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Zero-Sum Coefficient Derivations in Three Variables of Triangular Algebras

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Abstract

Under mild assumptions Benkovič showed that an f-derivation of a triangular algebra is a derivation when the sum of the coefficients of the multilinear polynomial f is nonzero. We investigate the structure of f-derivations of triangular algebras when f is of degree 3 and the coefficient sum is zero. The zero-sum coeffient derivations include Lie derivations (degree 2) and Lie triple derivations (degree 3), which have been previously shown to be not necessarily derivations but in standard form, i.e., the sum of a derivation and a central map. In this paper, we present sufficient conditions on the coefficients of f to ensure that any f-derivations are derivations or are in standard form.

Keywords: derivation, f-derivation, Lie derivation, Lie triple derivation, triangular algebra

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ and $\mathcal{B}$ two algebras over $\mathcal{R}$ with units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively, and let $\mathcal{M}$ be an $(\mathcal{A},\mathcal{B})$-bimodule. We assume throughout the article that $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module and as a right $\mathcal{B}$-module. Let $\mathcal{T}$ be the matrix algebra

$$\mathcal{T} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| \, a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}.$$

An algebra isomorphic to $\mathcal{T}$ is called a *triangular algebra*. We assume $\mathcal{T}$ is 2-torsion free for the purpose of this article. Upper triangular matrix rings and nest algebras are typical examples of triangular algebras. See the references for recent results on maps of triangular algebras.

The structure of various types of derivations of $\mathcal{T}$ has been studied in a series of papers: (Benkovič, 2015), (Benkovič, 2016), (Benkovič & Eremita, 2012), (Cheung, 2003), (Ji, Liu, & Zhao, 2012), (Wang, Wang, & Du, 2013), (Xiao & Wei, 2012), (Yu & Zhang, 2010), and (Zhang & Yu, 2006). A derivation of $\mathcal{T}$ is an $\mathcal{R}$-linear map d such that $d(xy) = d(x)y + xd(y)$ for any $x, y \in \mathcal{T}$. There are variations of this definition. For example, a Jordan derivation J is defined by the property

$$J(xy + yx) = J(x)y + xJ(y) + J(y)x + yJ(x),$$

a Lie derivation L by

$$L([x, y]) = [L(x), y] + [x, L(y)],$$

and a Lie triple derivation L by

$$L([[x, y], z]) = [[L(x), y], z] + [[x, L(y)], z] + [[x, y], L(z)].$$

The most general notion of this type is that of f-derivations. Let f be a multilinear polynomial of degree $n \geq 2$ over $\mathcal{R}$ with noncommutative indeterminate variables.

$$f(x_1, x_2, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\pi x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}, \quad \alpha_\pi \in \mathcal{R}$$

where the sum is over S_n, the symmetric group. An $\mathcal{R}$-linear map $L : \mathcal{T} \to \mathcal{T}$ satisfying

$$d(f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, d(x_i), x_{i+1}, \ldots, x_n)$$

is called an *f-derivation* or a *derivation with respect to f*. We get the usual notion of derivation when $f = x_1x_2$, and the notion of Lie derivation when $f = x_1x_2 - x_2x_1$, and so on. Obviously, a derivation is an f-derivation for any f. The converse is true for certain classes of f.

Let α be the sum of all coefficients of f. Benkovič(Benkovič, 2015) proved that in case $\alpha \neq 0$, every f-derivation is a derivation if α is *$\mathcal{T}$-regular* and $\mathcal{T}$ is $(n-1)$-torsion free. He left the case $\alpha = 0$ as an open problem. In the latter case, an f-derivation L need not be a derivation, but it could be in standard form, that is, $L = d + h$, where d is a derivation and h is a linear map into the center $\mathcal{Z}(\mathcal{T})$ satisfying $h(f(\mathcal{T}, \ldots, \mathcal{T})) = 0$. Special cases of this problem have been previously studied in (Benkovič & Eremita, 2012; Cheung, 2003; Ji et al., 2012; Xiao & Wei, 2012; Zhang & Yu, 2006). Under mild assumptions, Cheung (Cheung, 2003) proved that a Lie derivation is of standard form and Xiao and Wei (Xiao & Wei, 2012) proved that Lie triple derivations are of standard form. In (Xiao & Wei, 2012), the following conditions are assumed. Refer to (Xiao & Wei, 2012) and (Cheung, 2003) for the discussion on these conditions.

(♣) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$.

(♠) $[a, a'] \in \mathcal{Z}(\mathcal{A})$ for all $a' \in \mathcal{A}$ implies $a \in \mathcal{Z}(\mathcal{A})$ and $[b, b'] \in \mathcal{Z}(\mathcal{B})$ for all $b' \in \mathcal{B}$ implies $b \in \mathcal{Z}(\mathcal{B})$.

In this article, under similar assumptions as in the aforementioned papers, we will discuss the structure of f-derivations when f is of degree 3 and $\alpha = 0$, that is, when

$$f(x, y, z) = rxyz + sxzy + tyxz + uyzx + vzxy + wzyx$$

with $r, s, t, u, v, w \in \mathcal{R}$, $r + s + t + u + v + w = 0$. We will examine sufficient coefficient conditions on which an f-derivation is a derivation or is in standard form. In the process, we leave certain special cases unsolved.

2. Preliminaries

We will identify $\mathcal{A}$ with the subalgebra of $\mathcal{T}$ with elements of the form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. Similarly, we will identify $m \in \mathcal{M}$ with $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $b \in \mathcal{B}$ with $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$. Under this identification, $\mathcal{T} = \mathcal{A} + \mathcal{M} + \mathcal{B}$ and every element of $\mathcal{T}$ is written uniquely as $a + m + b$ for some $a \in \mathcal{A}$, $m \in \mathcal{M}$, and $b \in \mathcal{B}$. We denote the projections from the triangular algebra $\mathcal{T}$ to $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{B}$ by $\pi_{\mathcal{A}}$, $\pi_{\mathcal{M}}$, and $\pi_{\mathcal{B}}$, respectively. We have the following evident rules, which will be used extensively throughout the article.

