
	-0.12998632621606		-1.38081160962	
	-0.12998632621724	2.366 E-12	-1.38081160963	1.967E-11
0.16	-0.18929122697778		-1.58389434174	
	-0.18929122697935		-1.58389434175	
	-0.18929122698093	3.152E-12	-1.58389434176	1.962E-11
0.20	-0.2565931464577		-1.7790458154436	
	-0.2565931464596		-1.7790458154534	
	-0.2565931464616	3.936E-12	-1.7790458154632	1.955E-11
0.24	-0.3313649805223		-1.95536897078	
	-0.3313649805246		-1.95536897079	
	-0.3313649805270	4.716E-12	-1.95536897080	1.947E-11
0.28	-0.412593619239		-2.0994214225602	
	-0.412593619242		-2.0994214225699	
	-0.412593619245	5.493E-12	-2.0994214225796	1.938E-11
0.32	-0.49867362353457		-2.194995066262	
	-0.49867362353769		-2.194995066272	
	-0.49867362354084	6.266E-12	-2.194995066281	1.928E-11
0.36	-0.587292786148		-2.222932856825	
	-0.587292786151		-2.222932856835	
	-0.587292786155	7.036E-12	-2.222932856844	1.921E-11
0.40	-0.6753113307156		-2.160996258293	
	-0.6753113307194		-2.160996258302	
	-0.6753113307234	7.805E-12	-2.160996258312	1.9356E-11
0.44	-0.75863705860610		-1.983797593253	
	-0.75863705861038		-1.983797593263	
	-0.75863705861469	8.596E-12	-1.983797593274	2.054E-11
0.48	-0.83209932999461		-1.662811827770	
	-0.83209932999933		-1.662811827782	
	-0.83209933000411	9.497E-12	-1.662811827796	2.604E-11
0.52	-0.88932534483397		-1.16648212262	
	-0.88932534483933		-1.16648212264	
	-0.88932534484483	1.085E-11	-1.16648212266	4.745E-11
0.56	-0.92262274878995		-0.46043268601	
	-0.92262274879667		-0.46043268607	
	-0.92262274880380	1.385E-11	-0.46043268613	1.221E-10
0.60	-0.92287310385496		0.49219898889	
	-0.92287310386548		0.49219898873	
	-0.92287310387713	2.217E-11	0.49219898853	3.604E-10
0.64	-0.87944120598574		1.73030057332	
	-0.87944120600750		1.73030057282	
	-0.87944120603227	4.654E-11	1.73030057225	1.066E-9
0.68	-0.78010557407337		3.2941869905	
	-0.78010557412757		3.2941869891	
	-0.78010557418929	1.159E-10	3.2941869875	3.031E-9
0.72	-0.61101564685		5.2248626925	
	-0.61101564701		5.2248626883	
	-0.61101564719	3.317E-10	5.2248626835	8.963E-9
0.76	-0.35668127899858		7.5631449532	
	-0.35668127946535		7.5631449406	
	-0.35668127997107	9.725E-10	7.5631449271	2.612E-8
0.80	1.3688564996066E-9		10.348653889	
	0		10.348653852	
	-1.4497281379915E-9	2.819E-9	10.348653814	7.478E-8

Table 3. (Example 3). This problem was solved by Tau Method with Legendre $L_6(x)$. This table lists the exact errors in the function and its derivative y_i, $y_i^{'}$ at some points x_i as well as their respective upper and lower bounds.

