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Approximation of a Second-order Elliptic Equation with Discontinuous and Highly Oscillating Coefficients by Finite Volume Methods

Bienvenu Ondami[1]

[1] Univesrité Marien Ngouabi, Brazzaville, Congo

Correspondence: Université Marien Ngouabi, Faculté des Sciences et Techniques, BP. 69, Brazzaville, Congo. E-mail: bondami@gmail.com

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Abstract

In this paper we consider the numerical approximation of a class of second order elliptic boundary value problems with discontinuous and highly periodically oscillating coefficients. We apply both classical and modified finite volume methods for the approximate solution of this problem. Error estimates depending on ε the parameter involved in the periodic homogenization are established. Numerical simulations for one-dimensional problem confirm the theorical results and also show that the modified scheme has a smaller constant of convergence than the classical scheme based on harmonic averaging for this class of equations.

Keywords: homogenization, elliptic equations, oscillating coefficients, finite volume method, finite difference

1. Introduction

There are many practical computational problems with highly oscillatory solutions e.g. computation of flow in heterogeneous porous media for petroleum and groundwater reservoir simulation (see, e.g., (Hornung, 1997) and the bibliographies therein). If a porous medium with a periodic structure is considered, with the size of the period is small enough compared to the size of the reservoir, and denoting their ratio by ε ($0 < \varepsilon << 1$) an asymptotic analysis, as $\varepsilon \longrightarrow 0$, is required.
In this paper we will consider problems that are described by a linear elliptic equation in divergence form with highly periodically oscillating coefficients. Especially we will consider the following model problem:

$$(P_\varepsilon) \qquad \left\{ \begin{array}{rcll} -\mathrm{div}\,(K^\varepsilon(x)\,\nabla u^\varepsilon) & = & f & \text{in } \Omega, \\ u_\varepsilon & = & 0 & \text{on } \Gamma. \end{array} \right.$$

$\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) is a bounded polygonal convex domain with a periodic structure and smooth boundary Γ, $K^\varepsilon(x) = K(x\ /\varepsilon)$, K is a symmetric and uniformly positive definite matrix in Ω which has jumps discontinuities across a given interface. The case of piecewise constant coefficient K^ε is very important for the applications.
In porous medium flow, the problem (P_ε) results from Darcy's law and continuity for a single phase, incompressible flow through a horizontal heterogeneous porous medium with periodic structure.

Using the homogenization tools (see, e.g., (Bakhvalov & Panasenko, 1989), (Bensoussan, Lions & Papanicolaous, 1987), (Jikov, Kozlov & Oleinik, 1994) and (Sanchez-Palancia, 1980)) original Problem (P_ε) can be replaced by homogenized Problem, modeling some average quantity without the oscillations.

Whenever effective equations are applicable they are very useful for computational purposes. There are however many situations for which ε is not sufficiently small so that the effective equations are not practical. In this cases the original equation has to be approximated directly.
The numerical approximation partial differential equations with highly oscillating coefficients has been a problem of interest for many years and many methods have been developed (see, e.g., (Amaziane & Ondami, 1999), (Chen & Hou, 2002), (Versieux & Sarkis, 2008) and (Ondami, 2001, 2015) and the bibliographies therein).

The case where K^ε has continuous coeffients is the most studied. The case of discontinuous coefficients has been addressed in some work as in (Bourgat, 1978) and (Bourgat & Dervieux, 1978) where Authors use a double-scale asymptotic expansion. In (Amaziane & Ondami, 1999) and (Ondami, 2001), the numerical approximation of the problem, in the case of discontinuous coefficients was done by finite elements methods. However no error estimate has been established and is still, to our knowledge, an opened issue.

Elliptic problems with discontinuous coefficients (often called interface problems) arise naturally in mathematical modeling processes in heat and mass transfer, diffusion in composite media flows in porous media etc.

In this paper the approximation will be done by finite volume methods (see, e.g., (Eymard, Gallouët & Herbin, 2000) and (Chernogorova, Ewing, Iliev & Lazarov, 2001)) and the study is limited to one-dimensional problem (1-D problem). This 1-D problem illustrates very clearly the dependence of numerical results to ε, the parameter of homogenization. Error estimates are established. The obtained results can be generalized in the two-dimensional and three-dimensional problems.
The paper is organized as follows. In section 2, a description of methods used is presented as well as the obtained error estimates. Section 3 is devoted to numerical simulations. Lastly, some concluding remarks are presented in section 4.

2. Methods

Our study will focus on the one-dimensional problem and we assume (without loss of generality) that $\Omega =]0,1[$. In this case the problem (P_ε) is written simply

$$\begin{cases} -\frac{d}{dx}\left(k^{\varepsilon}(x)\frac{du^{\varepsilon}}{dx}\right) = f \text{ in } (0,1), \\ u^{\varepsilon}(0) = u^{\varepsilon}(1) = 0. \end{cases} \tag{1}$$

where $k^{\varepsilon}(x) = k(x/\varepsilon) = k(y)$, with $y = x/\varepsilon$, k is a discontinuous and periodic function of period 1, on $]0,1[$.
In all this paper we make the following assumptions

$(A1)$ $\alpha < k(y) \leqslant \beta$, a.e., in $]0,1[$, with some $\alpha, \beta \in \mathbb{R}_+^*$,

$(A2)$ $f \in L^2(]0,1[)$.

The assumptions $(A1)$ and $(A2)$ ensure the existence and uniqueness of the solution of the problem (1). From homogenization theory (see, e.g., (Sanchez-Palancia, 1980), (Bensoussan, Lions & Papanicolaous, 1987) and (Jikov, Kozlov & Oleinik, 1994)) follows

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega) \text{ (consists of functions in Sobolev space } H^1(\Omega) \text{ that vanish on 0 and 1) weakly,}$$

where u (homogenized solution) satisfies the following homogenized problem,

$$\begin{cases} -\frac{d}{dx}\left(k^{*}\frac{du}{dx}\right) = f \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

and the constant k^* is the mean harmonic value of $k(y)$ on $(0,1)$.