Lemma 1. *For any $a \in \mathcal{A}$, $m, m' \in \mathcal{M}$, $b \in \mathcal{B}$, and $t, t' \in \mathcal{T}$,*

1. $ab = ba = [a, b] = [b, a] = 0$,
2. $ma = 0$ *and* $[a, m] = am$,
3. $bm = 0$ *and* $[m, b] = mb$,
4. $mm' = 0$ *and* $[m, m'] = 0$,
5. $1_{\mathcal{T}} = 1_{\mathcal{A}} + 1_{\mathcal{B}}$,
6. $[1_{\mathcal{A}}, a + m + b] = m$ *and* $[a + m + b, 1_B] = m$,
7. $[a + m + b, m'] = am' - m'b \in \mathcal{M}$,
8. $[a, t] \in \mathcal{A} + \mathcal{M}$, *or equivalently* $\pi_{\mathcal{B}}[a, t] = 0$,
9. $[t, b] \in \mathcal{M} + \mathcal{B}$, *or equivalently* $\pi_{\mathcal{A}}[t, b] = 0$,
10. $\pi_{\mathcal{A}}(tt') = \pi_{\mathcal{A}}(t)\pi_{\mathcal{A}}(t')$, $\pi_{\mathcal{B}}(tt') = \pi_{\mathcal{B}}(t)\pi_{\mathcal{B}}(t')$.

We will use the convention that, unless stated otherwise, elements in small letters belong to the sets in the corresponding capital letters. For example, a, a', a_1, a_2, etc. should be understood as elements of $\mathcal{A}$.

Proposition 2 (Proposition 3 (Cheung, 2003))**.** *The center $\mathcal{Z}(\mathcal{T})$ of the triangular algebra $\mathcal{T}$ is $\{a+b \mid am = mb$ for all $m \in \mathcal{M}\}$.*

Proof. Suppose $a + m' + b \in \mathcal{Z}(\mathcal{T})$, then we have $0 = [1_{\mathcal{A}}, a + m' + b] = m'$. We also have $0 = [a + m' + b, m] = am - mb$ for any $m \in \mathcal{M}$.

Conversely, suppose $am = mb$ for all $m \in \mathcal{M}$. Then $a \in \mathcal{Z}(\mathcal{A})$ because for any $a' \in \mathcal{A}$ and $m \in \mathcal{M}$,

$$(aa')m = a(a'm) = (a'm)b = a'(mb) = a'(am) = (a'a)m$$

and by the faithfulness of $\mathcal{M}$, $aa' = a'a$. Similarly, $b \in \mathcal{Z}(\mathcal{B})$. Now for any $a' + m' + b' \in \mathcal{T}$,

$$\begin{aligned}[a + b, a' + m' + b'] &= [a, a'] + [a, m'] + [a, b'] + [b, a'] + [b, m'] + [b, b'] \\ &= [a, m'] + [b, m'] = am' - m'b = 0.\end{aligned}$$

□

For $a + b \in \mathcal{Z}(\mathcal{T})$, the elements a and b are a pair. If $a + b_1 \in \mathcal{Z}(\mathcal{T})$ and $a + b_2 \in \mathcal{Z}(\mathcal{T})$, then $am = mb_1 = mb_2$ for any $m \in \mathcal{M}$, and by faithfulness of $\mathcal{M}$, we have $b_1 = b_2$. Using this property, we can construct an isomorphism

$$\phi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$$

by sending an element to the other element in the pair. It is straightforward to verify that ϕ respects algebra operations. See Proposition 3 in (Cheung, 2003) for the proof. The following formulas follow from the definition.

Lemma 3. *For any $a \in \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})), b \in \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$, and $m \in \mathcal{M}$,*

1. $am = m\phi(a)$,
2. $\phi^{-1}(b)m = mb$.

Lemma 4. *Let $t \in \mathcal{Z}(\mathcal{T})$. If $tm = mt = 0$ for all $m \in \mathcal{M}$, then $t = 0$.*

Proof. Since $t \in \mathcal{Z}(\mathcal{T})$, $t = a + b$ for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If $tm = mt = 0$, then $am = 0$ and $mb = 0$. Since m is arbitrary and $\mathcal{M}$ is faithful, $a = 0$ and $b = 0$. □

Definition 5. An element $r \in \mathcal{R}$ is called *$\mathcal{T}$-regular* or simply *regular* if $rt = 0$, $t \in \mathcal{T}$ implies $t = 0$. Equivalently, r is regular if $rt_1 = rt_2$, $t_1, t_2 \in \mathcal{T}$ implies $t_1 = t_2$.

3. Discussion of the Problem

Now we discuss the main problem. Suppose L is an f-derivation where

$$f(x, y, z) = rxyz + sxzy + tyxz + uyzx + vzxy + wzyx$$

with $r, s, t, u, v, w \in \mathcal{R}$, and $r + s + t + u + v + w = 0$. We assume the sum or difference of any combination of these coefficients is either 0 or regular. The regularity condition is necessary to cancel coefficients. For example, if r is regular, then L is a derivation if and only if L is a derivation with respect to $f(x, y) = rxy$.