x_i	$\overline{y}_i$ y_i $\underline{y}_i$	$\overline{y}_i - \underline{y}_i$	$\overline{y'}_i$ y_i^1 $\underline{y'}_i$	$\overline{y'}_i - \underline{y'}_i$
0	1.0000000000000		1.000000010	
	1.0000000000000		1.000000000	
	1.0000000000000	0	0.999999992	1.836E-8
0.05	1.05390256260		1.1592928287	
	1.05390256208		1.1592928183	
	1.05390256168	9.190E-10	1.1592928103	1.843E-8
0.10	1.11627807150		1.3395336951	
	1.11627807046		1.3395336845	
	1.11627806966	1.845E-9	1.3395336765	1.866E-8
0.15	1.18827182219		1.5447533776	
	1.18827182061		1.5447533668	
	1.18827181940	2.788 E-9	1.5447533585	1.907E-8
0.20	1.27124915245		1.7797488216	
	1.27124915032		1.7797488104	
	1.27124914869	3.756E-9	1.7797488019	1.970E-8
0.25	1.36683794387		2.0502569234	
	1.36683794117		2.0502569118	
	1.36683793911	4.762E-9	2.0502569028	2.057E-8
0.30	1.47698079718		2.3631692825	
	1.47698079388		2.3631692702	
	1.47698079136	5.818E-9	2.3631692608	2.174E-8
0.35	1.6039991868		2.7267986240	
	1.6039991828		2.7267986108	
	1.6039991798	6.942E-9	2.7267986008	2.324E-8
0.40	1.7506725049		3.1512105148	
	1.7506725003		3.1512105005	
	1.7506724968	8.150E-9	3.1512104896	2.515E-8
0.45	1.9203356775		3.6486377927	
	1.9203356721		3.6486377771	
	1.9203356680	9.465E-9	3.6486377651	2.754E-8
0.50	2.1170000228		4.2340000505	
	2.1170000166		4.2340000332	
	2.1170000119	1.091E-8	4.2340000200	3.050E-8
0.55	2.3455032936		4.9255569211	
	2.3455032865		4.9255569017	
	2.3455032811	1.253E-8	4.9255568869	3.413E-8
0.60	2.6116964815		5.7457322634	
	2.6116964734		5.7457322415	
	2.6116964672	1.434E-8	5.7457322248	3.858E-8
0.65	2.9226770762		6.7221572787	
	2.9226770669		6.7221572538	
	2.9226770598	1.640E-8	6.7221572347	4.400E-8
0.70	3.2870812180		7.8889949264	
	3.2870812074		7.8889948977	
	3.2870811992	1.876E-8	7.8889948758	5.061E-8
0.75	3.7154507501		9.2886268781	
	3.7154507379		9.2886268448	
	3.7154507286	2.149E-8	9.2886268194	5.869E-8
0.80	4.2206958310		10.973809163	
	4.2206958170		10.973809124	
	4.2206958063	2.466E-8	10.973809094	6.871E-8
0.85	4.818679864		13.010435634	
	4.818679847		13.010435588	
	4.818679835	2.841E-8	13.010435552	8.197E-8
0.90	5.528961497		15.481092195	
	5.528961478		15.481092137	
	5.528961464	3.299E-8	15.481092090	1.045E-7
0.95	6.375738986		18.489643084	
	6.375738962		18.489642991	
	6.375738946	3.954E-8	18.489642906	1.781E-7
1.	7.389056134		22.167168806	
	7.389056099		22.167168297	
	7.389056075	5.920E-8	22.167167787	1.019E-6

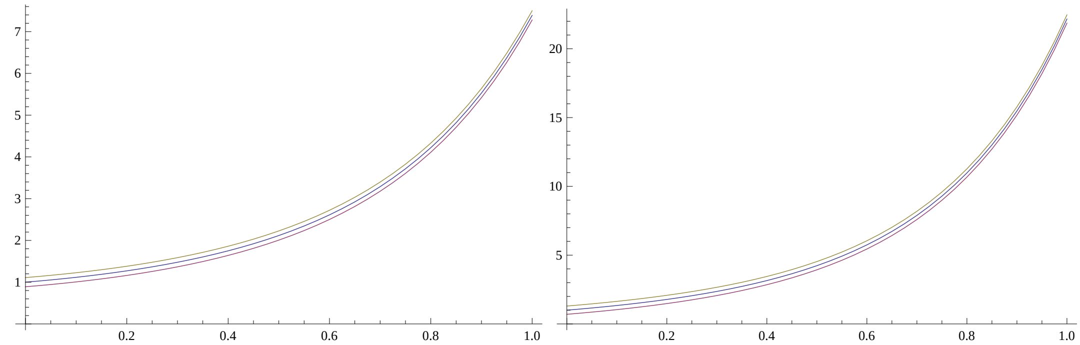


Figure 3. (Example 3). This problem was solved by Tau with Legendre $L_6^*(x)$. Plot of the exact solution, its upper and lower bounds for Example 3: (Left) the function, (Right) the first derivative

5. Conclusion

There are several results concerning the error estimates when a boundary value problem is solved by numerical methods. But very few of those results have a practical value because they contain unknown parameters related to the differential operator such as the conditioning number and the Lipschitz constant. In this paper, using a technique based on the Coefficients Perturbation Methods, we managed to avoid this drawback and to develop practical formulas that allow to compute upper and lower bounds for the error in the function and in its derivative. These computed bounds are expressed in terms of the perturbations introduced in the differential equation and in the prescribed boundary conditions associated with it. We demonstrated the sharpness of our results through some numerical examples.

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