This one-dimensional problem helps to clarify the eventual dependency of numerical results to the parameter ε. The error estimates obtained can be generalized in the two-dimensional and three-dimensional cases which uses, for instance, simplices or parallelepipedes mesh.
Two finite volume schemes will be compared: The classical scheme (see, e.g., (Eymard, Gallouët & Herbin, 2000)) and a modified scheme (see, (Ewing, Iliev & Lazarov, 2001)).

In order to compute a numerical approximation to the solution u^{ε}, let us define a mesh, denoted by $\mathcal{T}$, of the interval $(0,1)$ consisting of N cells (or control volumes), denoted by V_i, $i = 1,...,N$, and N points of $(0,1)$, denoted by x_i, $i = 1,...,N$, satisfying the following assumptions:

Definition 1. *An admissible mesh of* $(0,1)$*, denoted by* $\mathcal{T}$*, is given by a family* $(V_i)_{i=1,...,N}$*,* N *a positive integer, such that* $V_i = \left(x_{i-\frac{1}{2}},\ x_{i+\frac{1}{2}}\right)$*, and a family* $(x_i)_{i=0,...,N}$*, such that*

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < ... < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < ... < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1.$$

One sets

$$h_i = m(V_i) = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}},\ i = 1,...,N, \text{ and therefore } \sum_{i=1}^{N} h_i = 1,$$

$$h_i^- = x_i - x_{i-\frac{1}{2}},\quad h_i^+ = x_{i+\frac{1}{2}} - x_i,\ i = 1,...,N,$$

$$h_{i+\frac{1}{2}} = x_{i+1} - x_i,\ i = 0,...,N,$$

$$size(\mathcal{T}) = h = max\{h_i,\ i = 1,...,N\}.$$

2.1 Classical Finite Volume Scheme

Let $\mathcal{T} = (V_i)_{i=1,...,N}$ be an admissible mesh, in the sense of *Definition 1* , such that the discontinuities of k^ε coincide with the interfaces of the mesh.

Classical finite volume scheme consiste to integrate the first equation of the problem (1) over V_i (see,e.g., (Eymard, Gallouët & Herbin, 2000)). So we have:

$$-\left(k^\varepsilon(x)\frac{du^\varepsilon(x)}{dx}\right)(x_{i+\frac{1}{2}}) + \left(k^\varepsilon(x)\frac{du^\varepsilon(x)}{dx}\right)(x_{i-\frac{1}{2}}) = \int_{V_i} f(x)dx, \quad i = 1, ..., N. \tag{3}$$

Let

$$(u_i^\varepsilon)_{i=1,...,N} \text{ be the discrete unknows}$$

and let

$$k_i^\varepsilon = \frac{1}{h_i}\int_{V_i} k^\varepsilon(x)dx.$$

In order that the scheme be conservative, the discretization of the flux $-k^\varepsilon(x)\frac{du^\varepsilon(x)}{dx}$ at $x_{i+\frac{1}{2}}$ should have the same value on V_i and V_{i+1}. To this purpose, one introduces the auxiliary unknown $u^\varepsilon_{i+\frac{1}{2}}$ (approximation of u^ε at $x_{i+\frac{1}{2}}$). Since on V_i and V_{i+1}, k^ε is continuous, the approximation of $-k^\varepsilon(x)\frac{du^\varepsilon(x)}{dx}$ may be performe on each side of $x_{i+\frac{1}{2}}$ by using the finite difference principe:

$$H^\varepsilon_{i+\frac{1}{2}} = -k_i^\varepsilon \frac{u^\varepsilon_{i+\frac{1}{2}} - u_i^\varepsilon}{h_i^+} \quad \text{on} \quad V_i, \quad i = 1, ..., N,$$

$$H^\varepsilon_{i+\frac{1}{2}} = -k_{i+1}^\varepsilon \frac{u^\varepsilon_{i+1} - u^\varepsilon_{i+\frac{1}{2}}}{h_{i+1}^-} \quad \text{on} \quad V_{i+1}, \quad i = 0, ..., N-1,$$

with $u^\varepsilon_{1/2} = 0$ and $u^\varepsilon_{N+1/2} = 0$, for the boundary conditions. Requiring the two above approximation of $\left(k^\varepsilon\frac{du^\varepsilon}{dx}\right)\left(x_{i+\frac{1}{2}}\right)$ to be equal (conservativity of the flux) yields the value of $u^\varepsilon_{1+\frac{1}{2}}$ (for $i = 1, ..., N-1$) :

$$u^\varepsilon_{1+\frac{1}{2}} = \frac{u^\varepsilon_{i+1}\frac{k^\varepsilon_{i+1}}{h^-_{i+1}} + u_i^\varepsilon\frac{k_i^\varepsilon}{h_i^+}}{\frac{k^\varepsilon_{i+1}}{h^-_{i+1}} + \frac{k_i^\varepsilon}{h_i^+}}$$

which, in turn, allows to give expression of the approximation $H^\varepsilon_{i+\frac{1}{2}}$ of $\left(k^\varepsilon\frac{du^\varepsilon}{dx}\right)\left(x_{i+\frac{1}{2}}\right)$:

$$H^\varepsilon_{i+\frac{1}{2}} = -\tau^\varepsilon_{i+\frac{1}{2}}\left(u^\varepsilon_{i+1} - u_i^\varepsilon\right), \quad i = 1, ..., N-1, \tag{4}$$

$$H^\varepsilon_{\frac{1}{2}} = -\frac{k_1^\varepsilon}{h_1^-}u_1^\varepsilon, \tag{5}$$

$$H^\varepsilon_{N+\frac{1}{2}} = \frac{k_N^\varepsilon}{h_N^+}u_N^\varepsilon, \tag{6}$$

with

$$\tau^\varepsilon_{i+\frac{1}{2}} = \frac{k_i^\varepsilon\, k^\varepsilon_{i+1}}{h_i^+k^\varepsilon_{i+1} + h^-_{i+1}k_i^\varepsilon}, \quad i = 1, ..., N-1. \tag{7}$$