Lemma 6. *Let $x, y, z \in \mathcal{T}$.*

1. *If $x \in \mathcal{Z}(\mathcal{T})$, then $f(x, y, z) = (r + t + u)x[y, z]$.*
2. *If $y \in \mathcal{Z}(\mathcal{T})$, then $f(x, y, z) = (u + v + w)y[z, x]$.*
3. *If $z \in \mathcal{Z}(\mathcal{T})$, then $f(x, y, z) = (r + s + v)z[x, y]$.*

In particular, if $x = 1_{\mathcal{T}}$, $y = 1_{\mathcal{T}}$, or $z = 1_{\mathcal{T}}$, then $f(x, y, z)$ is the constant multiple of a commutator.

Proof. We prove the first one. Others are similar. If x is in the center, it can be factored out. Therefore,

$$\begin{aligned}f(x, y, z) &= x(ryz + szy + tyz + uyz + vzy + wzy) \\ &= x((r + t + u)yz + (s + v + w)zy) \\ &= x((r + t + u)yz - (r + t + u)zy) \\ &= (r + t + u)x[y, z].\end{aligned}$$

□

Lemma 7. *For any $x, y, z \in \mathcal{T}$,*

1. $(r + s - u - w)[x, L(1_{\mathcal{T}})] = 0$,
2. $(t + u - s - v)[y, L(1_{\mathcal{T}})] = 0$,
3. $(v + w - r - t)[z, L(1_{\mathcal{T}})] = 0$.

Proof. For any $x \in \mathcal{T}$, $f(x, 1_{\mathcal{T}}, 1_{\mathcal{T}}) = (r + s + t + u + v + w)x = 0$, thus,

$$\begin{aligned} 0 = L(f(x, 1_{\mathcal{T}}, 1_{\mathcal{T}})) &= f(L(x), 1_{\mathcal{T}}, 1_{\mathcal{T}}) + f(x, L(1_{\mathcal{T}}), 1_{\mathcal{T}}) + f(x, 1_{\mathcal{T}}, L(1_{\mathcal{T}})) \\ &= 0 + (r + s + v)[x, L(1_{\mathcal{T}})] + (u + v + w)[L(1_{\mathcal{T}}), x] \\ &= (r + s - u - w)[x, L(1_{\mathcal{T}})]. \end{aligned}$$

Others are derived similarly. □

If one of the coefficients in this lemma is nonzero and regular, then we can cancel it to say $L(1_{\mathcal{T}})$ is in the center.

Proposition 8. *The following are equivalent.*

1. $r + s = u + w, t + u = s + v$, *and* $v + w = r + t$.
2. $r + t + u = r + s + v = u + v + w$.

Proof. The equivalence is evident. □

By the proposition, we only need to study the following mutually exclusive cases.

- Case 1: $r + s \neq u + w$, $t + u \neq s + v$, or $v + w \neq r + t$.
- Case 2: $r + t + u = u + v + w = r + s + v = 0$.
- Case 3: $r + t + u = u + v + w = r + s + v \neq 0$.

The first case is examined in the following section. The second case results in a generalization of the theorem by (Xiao & Wei, 2012) on triple Lie derivations. It is discussed in Section 5. The third case will be left unsolved. The difficulty in the third case lies in the fact that $L(1_{\mathcal{T}})$ may not belong to the center. It complicates the effort to derive meaningful properties of f-derivations.

4. Case 1

In this case, Lemma 7 and the regularity of coefficients condition imply that $L(1_{\mathcal{T}}) \in \mathcal{Z}(\mathcal{T})$. Applying Lemma 6 to

$$L(f(1_{\mathcal{T}}, y, z)) = f(L(1_{\mathcal{T}}), y, z) + f(1_{\mathcal{T}}, L(y), z) + f(1_{\mathcal{T}}, y, L(z))$$

yields

$$(r + t + u)L([y, z]) = (r + t + u)(L(1_{\mathcal{T}})[y, z] + [L(y), z] + [y, L(z)]).$$

Similarly with $y = 1_{\mathcal{T}}$ or $z = 1_{\mathcal{T}}$, we get

$$\begin{aligned} (u + v + w)L([z, x]) &= (u + v + w)(L(1_{\mathcal{T}})[z, x] + [L(z), x] + [z, L(x)]), \\ (r + s + v)L([x, y]) &= (r + s + v)(L(1_{\mathcal{T}})[x, y] + [L(x), y] + [x, L(y)]). \end{aligned}$$

Since not all of $r + t + u$, $u + v + w$, and $r + s + v$ are zeros in Case 1, by the regularity of coefficients,

$$L([x, y]) = L(1_{\mathcal{T}})[x, y] + [L(x), y] + [x, L(y)].$$

The next proposition describes the structure of L.

Proposition 9. *Let $\mathcal{T}$ be a triangular algebra satisfying (♣). Suppose $c \in \mathcal{Z}(\mathcal{T})$ and $L : \mathcal{T} \to \mathcal{T}$ is a linear map such that*

$$L([x, y]) = c[x, y] + [L(x), y] + [x, L(y)]$$

for all $x, y \in \mathcal{T}$. Then $L = d + h - ci$ where d is a derivation, h is a central map vanishing on commutators, and i is the identity map.

Proof. Let $L' = L + ci$. Then L' is a Lie derivation since

$$\begin{aligned} L'([x, y]) =& L([x, y]) + c[x, y] \\ =& c[x, y] + [L(x), y] + [x, L(y)] + c[x, y] \\ =& [L(x) + cx, y] + [x, L(y) + cy] \\ =& [L'(x), y] + [x, L'(y)]. \end{aligned}$$

We apply Theorem 11 in (Cheung, 2003) assuming (♣). Then $L' = d + h$ where d is a derivation, h is a central map vanishing on commutators. Therefore, $L = d + h - ci$. □

Theorem 10. *Let L be an f-derivation on $\mathcal{T}$ where*

$$f(x, y, z) = rxyz + sxzy + tyxz + uyzx + vzxy + wzyx$$

with $r, s, t, u, v, w \in \mathcal{R}$, $r + s + t + u + v + w = 0$. Assume condition (♣). If $r + s \neq u + w$, $t + u \neq s + v$, or $v + w \neq r + t$, then $L(1_{\mathcal{T}}) \in \mathcal{Z}(\mathcal{T})$ and

$$L = d + h - L(1_{\mathcal{T}})i$$

where $d : \mathcal{T} \to \mathcal{T}$ is a derivation, $h : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ satisfies $h([x, y]) = 0$ for all $x, y \in \mathcal{T}$, and $i : \mathcal{T} \to \mathcal{T}$ is the identity map. Furthermore, in one of the following three cases, $L(1_{\mathcal{T}}) = 0$ and $h = 0$, thus L is a derivation.

1. *$r + s = u + w \neq 0$, $t + u \neq s + v$, and $v + w \neq r + t$.*
2. *$r + s \neq u + w$, $t + u = s + v \neq 0$, and $v + w \neq r + t$.*
3. *$r + s \neq u + w$, $t + u \neq s + v$, and $v + w = r + t \neq 0$.*

Proof. We have seen that $c = L(1_{\mathcal{T}}) \in \mathcal{Z}(\mathcal{T})$ and $L = d + h - ci$ by Proposition 9. It remains to show $L = d$ in one of three special cases. We may assume the second case where $t + u = s + v \neq 0$ without loss of generality. Note that for any $m \in \mathcal{M}$, $h(m) = h([1_{\mathcal{A}}, m]) = 0$ and $h(1_{\mathcal{T}}) = (L - d + ci)(1_{\mathcal{T}}) = c - 0 + c = 2c$.

Since both L and d are f-derivations, so is $L - d = h - ci$. For any $x, y, z \in \mathcal{T}$,

$$(h - ci)(f(x, y, z) = f((h - ci)(x), y, z) + f(x, (h - ci)(y), z) + f(x, y, (h - ci)(z)).$$

Simplifying the equation, we get

$$h(f(x, y, z)) = f(h(x), y, z) + f(x, h(y), z) + f(x, y, h(z)) - 2cf(x, y, z).$$

Substitute $x = 1_{\mathcal{A}}, y = m, z = 1_{\mathcal{B}}$ where m is an arbitrary element of $\mathcal{M}$. Then $f(1_{\mathcal{A}}, m, 1_{\mathcal{B}}) = rm \in \mathcal{M}$. Hence, the left-hand side is zero. We can simplify the right-hand side using Lemma 6. Then we have

$$\begin{aligned} 0 &= (r + t + u)h(1_{\mathcal{A}})[1_{\mathcal{A}}, m] + (r + s + v)h(1_{\mathcal{B}})[1_{\mathcal{A}}, m] - 2c(rm) \\ &= (r + t + u)h(1_{\mathcal{A}})m + (r + s + v)h(1_{\mathcal{B}})m - (h(1_{\mathcal{A}}) + h(1_{\mathcal{B}}))(rm) \\ &= ((t + u)h(1_{\mathcal{A}}) + (s + v)h(1_{\mathcal{B}}))m. \end{aligned}$$

By Lemma 4, we get $(t+u)h(1_{\mathcal{A}})+(s+v)h(1_{\mathcal{B}}) = 0$. Since $t+u = s+v \neq 0$ and they are regular, we have $h(1_{\mathcal{A}})+h(1_{\mathcal{B}}) = 0$, or $2c = h(1_{\mathcal{T}}) = 0$. Since $\mathcal{T}$ is 2-torsion free, $c = 0$.

Next, we substitute $x = 1_{\mathcal{A}}$, $y = 1_{\mathcal{A}}$, and $z = m$. Then $f(1_{\mathcal{A}}, 1_{\mathcal{A}}, m) = rm + tm \in \mathcal{M}$. So the left hand side is zero again. Then

$$\begin{aligned} 0 &= (r + t + u)h(1_{\mathcal{A}})[1_{\mathcal{A}}, m] + (u + v + w)h(1_{\mathcal{A}})[m, 1_{\mathcal{A}}] \\ &= (r + t - v - w)h(1_{\mathcal{A}})m, \end{aligned}$$

which implies $h(1_{\mathcal{A}}) = 0$ since $r + t - v - w$ is nonzero, $h(1_{\mathcal{A}}) \in \mathcal{Z}(\mathcal{T})$, and m is arbitrary. We also have $h(1_{\mathcal{B}}) = h(1_{\mathcal{T}}) - h(1_{\mathcal{A}}) = 0$.

Finally, we show that $h(a) = h(b) = 0$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If we substitute $x = a$, $y = 1_{\mathcal{A}}$, and $z = m$, then $(r + t + u)h(a)[1_{\mathcal{A}}, m] = (r + t + u)h(a)m = 0$, which implies $(r + t + u)h(a) = 0$ by Lemma 4. Rotating the values of x, y, and z, we also get $(u + v + w)h(a) = 0$ and $(r + s + v)h(a) = 0$. Not all three of the coefficients are zeros, therefore $h(a) = 0$. Similarly, $h(b) = 0$. □

Example 11. Let $f(x, y, z) = x[y, z] + [x, y]z$. Any f-derivation on $\mathcal{T}$ is a derivation because $r + s = -2$, $u + w = 0$, $t + u = s + v = 1$, $v + w = 0$, and $r + t = -1$.