Remark 1. If $h_i = h$, for all $i \in 1, ..., N$, and x_i is assumed to be center of V_i, then $h_i^- = h_i^+ = \frac{h}{2}$, so that

$$H^\varepsilon_{i+\frac{1}{2}} = -\frac{2k_i^\varepsilon k^\varepsilon_{i+1}}{k_i^\varepsilon + k^\varepsilon_{i+1}}\frac{\left(u^\varepsilon_{i+1} - u_i^\varepsilon\right)}{h},$$

and therefore the mean harmonic value of k^ε is involved.

The numerical scheme for the approximation of Problem (1) is therefore,

$$H^\varepsilon_{i+\frac{1}{2}} - H^\varepsilon_{i-\frac{1}{2}} = h_if_i, \quad \forall i \in 1, ..., N. \tag{8}$$

with

$$f_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx, \text{ for } i = 1, ..., N, \text{ and where } \left(H_{i+\frac{1}{2}}\right)_{i\in\{0,...,N\}} \text{ is defined by (4)-(6)}$$

Taking (7) and 4)-(6) into account, the scheme (8) yields a system of N equations with N unknowns $u_1^\varepsilon, ..., u_N^\varepsilon$.

Remark 2. The fact that k^ε is discontinuous, periodic (period ε) and with discontinuities that coincide with the interfaces of the mesh $\mathcal{T}$ leads to say that:

$\varepsilon = \frac{1}{n_p}$, where n_p is a positive integer (periods number), and if $h_i = h = size(\mathcal{T})$, for all $i \in \{1, ..., N\}$ then $h = \frac{\varepsilon}{m}$, where $1 < m < N$.

Note. Throughout the paper, we will denote by c generic constants, even if they take different values at different places.

We now state the main result of this section.

Theorem 1. *Let $\mathcal{T} = (V_i)_{i=1,...,N}$ be an admissible mesh of $(0,1)$, in the sens of Definition 1, and uniform (i.e. $h_i = h,\ \forall i \in \{1, ...N\}$) such that*
1) x_i is the center of $(V_i)_{i=1,...,N}$, and the discontinuities of k^ε coincide with the interfaces of the mesh,
2) $k^\varepsilon \in C^1\left(\overline{V_i}\right)$, and $f \in C^1\left(\overline{V_i}\right)$, for all $i = 1, ..., N$.

Let $e_i^\varepsilon = u^\varepsilon(x_i) - u_i^\varepsilon$, $e_i = u(x_i) - u_i^\varepsilon$ where u^ε is the solution of Problem (1), u is the homogenized solution and $u^{\varepsilon h} = (u_i^\varepsilon)_{i=1,...,N}$ is the solution of (4)-(8). Then there exists a constant c independent of ε and h such that

$$\left\|u^\varepsilon - u^{\varepsilon h}\right\|_{H^1(\Omega)}^2 \equiv \sum_{i=1}^{N} \tau_{i+\frac{1}{2}}^\varepsilon \left(e_{i+1}^\varepsilon - e_i^\varepsilon\right)^2 \leqslant \frac{ch^2}{\varepsilon}, \tag{9}$$

where $\tau_{i+\frac{1}{2}}^\varepsilon$ is defined in (7), and

$$\left\|u^\varepsilon - u^{\varepsilon h}\right\|_{L^\infty(\Omega)} \equiv \max_{1\le i\le N} | e_i^\varepsilon | \leqslant c\,h, \tag{10}$$

$$\left\|u - u^{\varepsilon h}\right\|_{L^\infty(\Omega)} \equiv \max_{1\le i\le N} | e_i | \leqslant c\,h + c\varepsilon. \tag{11}$$

Proof.

The proof of the Theorem 1 is obtained by using the same gait as in (Eymard, Gallouët, & Herbin, 2000). Let

$$\overline{H}_{i+\frac{1}{2}}^\varepsilon = -\left(k^\varepsilon \frac{du^\varepsilon}{dx}\right)\left(x_{i+\frac{1}{2}}\right) \text{ and } H_{i+\frac{1}{2}}^{*,\varepsilon} = -\tau_{i+\frac{1}{2}}^\varepsilon\left(u^\varepsilon(x_{i+1}) - u^\varepsilon(x_i)\right), \text{ for } i = 0, ..., N, \text{ with } \tau_{\frac{1}{2}}^\varepsilon = \frac{k_1^\varepsilon}{h_1^-} \text{ and } \tau_{N+\frac{1}{2}}^\varepsilon = \frac{k_N^\varepsilon}{h_N^+}. \tag{12}$$

Let us first show there exists c independent of ε and h such that

$$H_{i+\frac{1}{2}}^{*,\varepsilon} = \overline{H}_{i+\frac{1}{2}}^\varepsilon + T_{i+\frac{1}{2}}^\varepsilon, \qquad \left|T_{i+\frac{1}{2}}^\varepsilon\right| \le \frac{ch}{\varepsilon},\ i = 0, ..., N. \tag{13}$$

In order to show this, let us introduce

$$H_{i+\frac{1}{2}}^{*,-,\varepsilon} = -k_i^\varepsilon \frac{u^\varepsilon(x_{i+\frac{1}{2}}) - u^\varepsilon(x_i)}{h_i^+} \text{ and } H_{i+\frac{1}{2}}^{*,+,\varepsilon} = -k_{i+1}^\varepsilon \frac{u^\varepsilon(x_{i+1}) - u^\varepsilon(x_{i+\frac{1}{2}})}{h_{i+1}^-} \tag{14}$$

Since $k^\varepsilon \in C^1\left(\overline{V_i}\right)$, one has $u^\varepsilon \in C^2\left(\overline{V_i}\right)$. By developing the first equation of (1) in $\overline{V_i}$ ($i = 1, ..., N$), one obtains

$$-k^\varepsilon \frac{d^2u^\varepsilon}{d^2x} - \frac{1}{\varepsilon}\frac{dk\left(\frac{x}{\varepsilon}\right)}{dx}\frac{du^\varepsilon}{dx} = f \text{ in } \overline{V_i},\quad i = 1, ..., N. \tag{15}$$

According to (Amaziane & Ondami, 1999) and (Ondami, 2001), one has

$$\|u^\varepsilon\|_{H^1(\Omega)} \le c,\quad i = 1, ..., N.$$

Hence

$$\left\| \frac{du^\varepsilon}{dx} \right\|_{L^\infty(V_i)} \leq c, \quad i = 1, ..., N, \tag{16}$$

and

$$\left\| \frac{d^2u^\varepsilon}{d^2x} \right\|_{L^\infty(V_i)} \leq \frac{c}{\varepsilon}, \quad i = 1, ..., N. \tag{17}$$

By using (A1) and (14)-(17), one deduces that there exists c independent of ε and h such that

$$H^{*,-,\varepsilon}_{i+\frac{1}{2}} = \overline{H}^{\varepsilon}_{i+\frac{1}{2}} + R^{-,\varepsilon}_{i+\frac{1}{2}}, \quad \text{where} \quad \left| R^{-,\varepsilon}_{i+\frac{1}{2}} \right| \leqslant \frac{c\,h}{\varepsilon}, \quad i = 1, ..., N, \tag{18}$$

$$H^{*,+,\varepsilon}_{i+\frac{1}{2}} = \overline{H}^{\varepsilon}_{i+\frac{1}{2}} + R^{+,\varepsilon}_{i+\frac{1}{2}}, \quad \text{where} \quad \left| R^{+,\varepsilon}_{i+\frac{1}{2}} \right| \leqslant \frac{c\,h}{\varepsilon}, \quad i = 0, ..., N-1. \tag{19}$$

This yields (13) for $i = 0$ and $i = N$.

The following equality:

$$\overline{H}^{\varepsilon}_{i+\frac{1}{2}} = H^{*,-,\varepsilon}_{i+\frac{1}{2}} - R^{-,\varepsilon}_{i+\frac{1}{2}} = H^{*,+,\varepsilon}_{i+\frac{1}{2}} - R^{+,\varepsilon}_{i+\frac{1}{2}}, \quad i = 1, ..., N-1 \tag{20}$$

yields that

$$u^\varepsilon\left(x_{i+\frac{1}{2}}\right) = \frac{\frac{k^\varepsilon_{i+1}}{h^-_{i+1}} u^\varepsilon(x_{i+1}) + \frac{k^\varepsilon_i}{h^+_i} u^\varepsilon(x_i)}{\frac{k^\varepsilon_i}{h^+_i} + \frac{k^\varepsilon_{i+1}}{h^-_{i+1}}} + S^\varepsilon_{i+\frac{1}{2}}, \; i = 1, ..., N-1, \tag{21}$$

where

$$S^\varepsilon_{i+\frac{1}{2}} = \frac{R^{+,\varepsilon}_{i+\frac{1}{2}} - R^{-,\varepsilon}_{i+\frac{1}{2}}}{\frac{k^\varepsilon_i}{h^+_i} + \frac{k^\varepsilon_{i+1}}{h^-_{i+1}}}$$

So that

$$\left| S^\varepsilon_{i+\frac{1}{2}} \right| \leqslant \frac{1}{\alpha} \frac{h^+_i h^-_{i+1}}{h^+_i + h^-_{i+1}} \left| R^{+,\varepsilon}_{i+\frac{1}{2}} - R^{-,\varepsilon}_{i+\frac{1}{2}} \right|$$

Let us replace the expression (21) of $u^\varepsilon\left(x_{i+\frac{1}{2}}\right)$ in $H^{*,-,\varepsilon}_{i+\frac{1}{2}}$ defined by (14); this yields

$$H^{*,-,\varepsilon}_{i+\frac{1}{2}} = -\tau_{i+\frac{1}{2}} \left(u^\varepsilon(x_{i+1}) - u^\varepsilon(x_i)\right) - \frac{k^\varepsilon_i}{h^+_i} S^\varepsilon_{i+\frac{1}{2}}, \; i = 1, ..., N-1. \tag{22}$$

Using (20), this implies that

$$H^{*,\varepsilon}_{i+\frac{1}{2}} = \overline{H}^{\varepsilon}_{i+\frac{1}{2}} + T^\varepsilon_{i+\frac{1}{2}}$$

where

$$\left| T^\varepsilon_{i+\frac{1}{2}} \right| \leqslant \left| R^{-,\varepsilon}_{i+\frac{1}{2}} \right| + \left| R^{+,\varepsilon}_{i+\frac{1}{2}} - R^{-,\varepsilon}_{i+\frac{1}{2}} \right| \frac{\beta}{2\alpha}. \tag{23}$$

Using (18) and (19), this last inequality yields that there exists c, independent of ε and h such that

$$\left| H^{*,\varepsilon}_{i+\frac{1}{2}} - \overline{H}^{\varepsilon}_{i+\frac{1}{2}} \right| = \left| T^\varepsilon_{i+\frac{1}{2}} \right| \leqslant \frac{c\,h}{\varepsilon}, \quad i = 1, ..., N-1.$$

Therefore (13) is proved.