5. Case 2

In this case $r + t + u = u + v + w = r + s + v = 0$ and $r + s + t + u + v + w = 0$. Solving the system of linear equations we get $v = t, w = r, u = s$, and $t = -r - s$. Therefore,

$$\begin{aligned} f(x, y, z) &= r(xyz - yxz - zxy + zyx) + s(xzy - yxz + yzx - zxy) \\ &= r[[x, y], z] + s[[x, z], y]. \end{aligned}$$

If $r + s = 0$, then $f(x, y, z) = r([[x, y], z] - [[x, z], y]) = r[[z, y], x]$ by the Jacobi identity. Then L is a Lie triple derivation. So this case is resolved by Theorem 2.1 in (Xiao & Wei, 2012). The case when $r + s \neq 0$ is resolved by the next theorem, which is a generalization of Theorem 2.1 in (Xiao & Wei, 2012).

Theorem 12. *Let $\mathcal{T}$ be a 2-torsion free triangular algebra $\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$, and assume $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module and as a right $\mathcal{B}$-module. Suppose that $\mathcal{T}$ satisfies conditions (♣) and (♠). Let $r, s \in \mathcal{R}$ and assume r, s, and $r+s$ are $\mathcal{T}$-regular. If L is an f-derivation where $f(x, y, z) = r[[x, y], z] + s[[x, z], y]$, then L is of standard form, that is, there exist a derivation $d : \mathcal{T} \to \mathcal{T}$ and a linear map $h : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ such that $L = d + h$ and $h(r[[x, y], z] + s[[x, z], y]) = 0$ for all $x, y, z \in \mathcal{T}$.*

The theorem is proved after a series of lemmas, which are direct extensions of results by Xiao and Wei. We assume all conditions in the theorem for all subsequent lemmas, and we let $\alpha = \pi_{\mathcal{A}}L$, $\mu = \pi_{\mathcal{M}}L$, and $\beta = \pi_{\mathcal{B}}L$. Then $L = \alpha + \mu + \beta$.

Lemma 13. $L(1_{\mathcal{T}}) \in \mathcal{Z}(\mathcal{T})$

Proof. Since $1_{\mathcal{T}}$ is in the center, $[1_{\mathcal{T}}, x] = 0$ for all $x \in \mathcal{T}$. Therefore,

$$\begin{aligned} 0 =& L(r[[1_{\mathcal{T}}, y], z] + s[[1_{\mathcal{T}}, z], y]) \\ =& r[[L(1_{\mathcal{T}}), y], z] + r[[1_{\mathcal{T}}, L(y)], z] + r[[1_{\mathcal{T}}, y], L(z)] \\ &\quad + s[[L(1_{\mathcal{T}}), z], y] + s[[1_{\mathcal{T}}, L(z)], y] + s[[1_{\mathcal{T}}, z], L(y)] \\ =& r[[L(1_{\mathcal{T}}), y], z] + s[[L(1_{\mathcal{T}}), z], y]. \end{aligned}$$

Let $L(1_{\mathcal{T}}) = a + m' + b$. We want to show that $m' = 0$ and $am = mb$ for all $m \in \mathcal{M}$. Substitute $y = z = 1_{\mathcal{B}}$. Then

$$0 = r[[a + m' + b, 1_{\mathcal{B}}], 1_{\mathcal{B}}] + s[[a + m' + b, 1_{\mathcal{B}}], 1_{\mathcal{B}}] = (r + s)m'.$$

Therefore, $m' = 0$ since $r + s$ is regular. Next, let $y = m$ and $z = 1_{\mathcal{B}}$. Then

$$\begin{aligned} 0 =& r[[a + b, m], 1_{\mathcal{B}}] + s[[a + b, 1_{\mathcal{B}}], m] \\ =& r[am - mb, 1_{\mathcal{B}}] + s[0, m] \\ =& r(am - mb). \end{aligned}$$

Since r is regular, $am = mb$ for all $m \in \mathcal{M}$. □

Lemma 14. *For any $m \in \mathcal{M}$, $L(m) \in \mathcal{M}$.*

Proof. For $x = m$, $y = 1_{\mathcal{B}}$, and $z = 1_{\mathcal{B}}$,

$$L(f(m, 1_{\mathcal{B}}, 1_{\mathcal{B}})) = f(L(m), 1_{\mathcal{B}}, 1_{\mathcal{B}}) + f(m, L(1_{\mathcal{B}}), 1_{\mathcal{B}}) + f(m, 1_{\mathcal{B}}, L(1_{\mathcal{B}})).$$

The left-hand side is $(r + s)L(m)$ since $[[m, 1_{\mathcal{B}}], 1_{\mathcal{B}}] = m$. On the other hand, $f(L(m), 1_{\mathcal{B}}, 1_{\mathcal{B}})$, $f(m, L(1_{\mathcal{B}}), 1_{\mathcal{B}})$, and $f(m, 1_{\mathcal{B}}, L(1_{\mathcal{B}}))$ belong to $\mathcal{M}$ because $[t, 1_{\mathcal{B}}] \in \mathcal{M}$ and $[m, t] \in \mathcal{M}$ for any $t \in \mathcal{T}$ by Lemma 1 (6) and (7). Since $r + s$ is regular, $L(m) \in \mathcal{M}$. □

Lemma 15. *$\alpha(1_{\mathcal{A}}) + \beta(1_{\mathcal{A}}) \in \mathcal{Z}(\mathcal{T})$ and $\alpha(1_{\mathcal{B}}) + \beta(1_{\mathcal{B}}) \in \mathcal{Z}(\mathcal{T})$. Thus, $\phi\alpha(1_{\mathcal{A}}) = \beta(1_{\mathcal{A}})$ and $\phi\alpha(1_{\mathcal{B}}) = \beta(1_{\mathcal{B}})$.*