Now, from (3) and (12), one has

$$\overline{H}^{\varepsilon}_{i+\frac{1}{2}} - \overline{H}^{\varepsilon}_{i-\frac{1}{2}} = h_i f_i, \quad \forall i \in \{1, ..., N\}. \tag{24}$$

Using (13) yields that

$$\overline{H}^{*,\varepsilon}_{i+\frac{1}{2}} - \overline{H}^{*,\varepsilon}_{i-\frac{1}{2}} = h_i f_i + T^\varepsilon_{i+\frac{1}{2}} - T^\varepsilon_{i-\frac{1}{2}}, \quad \forall i \in \{1, ..., N\}. \tag{25}$$

Let $e_i^\varepsilon = u^\varepsilon(x_i) - u_i^\varepsilon$ for $i = 1, ..., N$, and $e_0^\varepsilon = e_{N+1}^\varepsilon = 0$. Substracting (8) from (25) yields

$$-\tau^\varepsilon_{i+\frac{1}{2}}\left(e^\varepsilon_{i+1} - e^\varepsilon_i\right) + \tau^\varepsilon_{i-\frac{1}{2}}\left(e^\varepsilon_i - e^\varepsilon_{i-1}\right) = T^\varepsilon_{i+\frac{1}{2}} - T^\varepsilon_{i-\frac{1}{2}}, \quad \forall i \in \{1, ..., N\}.$$

Let us multiply this equation by e_i^ε, sum for $i = 1, ..., N$, reorder the summation. Therefore

$$\sum_{i=0}^{N} \tau^\varepsilon_{i+\frac{1}{2}}\left(e^\varepsilon_i - e^\varepsilon_{i-1}\right)^2 = \sum_{i=1}^{N} T^\varepsilon_{i+\frac{1}{2}}\left(e^\varepsilon_{i+1} - e^\varepsilon_i\right)$$

Thanks to (13), one has

$$\sum_{i=0}^{N} \tau^\varepsilon_{i+\frac{1}{2}}\left(e^\varepsilon_i - e^\varepsilon_{i-1}\right)^2 \leqslant \sum_{i=1}^{N} \frac{c\,h}{\varepsilon}\left|e^\varepsilon_{i+1} - e^\varepsilon_i\right|.$$

Denote by

$$A = \left(\sum_{i=0}^{N} \tau^\varepsilon_{i+\frac{1}{2}}\left(e^\varepsilon_{i+1} - e^\varepsilon_i\right)^2\right)^{\frac{1}{2}} \quad \text{and} \quad B = \left(\sum_{i=0}^{N} \frac{1}{\tau^\varepsilon_{i+\frac{1}{2}}}\right)^{\frac{1}{2}}$$

The Cauhy-Schwarz inequality yields

$$A^2 \leqslant \frac{c\,h}{\varepsilon} AB.$$

Now, since the mesh is uniform (i.e. $h_i = h, \forall i \in \{1, ..., N\}$), one has

$$\frac{1}{\tau^\varepsilon_{i+\frac{1}{2}}} \leqslant \frac{\beta}{\alpha^2}\left(h^-_{i+1} + h^+_i\right) = \frac{\beta}{\alpha^2} h.$$

Using Remark 2, one obtains

$$\frac{1}{\tau^\varepsilon_{i+\frac{1}{2}}} \leqslant \frac{\beta}{\alpha^2}\frac{\varepsilon}{m}, \quad \text{hence} \quad B \leqslant c\,\varepsilon^{\frac{1}{2}}.$$

Therefore

$$A \leqslant \frac{c\,h}{\varepsilon^{\frac{1}{2}}} \text{ which yields Estimation (9).}$$

Remark that

$$\left|e^\varepsilon_i\right| \leqslant \sum_{j=1}^{N}\left|e^\varepsilon_j - e^\varepsilon_{j-1}\right|$$

Applying once again the Cauchy-Schwarz inequatity, one obtains

$$\left|e^\varepsilon_i\right| \leqslant AB, \quad \text{which yields Estimation (10).}$$

Theory from (Bensoussan, Lions & Papanicolaous, 1987) and (Jikov, Kozlov & Oleinik, 1994) on the estimate of the difference $u^\varepsilon - u$, with u the homogenized solution implies

$$\|u - u^\varepsilon\|_{L^\infty(\Omega)} \leqslant c\varepsilon. \tag{26}$$

Using (26) and (10) we obtain (11). This completes the proof of Theorem 1. □

2.2 Modified Finite Volume Scheme

The modified finite volume approach (see, Ewing, Iliev & Lazarov, 2001) is to rewrite the problem (1) into its mixed form:

$$\begin{cases} q^\varepsilon = -k^\varepsilon(x)\frac{du^\varepsilon}{dx}, \ \frac{dq^\varepsilon}{dx} = f(x), \ 0 < x < 1 \\ u^\varepsilon(0) = u^\varepsilon(1) = 0. \end{cases} \tag{27}$$

$q^\varepsilon(x)$ is the flux dependent variable. Conditions for continuity of the fonction and the flux through interface points ξ are added:

$$[q^\varepsilon] = [u^\varepsilon] = 0, \text{ for } x = \xi. \tag{28}$$

Here $[u^\varepsilon]$ deontes the difference of the right and left limits of u^ε at the point of discontinuity. We introduce a standard uniform cell-centered grid $x_0 = 0$, $x_1 = \frac{h}{2}$, $x_i = x_{i-1} + h$, $i = 2, ..., N$, $x_{N+1} = 1$, where $h = \frac{1}{N}$. Note, that the endpoints $x_0 = 0$ and $x_{N+1} = 1$ are part of the grid, but they are at $\frac{h}{2}$ distance from their neighboring grid points.The internal grid points can be considered as centered around the volumes $V_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ where $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}h$, $x_{i-\frac{1}{2}} = x_i - \frac{1}{2}h$. The values of a funtion f defined at the grid points x_i are denoted by f_i. Non-uniform grids can be treated in a similar way.
The finite volume method exploits the idea of writing the balance equation over the finite volume V_i, i.e. integrating the first equation of Problem (1) over each volume V_i.

$$q^\varepsilon_{i+\frac{1}{2}} - q^\varepsilon_{i-\frac{1}{2}} = hf_i,\ f_i = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx,\ i = 1, 2, ..., N. \tag{29}$$

Next, we rewrite the flux equation in the form

$$-\frac{du^\varepsilon}{dx} = \frac{q^\varepsilon}{k^\varepsilon(x)}$$

and integrate this expression over the interval (x_i, x_{i+1}) :

$$\left(u^\varepsilon_{i+1} - u^\varepsilon_i\right) = -\int_{x_i}^{x_{i+1}} \frac{du^\varepsilon}{dx}dx = \int_{x_i}^{x_{i+1}} \frac{q^\varepsilon}{k^\varepsilon(x)}dx \tag{30}$$

One assumes that the flux is two times continuously differentiable on the interface, so it can be expanded around the point $x_{i+\frac{1}{2}}$ in th Taylor series

$$q^\varepsilon(x) = q_{i+\frac{1}{2}} + \left(x - x_{i+\frac{1}{2}}\right)\frac{dq_{i+\frac{1}{2}}}{dx} + \frac{\left(x - x_{i+\frac{1}{2}}\right)^2}{2}\frac{d^2q^\varepsilon(\eta)}{dx^2},\ \eta \in (x_i, x_{i+1}). \tag{31}$$

After replacing the first derivative of the flux at $x_{i+\frac{1}{2}}$ by a two-point backward difference one gets the following approximation of (30).

$$-\left(u^\varepsilon_{i+1} - u^\varepsilon_i\right) = q_{i+\frac{1}{2}}\int_{x_i}^{x_{i+1}} \frac{dx}{k^\varepsilon(x)} + \frac{q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}}{h}\int_{x_i}^{x_{i+1}} \frac{\left(x - x_{i+\frac{1}{2}}\right)}{k^\varepsilon(x)}dx + O(h^3). \tag{32}$$

Finally, by the same gait as in (Ewing, Iliev & Lazarov, 2001) we get the following finite difference approximation of the differential problem (27):

$$L_{\varepsilon h}u^{\varepsilon h}_i = f_i \ \text{ for }\ i = 1, ..., N. \tag{33}$$

where

$$L_{\varepsilon h}u^{\varepsilon h}_i \equiv \begin{cases} -\frac{4}{3}\frac{1}{h}\left(k^{\varepsilon h}_{\frac{3}{2}}\ \dfrac{u^{\varepsilon h}_2 - u^{\varepsilon h}_1}{h} - k^{\varepsilon h}_{\frac{1}{2}}\ \frac{2u^{\varepsilon h}_1}{h}\right) \ (\text{for }\ i = 1), \\ -\left(1 + a^\varepsilon_{i+\frac{1}{2}} - a^\varepsilon_{i-\frac{1}{2}}\right)^{-1}\frac{1}{h}\left(k^{\varepsilon h}_{i+\frac{1}{2}}\ \dfrac{u^{\varepsilon h}_{i+1} - u^{\varepsilon h}_i}{h} - k^{\varepsilon h}_{i-\frac{1}{2}}\ \frac{u^{\varepsilon h}_i - u^{\varepsilon h}_{i-1}}{h}\right),\ i \neq 1, N, \\ -\frac{4}{3}\frac{1}{h}\left(-k^{\varepsilon h}_{N+\frac{1}{2}}\ \dfrac{2u^{\varepsilon h}_N}{h} - k^{\varepsilon h}_{N-\frac{1}{2}}\ \frac{u^{\varepsilon h}_N - u^{\varepsilon h}_{N-1}}{h}\right) \ (\text{for }\ i = N). \end{cases} \tag{34}$$

where

$$k^{\varepsilon h}_{i+\frac{1}{2}} = \left(\frac{1}{h}\int_{x_i}^{x_{i+1}} \frac{dx}{k^\varepsilon(x)}\right)^{-1}. \tag{35}$$

$$a^\varepsilon_{i+\frac{1}{2}} = k^{\varepsilon h}_{i+\frac{1}{2}}\frac{1}{h^2}\int_{x_i}^{x_{i+1}} \frac{\left(x - x_{i+\frac{1}{2}}\right)}{k^\varepsilon(x)}dx, \tag{36}$$

and $u^{\varepsilon h}_i$ denotes the approximation values of the exact solution. $k^{\varepsilon h}_{i+\frac{1}{2}}$ is the well known harmonic averaging of the coefficient $K^\varepsilon(x)$ over the cell (x_i, x_{i+1}) , which has played a fundamental role in deriving accurate schemes for discontinuous coefficients (see e.g, (Samarskii, 1977) and (Samarskii & Andréev, 1978)).

We now state the main result of this section.