Proof. Let $L(1_{\mathcal{B}}) = a + m' + b$ where $a = \alpha(1_{\mathcal{B}})$, $m' = \mu(1_{\mathcal{B}})$, and $b = \beta(1_{\mathcal{B}})$. As in the previous lemma, let $x = m$, $y = 1_{\mathcal{B}}$, and $z = 1_{\mathcal{B}}$. Then

$$L(f(m, 1_{\mathcal{B}}, 1_{\mathcal{B}})) = f(L(m), 1_{\mathcal{B}}, 1_{\mathcal{B}}) + f(m, L(1_{\mathcal{B}}), 1_{\mathcal{B}}) + f(m, 1_{\mathcal{B}}, L(1_{\mathcal{B}})).$$

The left-hand side is $(r + s)L(m)$. On the other side, by Lemma 1 and 14,

$$\begin{aligned} f(L(m), 1_{\mathcal{B}}, 1_{\mathcal{B}}) &= (r + s)[[L(m), 1_{\mathcal{B}},]1_{\mathcal{B}}] = (r + s)L(m), \\ f(m, L(1_{\mathcal{B}}), 1_{\mathcal{B}}) &= r[[m, a + m' + b], 1_{\mathcal{B}}] + s[[m, 1_{\mathcal{B}}], a + m' + b] \\ &= (r + s)(-am + mb), \\ f(m, 1_{\mathcal{B}}, L(1_{\mathcal{B}})) &= r[[m, 1_{\mathcal{B}}], a + m' + b] + s[[m, a + m' + b], 1_{\mathcal{B}}] \\ &= (r + s)(-am + mb). \end{aligned}$$

Thus, $(r + s)L(m) = (r + s)(L(m) - 2(am - mb))$. Since $r + s$ is regular and $\mathcal{T}$ is 2-torsion free, $am = mb$ for all $m \in \mathcal{M}$, i.e., $a + b = \alpha(1_{\mathcal{B}}) + \beta(1_{\mathcal{B}}) \in \mathcal{Z}(\mathcal{T})$. Similarly, we can prove $\alpha(1_{\mathcal{A}}) + \beta(1_{\mathcal{A}}) \in \mathcal{Z}(\mathcal{T})$ by substituting $x = m$, $y = 1_{\mathcal{A}}$, and $z = 1_{\mathcal{A}}$. □

Lemma 16. *$\beta(a) \in \mathcal{Z}(\mathcal{B})$ and $\alpha(b) \in \mathcal{Z}(\mathcal{A})$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Proof. Note that $f(a, m, 1_{\mathcal{B}}) = r[[a, m], 1_{\mathcal{B}}] + s[[a, 1_{\mathcal{B}}], m] = ram$. Then

$$L(ram) = r\left([[L(a), m], 1_{\mathcal{B}}] + [[a, L(m)], 1_{\mathcal{B}}] + [[a, m], L(1_{\mathcal{B}})]\right) + s\left([[L(a), 1_{\mathcal{B}}], m] + [[a, L(1_{\mathcal{B}})], m] + [[a, 1_{\mathcal{B}}], L(m)]\right).$$

Each component on the right side is computed as follows.

$$\begin{aligned} &[[L(a), m], 1_{\mathcal{B}}] = [[\alpha(a) + \mu(a) + \beta(a), m], 1_{\mathcal{B}}] = \alpha(a)m - m\beta(a), \\ &[[a, L(m)], 1_{\mathcal{B}}] = aL(m) \text{ by Lemma 14}, \\ &[[a, m], L(1_{\mathcal{B}})] = [am, \alpha(1_{\mathcal{B}}) + \mu(1_{\mathcal{B}}) + \beta(1_{\mathcal{B}})] = [am, \mu(1_{\mathcal{B}})] = 0 \text{ by Lemma 15}, \end{aligned}$$

$$\begin{aligned} &[[L(a), 1_{\mathcal{B}}], m] = [[\alpha(a) + \mu(a) + \beta(a), 1_{\mathcal{B}}], m] = [\mu(a), m] = 0, \\ &[[a, L(1_{\mathcal{B}})], m] = 0 \text{ since } [a, L(1_{\mathcal{B}})] = [a, \mu(1_{\mathcal{B}})] \in \mathcal{M} \text{ by Lemma 15}, \\ &[[a, 1_{\mathcal{B}}], L(m)] = [0, L(m)] = 0. \end{aligned}$$

Then $rL(am) = L(ram) = r\left(\alpha(a)m - m\beta(a) + aL(m)\right)$. Since r is regular,

$$L(am) = \alpha(a)m - m\beta(a) + aL(m).$$

Similarly, with $x = b$, $y = m$, and $z = 1_{\mathcal{A}}$, we get

$$L(mb) = m\beta(b) - \alpha(b)m + L(m)b.$$

Now we compute $L(amb)$ in two ways. First,

$$\begin{aligned} L(a(mb)) =& \alpha(a)mb - mb\beta(a) + aL(mb) \\ =& \alpha(a)mb - mb\beta(a) + am\beta(b) - a\alpha(b)m + aL(m)b. \end{aligned}$$

Similarly,

$$\begin{aligned} L((am)b) =& am\beta(b) - \alpha(b)am + L(am)b \\ =& am\beta(b) - \alpha(b)am + \alpha(a)mb - m\beta(a)b + aL(m)b. \end{aligned}$$

Comparing two results, we obtain

$$\alpha(b)am - a\alpha(b)m = mb\beta(a) - m\beta(a)b$$

or

$$[\alpha(b), a]m = m[b, \beta(a)],$$

which leads to $[\alpha(b), a] + [b, \beta(a)] \in \mathcal{Z}(\mathcal{T})$. Then, $[\alpha(b), a] \in \mathcal{Z}(\mathcal{A})$ and $[b, \beta(a)] \in \mathcal{Z}(\mathcal{B})$. Since a and b are arbitrary, $\alpha(b) \in \mathcal{Z}(\mathcal{A}), \beta(a) \in \mathcal{Z}(\mathcal{B})$ by condition (♠). □