Theorem 2. *Assume that the coefficient $k^\varepsilon(x)$ is a piecewise C^1-function and has a finite number of jump discontinuities, the grid is such that the discontinuities are at the points $x_{i+\frac{1}{2}}$, and the source term $f(x)$ is a C^1-function on $(0,1)$. Then the following estimate is valid:*

$$\left\|u^\varepsilon - u^{\varepsilon h}\right\|^2_{H^1(\Omega)} \equiv \sum_{i=1}^{N} k^{\varepsilon h}_{i+\frac{1}{2}} \left(e^\varepsilon_{i+1} - e^\varepsilon_i\right)^2 /h \leqslant \frac{ch^2}{\varepsilon}, \tag{37}$$

where $k^{\varepsilon h}_{i+\frac{1}{2}}$ is given by (35), and c is a constant independent of ε and h.

Proof. The proof of Theorem 2 is the same as that of Theorem 1. □

3. Numerical Results

In this section, one presents numerical results, comparing the approximations described in this paper and an example of exact solution. More especially, we shall present numerical results obtained with following data of Problem (1):

$$k(y) = \begin{cases} k_1 & \text{if} \quad 0 < y < \frac{1}{2}, \\ k_2 & \text{if} \quad \frac{1}{2} < y < 1, \end{cases}$$

$k_1,\ k_2 \in \mathbb{R}^*_+$,

$$k^\varepsilon(x) = \begin{cases} k_1, & \text{if} \quad p\varepsilon < x < \left(p+\frac{1}{2}\right)\varepsilon, \\ \\ k_2, & \text{if} \quad \left(p+\frac{1}{2}\right)\varepsilon < x < (p+1)\varepsilon, \end{cases}$$

$\varepsilon = \frac{1}{n_p}$, where n_p is a positive integer; $0 < p < n_p - 1$ and the source function is $f = 1$.
Therefore the exact solution u^ε is

$$u^\varepsilon(x) = \begin{cases} \frac{-x^2}{2k_1} + \frac{x}{2k_1} + \frac{(k_1-k_2)\varepsilon x}{4k_1(k_1+k_2)} - \frac{(k_1-k_2)p\varepsilon^2}{4k_1(k_1+k_2)} + \frac{(k_1-k_2)p\varepsilon}{4k_1k_2} - \frac{(k_1-k_2)p^2\varepsilon^2}{4k_1K_2}, & \text{if} \quad p\varepsilon < x < \left(p+\frac{1}{2}\right)\varepsilon, \\ \\ \frac{-x^2}{2k_2} + \frac{x}{2k_2} + \frac{(k_1-k_2)\varepsilon x}{4k_2(k_1+k_2)} - \frac{(k_1-k_2)(p+1)\varepsilon^2}{4k_2(k_1+k_2)} - \frac{(k_1-k_2)(p+1)\varepsilon}{4k_1k_2} + \frac{(k_1-k_2)(p+1)^2\varepsilon^2}{4k_1k_2}, & \text{if} \quad \left(p+\frac{1}{2}\right)\varepsilon < x < (p+1)\varepsilon, \end{cases}$$

and the homogenized solution is

$$u(x) = \frac{(k_1+k_2)\,x\,(1-x)}{4k_1k_2}.$$

All the simulations presented have been done with uniform grids.

The first test problem involved simulation with $k_1 = 10^3$ and $k_2 = 1$.

Table 1. Error table with the classical finite volume method (CFVM) when $\varepsilon \to 0$.

Volumes number=256	$\varepsilon = 0.5$	$\varepsilon = 0.25$	$\varepsilon = 0.125$	$\varepsilon = 0.0625$
$\left\|u^\varepsilon - u^{\varepsilon h}\right\|_{L^\infty(\Omega)}$	1.907350e-06	1.907350e-06	1.907350e-06	1.907351e-06
$\left\|u^\varepsilon - u^{\varepsilon h}\right\|_{L^2(\Omega)}$	1.348700e-06	1.348700e-06	1.348701e-06	1.348701e-06
$\left\|u^\varepsilon - u^{\varepsilon h}\right\|_{H^1(\Omega)}$	1.543002e-03	2.069417e-03	2.843546e-03	3.961350e-03

Test Problem 1: This error table confirms the estimates of Theorem 1.

In the following graphics, Homog denotes the homogenized solution and Exact denotes the exact solution.

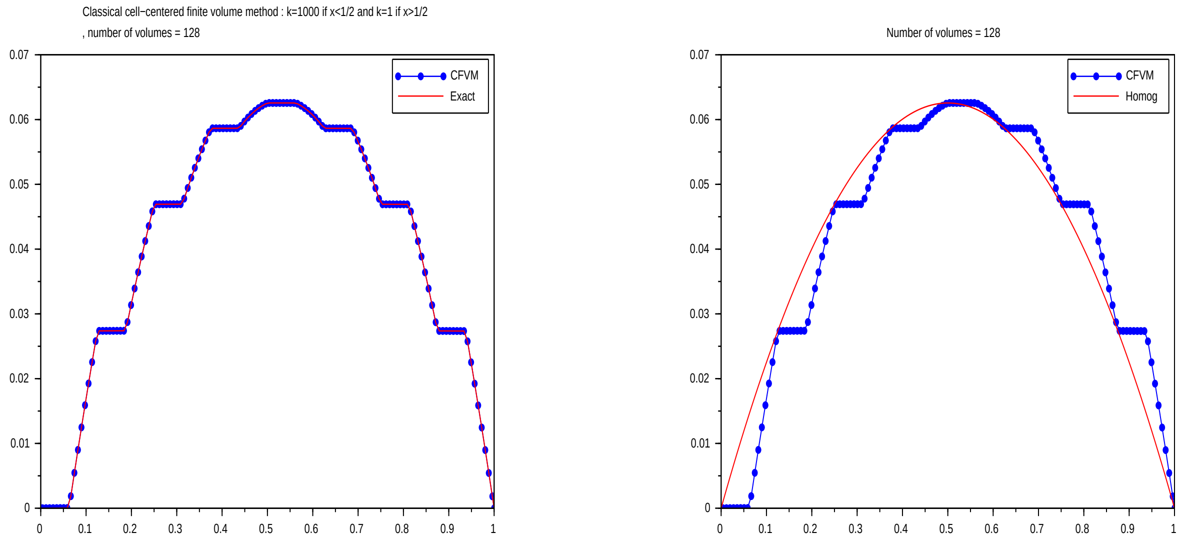

Figure 1. Test problem 1: $\varepsilon = 1/8$

Table 2. Error table with the modified finite volume method (MFVM) when $\varepsilon \to 0$.

Volumes number=256	$\varepsilon = 0.5$	$\varepsilon = 0.25$	$\varepsilon = 0.125$	$\varepsilon = 0.0625$
$\left\Vert u^{\varepsilon} - u^{\varepsilon h}\right\Vert_{L^{\infty}(\Omega)}$	1.897054e-06	1.897530e-06	1.897768e-06	1.897887e-06
$\left\Vert u^{\varepsilon} - u^{\varepsilon h}\right\Vert_{L^{2}(\Omega)}$	9.119900e-07	1.001070e-06	1.049512e-06	1.074557e-06
$\left\Vert u^{\varepsilon} - u^{\varepsilon h}\right\Vert_{H^{1}(\Omega)}$	1.193472e-03	1.823735e-03	2.670071e-03	3.838726e-03

Test Problem 1: This error table confirms the estimate of Theorem 2.

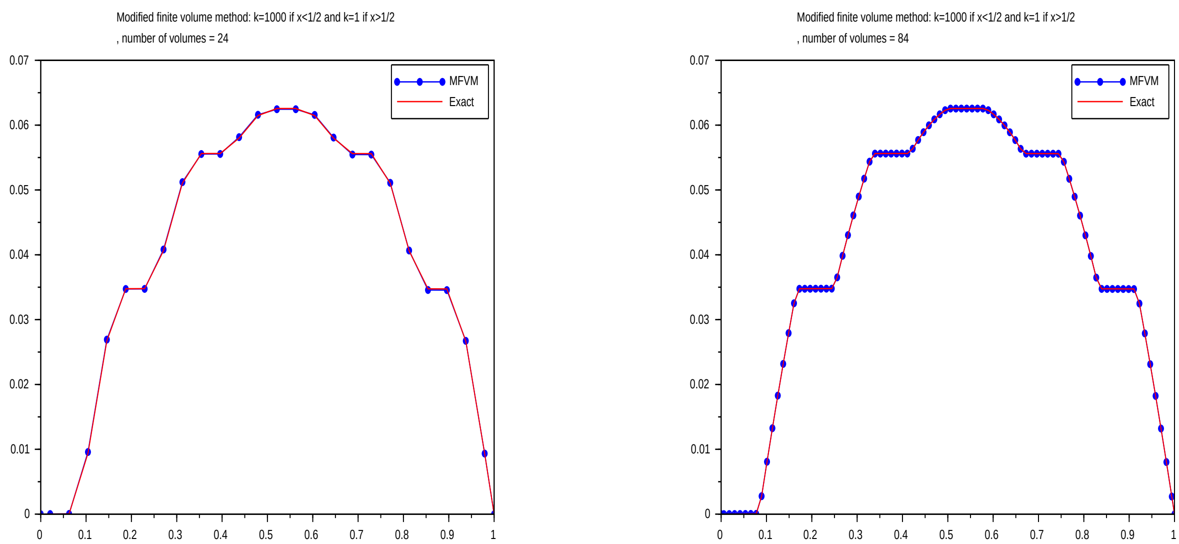

Figure 2. Test problem 1: $\varepsilon = 1/6$

The second test problem involved simulation with $k_1 = 1$ and $k_2 = 10^4$.

Table 3. Convergence test of the classical finite volume

ε=0.1	$h = 1/20$	$h = 1/40$	$h = 1/100$	$h = 1/200$
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{L^{\infty}(\Omega)}$	3.125000e-04	7.812500e-05	1.250000e-05	3.125010e-06
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{L^{2}(\Omega)}$	2.209709e-04	5.524272e-05	8.838836e-06	2.209713e-06
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{H^{1}(\Omega)}$	4.049933e-02	2.024966e-02	8.099865e-03	4.049933e-03

Table 4. Convergence test of the modified finite volume method

ε=0.1	$h = 1/20$	$h = 1/40$	$h = 1/100$	$h = 1/200$
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{L^{\infty}(\Omega)}$	2.968438e-04	7.616367e-05	1.237365e-05	3.109035e-06
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{L^{2}(\Omega)}$	1.736241e-04	4.342797e-05	6.949458e-06	1.737400e-06
$\left\|u^{\varepsilon} - u^{\varepsilon h}\right\|_{H^{1}(\Omega)}$	3.847363e-02	1.924855e-02	7.702354e-03	3.851676e-03

The tables 3 and 4 demonstrate, as in (Ewing, Iliev & Lazarov, 2001) that the modified scheme has a smaller constant of convergence than the classical scheme based on harmonic averaging for this class of equations.

4. Concluding remarks

The purpose of this paper was to resolve, by finite volume methods, a class of second-order elliptic problems, with discontinuous and highly oscillating coefficients, in order to evaluate the effect of ε the parameter involved in the periodic homogenization on the approximate solution. Study was limited to the academic Dirichlet problem and enabled to clarify the dependence of numerical experiments to ε. Error estimates were obtained. Numerical simulations confirm theorical results and also show that the modified finite volume scheme is much more accurate than the classical scheme in solving these problems.
The extension of these results to two-dimensional and three-dimensional problems is currently underway.

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