The consequence of the lemma is that $\alpha(b) \in \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T}))$ and $\beta(a) \in \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$ by condition (♣). Therefore, $\alpha\pi_{\mathcal{B}} + \phi\alpha\pi_{\mathcal{B}}$ and $\phi^{-1}\beta\pi_{\mathcal{A}} + \beta\pi_{\mathcal{A}}$ are maps into the center $\mathcal{Z}(\mathcal{T})$. Now we define $h : T \to \mathcal{Z}(\mathcal{T})$ to be the sum of those central maps

$$h = \alpha\pi_B + \phi\alpha\pi_B + \phi^{-1}\beta\pi_A + \beta\pi_A,$$

and define $d = L - h$. It remains to prove that h vanishes on $f(x, y, z)$ and d is a derivation.

Lemma 17. *$h(f(x, y, z)) = 0$ for any $x, y, z \in \mathcal{T}$*

Proof. By Lemma 1(10), $\pi_{\mathcal{B}}(f(x, y, z)) = f(\pi_{\mathcal{B}}x, \pi_{\mathcal{B}}y, \pi_{\mathcal{B}}z)$. Then

$$\begin{aligned}
\alpha\pi_{\mathcal{B}}(f(x, y, z)) &= \pi_{\mathcal{A}}L\pi_{\mathcal{B}}(f(x, y, z)) \\
&= \pi_{\mathcal{A}}L(f(\pi_{\mathcal{B}}x, \pi_{\mathcal{B}}y, \pi_{\mathcal{B}}z)) \\
&= \pi_{\mathcal{A}}(f(L\pi_{\mathcal{B}}x, \pi_{\mathcal{B}}y, \pi_{\mathcal{B}}z) + f(\pi_{\mathcal{B}}x, L\pi_{\mathcal{B}}y, \pi_{\mathcal{B}}z) + f(\pi_{\mathcal{B}}x, \pi_{\mathcal{B}}y, L\pi_{\mathcal{B}}z)) \\
&= f(\pi_{\mathcal{A}}L\pi_{\mathcal{B}}x, 0, 0) + f(0, \pi_{\mathcal{A}}L\pi_{\mathcal{B}}y, 0) + f(0, 0, \pi_{\mathcal{A}}L\pi_{\mathcal{B}}z) \\
&= 0.
\end{aligned}$$

Similarly, other components of $h(f(x, y, z))$ are zeros as well. □

Lemma 18. *d has the following properties for any $a \in \mathcal{A}$, $m \in \mathcal{M}$, and $b \in \mathcal{B}$.*

1. $d(1_{\mathcal{T}}) = 0$,
2. $d(m) = L(m) \in \mathcal{M}$,
3. $d(a) \in \mathcal{A} + \mathcal{M}$,
4. $d(b) \in \mathcal{B} + \mathcal{M}$.

Proof. First, by Lemma 13,

$$L(1_{\mathcal{T}}) = \alpha(1_{\mathcal{T}}) + \beta(1_{\mathcal{T}}) = \alpha(1_{\mathcal{A}}) + \alpha(1_{\mathcal{B}}) + \beta(1_{\mathcal{A}}) + \beta(1_{\mathcal{B}}).$$

By Lemma 15, $\phi\alpha(1_{\mathcal{B}}) = \beta(1_{\mathcal{B}})$ and $\phi^{-1}\beta(1_{\mathcal{A}}) = \alpha(1_{\mathcal{A}})$, thus,

$$\begin{aligned}
h(1_{\mathcal{T}}) &= \alpha(1_{\mathcal{B}}) + \phi\alpha(1_{\mathcal{B}}) + \phi^{-1}\beta(1_{\mathcal{A}}) + \beta(1_{\mathcal{A}}) \\
&= \alpha(1_{\mathcal{B}}) + \beta(1_{\mathcal{B}}) + \alpha(1_{\mathcal{A}}) + \beta(1_{\mathcal{A}}).
\end{aligned}$$

Therefore, $d(1_{\mathcal{T}}) = 0$. Second, since $h(m) = 0$ by definition, $d(m) = L(m) \in \mathcal{M}$ by Lemma 14. Lastly,

$$\begin{aligned}
\pi_B d(a) &= \pi_B L(a) - \pi_B h(a) \\
&= \beta(a) - \pi_B(\phi^{-1}\beta(a) + \beta(a)) \\
&= \beta(a) - \beta(a) = 0.
\end{aligned}$$

Thus $d(a) \in \mathcal{A} + \mathcal{M}$. Similarly, $d(b) \in \mathcal{B} + \mathcal{M}$.

□

To prove that d is a derivation, we prove that it is a derivation component-wise, then we combine the results together.

Lemma 19. *$d(am) = d(a)m + ad(m)$ and $d(mb) = d(m)b + md(b)$ for any $a \in \mathcal{A}$, $m \in \mathcal{M}$, and $b \in \mathcal{B}$.*

Proof. As in the proof of Lemma 16,

$$L(am) = \alpha(a)m - m\beta(a) + aL(m)$$

and by Lemma 18 (2),

$$d(am) = \alpha(a)m - m\beta(a) + ad(m).$$

On the other hand,

$$\begin{aligned} d(a)m &= \pi_{\mathcal{A}} d(a)m && \text{(by Lemma 18 (3))} \\ &= \pi_{\mathcal{A}} L(a)m - \pi_{\mathcal{A}} h(a)m \\ &= \alpha(a)m - \pi_{\mathcal{A}}(\phi^{-1}\beta(a) + \beta(a))m \\ &= \alpha(a)m - \phi^{-1}\beta(a)m \\ &= \alpha(a)m - m\beta(a). && \text{(by Lemma 3)} \end{aligned}$$

Hence, $d(am) = d(a)m + ad(m)$. Similarly, $d(mb) = d(m)b + md(b)$.

□

Lemma 20. *$d(a_1a_2) = d(a_1)a_2 + a_1d(a_2)$ and $d(b_1b_2) = d(b_1)b_2 + b_1d(b_2)$ for any $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.*

Proof. By the previous lemma,

$$d((a_1a_2)m) = d(a_1a_2)m + a_1a_2d(m).$$

We also have

$$\begin{aligned} d(a_1(a_2m)) &= d(a_1)a_2m + a_1d(a_2m) \\ &= d(a_1)a_2m + a_1d(a_2)m + a_1a_2d(m). \end{aligned}$$

Comparing two results, we have

$$d(a_1a_2)m = (d(a_1)a_2 + a_1d(a_2))m.$$

By the faithfulness of $\mathcal{M}$,

$$d(a_1a_2) = d(a_1)a_2 + a_1d(a_2).$$

Similarly,

$$d(b_1b_2) = d(b_1)b_2 + b_1d(b_2).$$

□

Lemma 21. *$d(a)b + ad(b) = 0$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Proof. Obviously, $f(a, b, 1_{\mathcal{B}}) = r[[a, b], 1_{\mathcal{B}}] + s[[a, 1_{\mathcal{B}}], b] = 0$. Then

$$\begin{aligned} 0 = L(f(a, b, 1_{\mathcal{B}})) &= f(L(a), b, 1_{\mathcal{B}}) + f(a, L(b), 1_{\mathcal{B}}) + f(a, b, L(1_{\mathcal{B}})) \\ &= f(d(a), b, 1_{\mathcal{B}}) + f(a, d(b), 1_{\mathcal{B}}) + f(a, b, d(1_{\mathcal{B}})) \\ &\quad + f(h(a), b, 1_{\mathcal{B}}) + f(a, h(b), 1_{\mathcal{B}}) + f(a, b, h(1_{\mathcal{B}})) \\ &= f(d(a), b, 1_{\mathcal{B}}) + f(a, d(b), 1_{\mathcal{B}}) + f(a, b, d(1_{\mathcal{B}})) \end{aligned}$$

since h is a central map. We compute each component of the last line one by one. Note that for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $bd(a) = 0$, $d(b)a = 0$, and $d(b)1_{\mathcal{B}} = d(b)$ by Lemma 18 (3) and (4). With this,

$$\begin{aligned} f(d(a), b, 1_{\mathcal{B}}) &= r[[d(a), b], 1_{\mathcal{B}}] + s[[d(a), 1_{\mathcal{B}}], b] \\ &= r[d(a)b, 1_{\mathcal{B}}] + s[d(a)1_{\mathcal{B}}, b] \\ &= (r + s)d(a)b, \\ f(a, d(b), 1_{\mathcal{B}}) &= r[[a, d(b)], 1_{\mathcal{B}}] + s[[a, 1_{\mathcal{B}}], d(b)] \\ &= r[ad(b), 1_{\mathcal{B}}] \\ &= rad(b), \\ f(a, b, d(1_{\mathcal{B}})) &= r[[a, b], d(1_{\mathcal{B}})] + s[[a, d(1_{\mathcal{B}})], b] \\ &= sad(1_{\mathcal{B}})b. \end{aligned}$$

As $d(b) = d(1_{\mathcal{B}}b) = d(1_{\mathcal{B}})b + 1_{\mathcal{B}}d(b)$ by the previous lemma, the last one is equal to $sa(d(b) - 1_{\mathcal{B}}d(b)) = sad(b)$. Therefore, taking the sum of three equations, we have $(r + s)(d(a)b + ad(b)) = 0$. Since $r + s$ is regular, $d(a)b + ad(b) = 0$. □

Lemma 22. *$d(t_1t_2) = d(t_1)t_2 + t_1d(t_2)$ for any $t_1, t_2 \in \mathcal{T}$.*

Proof. Let $t_1 = a_1 + m_1 + b_1$ and $t_2 = a_2 + m_2 + b_2$. Then, by Lemma 19 and Lemma 20,

$$\begin{aligned} d(t_1t_2) =& d((a_1 + m_1 + b_1)(a_2 + m_2 + b_2)) \\ =& d(a_1a_2 + a_1m_2 + m_1b_2 + b_1b_2) \\ =& d(a_1)a_2 + a_1d(a_2) + d(a_1)m_2 + a_1d(m_2) \\ &+ d(m_1)b_2 + m_1d(b_2) + d(b_1)b_2 + b_1d(b_2). \end{aligned}$$

On the other hand, by Lemma 18 and Lemma 21,

$$\begin{aligned} d(t_1)t_2 + t_1d(t_2) =& (d(a_1) + d(m_1) + d(b_1))(a_2 + m_2 + b_2) \\ &+ (a_1 + m_1 + b_1)(d(a_2) + d(m_2) + d(b_2)) \\ =& d(a_1)a_2 + d(a_1)m_2 + d(a_1)b_2 + d(m_1)b_2 + d(b_1)b_2 \\ &+ a_1d(a_2) + a_1d(m_2) + a_1d(b_2) + m_1d(b_2) + b_1d(b_2) \\ =& d(a_1)a_2 + d(a_1)m_2 + d(m_1)b_2 + d(b_1)b_2 \\ &+ a_1d(a_2) + a_1d(m_2) + m_1d(b_2) + b_1d(b_2). \end{aligned}$$

Therefore, $d(t_1t_2) = d(t_1)t_2 + t_1d(t_2)$. □

This completes the proof of Theorem 12.

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